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Phase transitions in systems of coupled phase oscillators

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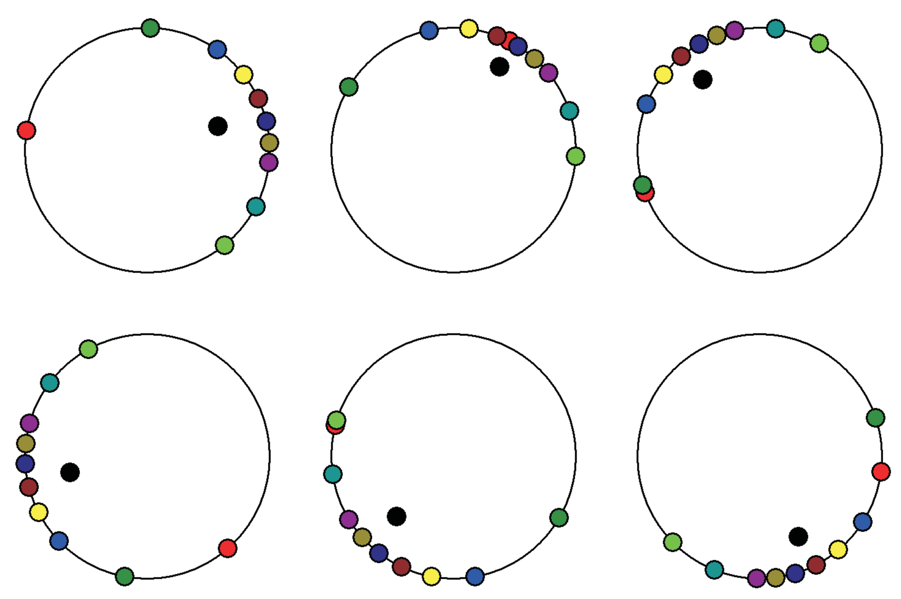
Coupled phase oscillators are used widely in describing cooperative phenomena in physics, biology, chemistry, engineering, and other fields. We study the models of Winfree and Kuramoto of synchronization of phase oscillators. Both models show that for typical distribution of natural frequencies a synchronous behaviour emerges when coupling intensity between oscillators exceeds certain value. A coherence is suitably described with an order parameter. The order parameter attains non-zero value for couplings stronger than the critical, thus making the onset of synchronization a phase transition.

For the Kuramoto model the phase transition is of first or second order depending on the type of distribution function of the natural frequencies of the oscillators. The transition is of first order when the distribution has a plateau where the seed of the synchronized cluster is formed. The exponents characterizing the dependence of the order parameter on the coupling strength are derived analytically for both first- and second-order phase transitions.

We also consider an analytically solvable version of the Winfree model of synchronization of phase oscillators. It is obtained that the transition from incoherence to partial death state is characterized by third or even higher order phase transitions according to Ehrenfest classification. The order depends on the type of distribution function of natural frequencies of the oscillators. The corresponding critical exponents are found analytically and confirmed numerically. The transition to partial death is considered also in more general setting when the interaction intensity depends on the Kuramoto order parameter r as Kr^{z-1}, where z is an additional parameter. If z is smaller than some particular value z_c dependent on the distribution of natural frequencies of the oscillators, the critical exponents remain unchanged. For $z>z_c$ there is a first order transition with hysteresis.

Introduction

Cooperative behaviour is ubiquitous in nature at different scales, ranging from coordinated pulsing of heart or circadian pacemaker cells, to populations of flashing fireflies or chorusing crickets, to unison clapping of theatre audience [1,2,3]. The models of Winfree and Kuramoto are paradigmatic models for studying the synchronization phenomena [4,5]. The models consider interacting units described with phase oscillators. When the coupling strength is weaker than the threshold value, the population is incoherent. Coherence emerges as a phase transition when the coupling intensity exceeds the critical value. The dynamics of the oscillators can be visualized as a population of small dots running around the circle.



Population of ten coupled oscillators, evolving according to the Kuramoto model. The black circle represents the order parameter.

The Kuramoto Model

The Kuramoto model considers population of N coupled phase oscillators with phase dynamics [5,6]

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i).$$

The natural frequencies are distributed according to an asymmetric unimodal (single-peaked) function. Phase coherence is described with the order parameter

$$re^{i\psi} = \frac{1}{N}\sum_{j=1}^{N} e^{i\theta_j}.$$

In an appropriately selected reference frame, drifting with some velocity ω_0, the order parameter has constant amplitude r and phase zero. Phase dynamics equations in the drifting reference frame are

$$\dot{\theta}_i = \omega_i - Kr\sin\theta_i .$$

In the limit $N\to\infty$ the population is more conveniently described with number density function $\rho(\omega,\theta,t)$ of oscillators with natural frequency ω and phase θ at the instant t. The order parameter is calculated from the density with an integral. By conservation principle, density evolves according to continuity equation

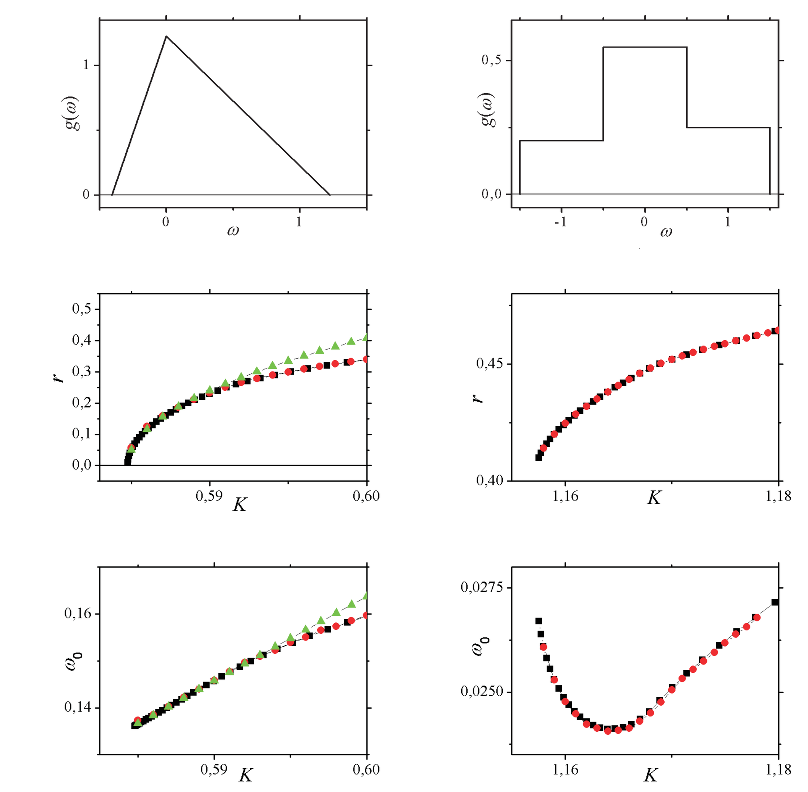
$$\frac{\partial\rho}{\partial t} + \frac{\partial[\rho(\omega - Kr\sin\theta)]}{\partial\theta} = 0.$$

From stationary solutions of the continuity equation one can obtain the frequency of synchronization ω_0 and the order parameter of the system. For asymmetric distribution of natural frequencies $g(\omega)$ and particular value of the coupling constant they can be calculated from the following self-consistent pair of equations [6]

$$r = \int_{-Kr}^{Kr} d\omega\, g(\omega+\omega_0)\sqrt{1-\left(\frac{\omega}{Kr}\right)^2},$$

$$0 = \int_{-\infty}^{\infty} d\omega\, g(\omega+\omega_0)\,\omega + \int_{-\infty}^{-Kr} d\omega\, g(\omega+\omega_0)\sqrt{\omega^2-(Kr)^2} - \int_{Kr}^{\infty} d\omega\, g(\omega+\omega_0)\sqrt{\omega^2-(Kr)^2}.$$

For the second-order phase transitions the order parameter grows continuously from zero at the critical coupling. First-order phase transitions appear when the distribution function has a flat section and the seed of synchronized cluster is formed inside that section.



Triangular and olympic distributions of natural frequencies, with corresponding graphs of $r(K)$ and $\omega_0(K)$ dependence. For the triangular distribution the transition is of second and for the olympic distribution the transition is of first order. The notation is following: analytical (black), asymptotic (green), and numerical (red).

Asymptotic relationships for the order parameter and frequency of synchronization are summarized in the following table [7, 8].

Critical exponents of the phase transitions in the Kuramoto model. The parameter m for second-order transitions is the degree of the first non-zero term in the Taylor expansion of $g(\omega)$ at the frequency of synchronization. For first-order transition it is the degree of the polynomial fall-off of the distribution in immediate vicinity of the plateau.

First – order transitions		
	Symmetric $g(\omega)$	Asymmetric $g(\omega)$
Sync frequency	-	$2 / (2m+3)$
Order parameter	$2 / (2m+3)$	$2 / (2m+3)$
Second – order transitions		
	Symmetric $g(\omega)$	Asymmetric $g(\omega)$
Sync frequency	-	1
Order parameter	$1/m$	1/2

The Winfree model

The phases of the oscillators in the Winfree model evolve according to the equation [9]

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N} P(\theta_j) R(\theta_i).$$

For general influence $P(\theta)$ and response function $R(\theta)$, the Winfree model is hard to analyze mathematically. Some analytical calculations can be done for the following choice of interaction functions [9]

$$P(\theta) = 1+\cos\theta, \quad R(\theta) = -\sin\theta .$$

The order parameter X and the effective coupling σ are defined as

$$X = \frac{1}{N}\sum_{j=1}^{N}\cos\theta_j, \qquad \sigma = \frac{K}{N}\sum_{j=1}^{N}(1+\cos\theta_j),$$

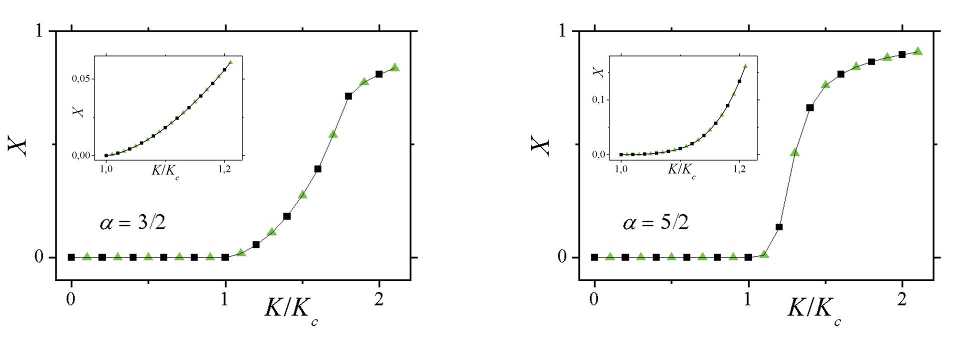
The natural frequencies are considered to be positive and symmetrically distributed around the mean. For different coupling strengths and different natural frequency dispersions, in the phase portrait exist incoherence, synchronization, oscillation death, and partially synchronized states. With similar reasoning as for the Kuramoto model it can be shown that in partial death state the order parameter is given by

$$X = \int_{\omega_{\min}}^{\sigma} d\omega\, g(\omega)\sqrt{1-\frac{\omega^2}{\sigma^2}}.$$

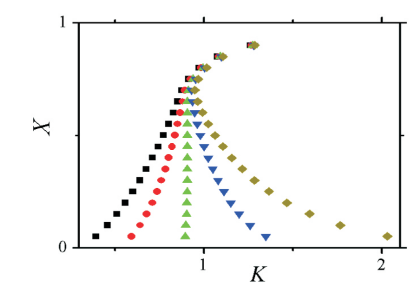
The phase transition from incoherence to partial death state is of third or higher order. The order parameter growth in vicinity of the critical point scales as

$$X \propto (K-K_c)^{\alpha}.$$

The value of the critical exponent α depends on the properties of the distribution function in immediate vicinity of the lowest frequency. The exponent α has the same value even for more general coupling among the oscillators defined as Kr^{z-1}, where r is the Kuramoto order parameter and z is an additional parameter smaller than some critical value z_c.



Order parameter X versus coupling parameter K for uniform and triangular shaped distributions of natural frequencies, numerical (black) and analytical(green) results.



Order parameter X as a function of the coupling parameter K for the generalized Winfree model for uniform distribution of natural frequencies $g(\omega)=1$; $0.5\le\omega\le 1.5$. Going from the leftmost curve to the right, the values of the parameter z are 0.5, 1, 1.5, 2, and 2.5.

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