Detecting stickiness in nonintegrable Hamiltonian systems

Marcus W. Beims[1,2], Cesar Manchein[1] and JM Rost[2]

[1]Departamento de Física
Universidade Federal do Paraná, Curitiba, Brazil

[2]Max-Planck Institute for the Physics of Complex Systems
Dresden, Germany

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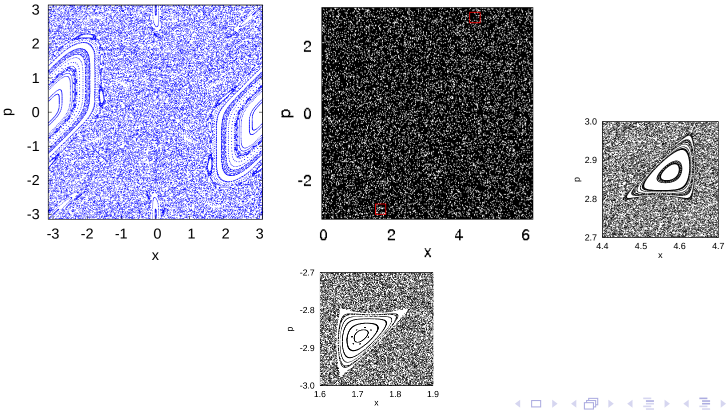
Outline

- Introduction
  - Motivation
  - Finite time Lyapunov exponents
- Model
  - Standard map ($2D$)
  - Coupled Standard maps ($4D \longrightarrow 20D$)
- Numerical results

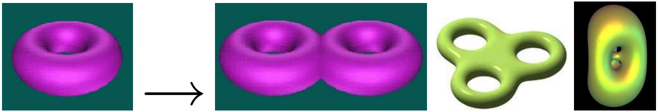
Stickiness in 2*D*

- Characterization of quasi-regular (stickiness) and chaotic motion in two dimensional Hamiltonian systems.



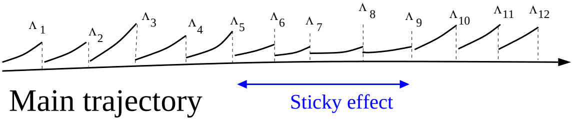
Stickiness in higher dimensions ?

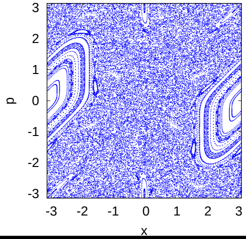
- Goal: Characterize the sticky motion in higher dimensions via the distribution of the finite time Lyapunov exponents.
- Anomalous transport of particles.
- How is the nonlinearity distributed along the different unstable directions ?

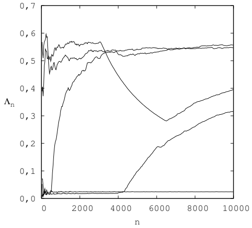


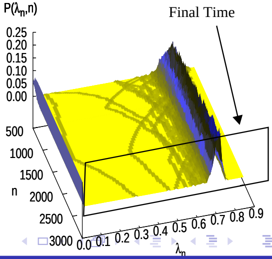
Finite Time Lyapunov Exponents (FTLEs)

$$\Lambda_n(\mathbf{x}) = \frac{1}{n} \log \parallel M(\mathbf{x}, n)\mathbf{e}(\mathbf{x}) \parallel,$$









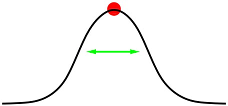
Four quantities to detect the sticky motion

- **(1)** P_Λ **Number of occurrencies** of the most probably FTLE. Sticky motion $\rightarrow P_\Lambda < 1$. [MWB, C Manchein and JM Rost, PRE (2007).]

$$\left.\frac{\partial P(\Lambda_n, K)}{\partial \Lambda_n}\right|_{\Lambda_n = \Lambda_n^p} = 0.$$

- **(2) Variance** σ: Increases at the sticky motion [Grassberger and Kantz, PLA (1987).]

$$\tilde{\sigma} = \left\langle (\Lambda_n - \langle \Lambda_n \rangle)^2 \right\rangle,$$

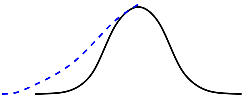


Four quantities to detect the sticky motion.

- **(3) Third Cummulant (Asymmetry)**

$$\kappa_3 = \frac{\left\langle (\Lambda_n - \langle \Lambda_n \rangle)^3 \right\rangle}{\tilde{\sigma}^{3/2}}.$$

$\kappa_3 = 0$: for normal distribution.
$\kappa_3 > 0$: asymmetric to the right.
$\kappa_3 < 0$: asymmetric to the left (sticky motion).



Four quantities to detect the sticky motion.

- **(4) Forth Cummulant (Flatness)**

$$\kappa_4 = \frac{\langle (\Lambda_n - \langle \Lambda_n \rangle)^4 \rangle}{\tilde{\sigma}^2} - 3.$$

$\kappa_4 = 0$: for regular distribution.
$\kappa_4 < 0$: for peakness.
$\kappa_4 > 0$: for flatness (sticky motion).

Higher cummulants FTLEs [Tomsovic and Lakshminarayan, PRE (2007); Froeschlé etal, Celestial Mechanics (1993).]

Our numerical investigation

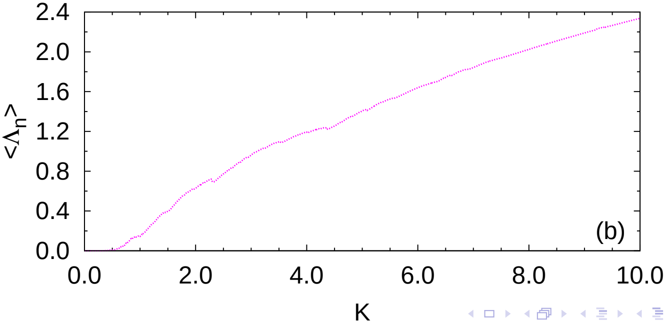
- 10^4 initial conditions in the whole phase space and $n = 10^7$ iterations.
- FTLEs spectrum [Benettin, Meccanica (1980)].
- What remains from the sticky effect on the FTLEs distribution after such times?

Standard Map

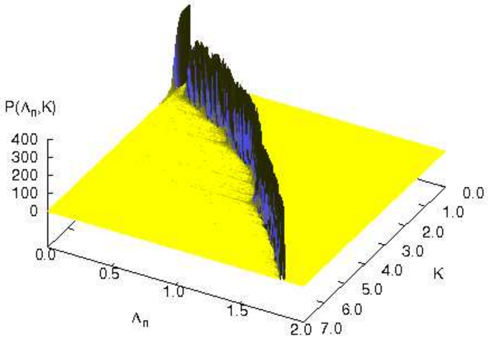
- The 2-dimensional model used here is the standard map

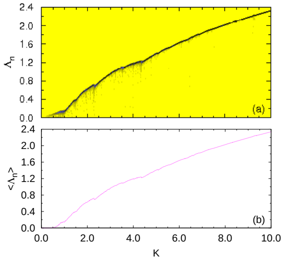
$$p_{n+1} = p_n + K \sin(q_n), \qquad \text{mod} 2\pi$$

$$q_{n+1} = q_n + p_{n+1}, \qquad \text{mod} 2\pi.$$

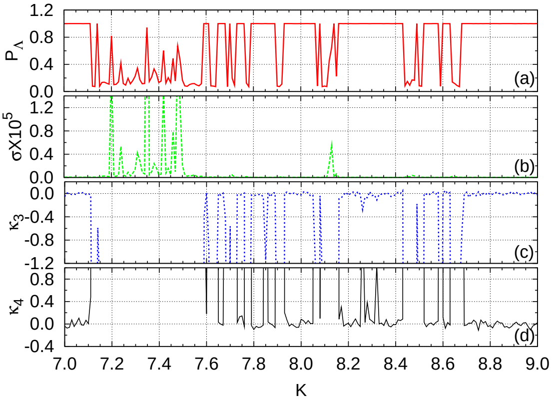


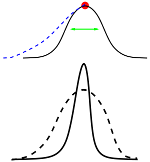
2-dimensional case





2-dimensional case





P_Λ, κ_3 and κ_4 are very sensitive to detect the sticky motion.

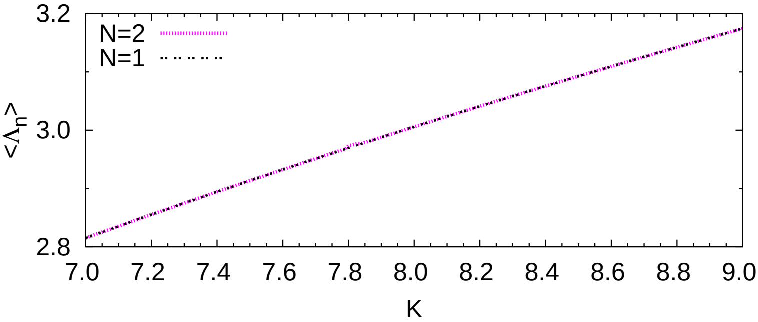
Coupled Standard Maps

- For higher dimensions we use the N coupled standard maps

$$
\begin{aligned}
p_{n+1}^{(i)} &= p_n^{(i)} + K\ \sin(q_n^{(i+1)} - q_n^{(i)}), \\
q_{n+1}^{(i)} &= q_n^{(i)} + p_{n+1}^{(i)},
\end{aligned}
$$

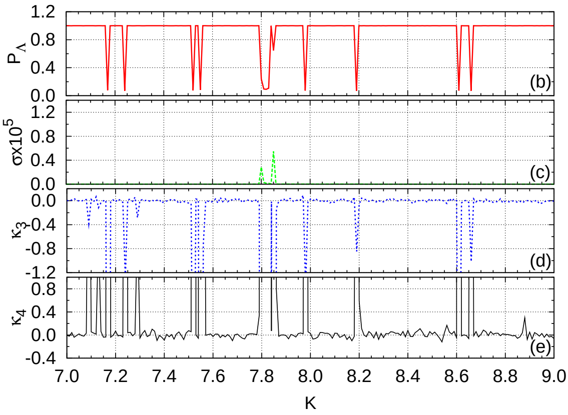
where $i = 1, \dots, N$, using periodic boundary conditions $p_{N+1} = p_1,\ q_{N+1} = q_1$ and next nearest neighbor coupling (unidirectional). Hamiltonian and translational invariance (total momentum is conserved).

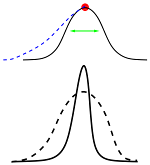
$N = 2$, 4-dimensional case - One positive FTLE



$$p_{\xi_{n+1}} = p_{\xi_n} + (2K)\sin(\xi_n), \quad \xi_{n+1} = \xi_n + p_{\xi_{n+1}}.$$

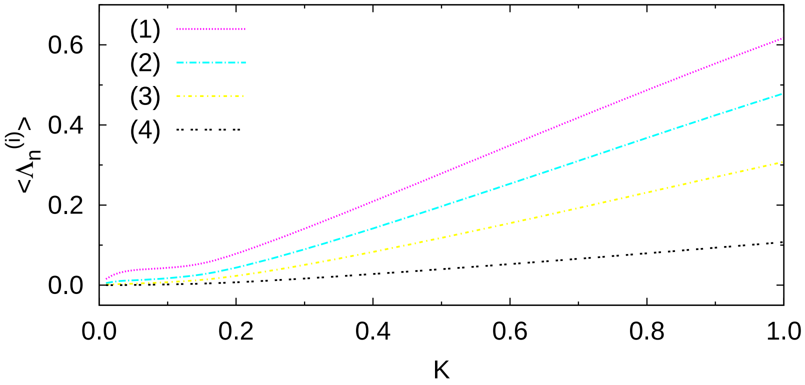
$N = 2$, 4-dimensional case - One positive FTLEs



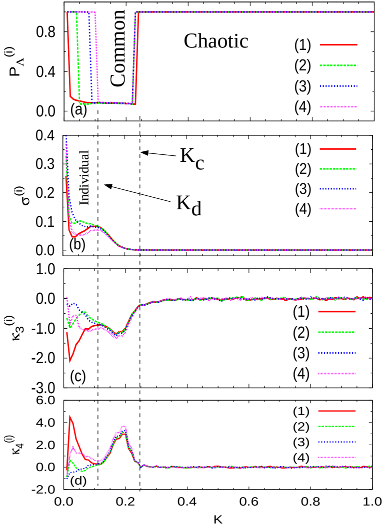


- Amount of sticky motion decreases with increasing K.

$N = 5, 10$-dimensional case - Four positive FTLEs



$N = 5$, 10-dimensional case - Four positive FTLEs



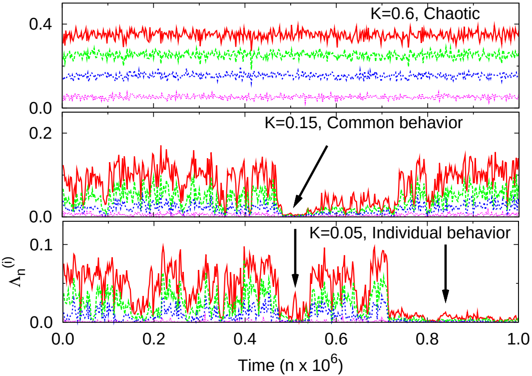
- $K_c \sim 0.25$ - largest FTLE
- $K_d \sim 0.13$- smallest FTLE.

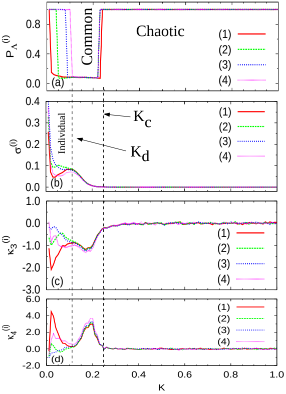
Three distinct regions:

1. $K > K_c$ (no sticky motion in all directions).
2. $K_d \lesssim K \lesssim K_c$ (sticky motion with common behavior).
3. $K < K_d$ (sticky motion with individual behavior).

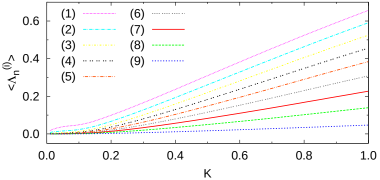
P_Λ contains more informations.

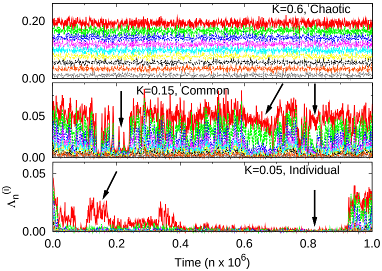
Time evolution

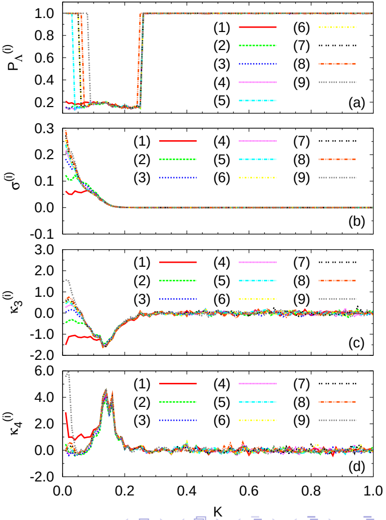




$N = 10$, 20-dimensional case - Nine positive FTLEs







Conclusions

- Suitable quantities to detect sticky motion: P_Λ and higher cummulants of the FTLEs distribution (Variance, Asymmetry and **Flatness**).
- Higher cummulants give essentially the same information. P_Λ gives additional informations.
- largest FTLE defines the limit where no sticky motion occurs.
- smallest FTLE defines the limit where a common effect of the sticky motion occurs in all unstable directions.
- **Acknowledgments**: CNPq and FINEP (under project CTINFRA-1).