Internal and external synchronization of self-oscillating oscillators with non-identical control parameters

Emelianova Yu.P., Kuznetsov A.P.

Department of Nonlinear Processes, Saratov State University, Russia
Institute of Radio-Engineering & Electronics, Russian Academy of Sciences,

yuliaro@mail.ru www.sgtnd.narod.ru

The system of two coupled van der Pol oscillators is the basic model of nonlinear dynamics demonstrating the phenomenon of mutual synchronization. There are many papers on this theme, because this system demonstrates a lot of interesting oscillation regimes and types of behavior, such as synchronous and quasiperiodic regimes, the "oscillator death" effect, etc.: A. Pikovsky, M. Rosenblum, J. Kurths [2001], D. G. Aronson , G. B. Ermentrout, N. Kopell [1990], R. H. Rand, P. J. Holmes [1980], D. W. Storti, R. H. Rand [1982], M. V. Ivanchenko, G. V. Osipov, V. D. Shalfeev, J. Kurths [2004]. In this work we carry out the investigation of the parameter space structure and of possible oscillation regimes in the dissipatively coupled van der Pol oscillators with non-identical controlling parameters.

1. Internal synchronization

The system describing the interaction between two van der Pol oscillators

$$\begin{aligned} &\ddot{x}-(\lambda_1 - x^2)\dot{x}+x+\mu(\dot{x}-\dot{y})=0, \\ &\ddot{y}-(\lambda_2 - \gamma y^2)\dot{y}+(1+\delta)y+\mu(\dot{y}-\dot{x})=0, \end{aligned} \tag{1}$$

λ_1 and λ_2 are controlling parameters in autonomous oscillators; γ is parameter of nonlinear dissipation; δ is the frequency mismatch between the autonomous second and first oscillators; μ is the coefficient of dissipative coupling.

1.1. Investigation by means of the method of dynamic regime chart construction

We use the method of dynamic regime chart construction. Within the framework of such a method we mark the oscillation period of the system of coupled oscillators by means of different colors on the parameter plane frequency mismatch δ – coupling value μ. White coloring corresponds to the chaotic or quasiharmonic motions. Cycle periods were calculated by means of the Poincare section method: this is the number of points of intersection of the phase trajectory on the attractor and the surface $\dot{y}=0$ selected as the Poincare section. Only those crossings were taken into account that correspond to the trajectories coming to the surface from the one side.

Boundary between the oscillator death area and the area of quasiperiodic regimes in case of non-identical in control parameters oscillators is not a line but a band of finite width in coupling parameter $\lambda_1 > \mu > \lambda_2$ which stretches infinitely into the area of increased frequency mismatch. We call an existence of synchronization in presence of arbitrarily large values of eigenfrequency mismatch as **"Broadband synchronization"**.

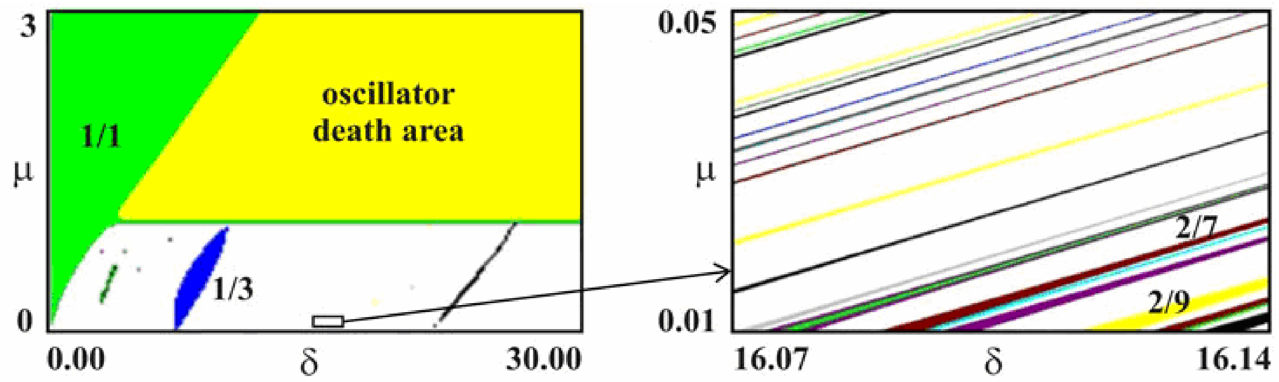
In case of distinctly different values of parameters λ_1 and λ_2 there is an explanation for the appearance of broadband synchronization. If μ exceeds both λ_1 and λ_2, both oscillators are behind the threshold of the "oscillator death" effect. In the range of

$\lambda_1 > \mu > \lambda_2$ only the 2-nd oscillator appears to be essentially dissipative. The 1-st oscillator appears to be leading and excites the 2-nd oscillator. So different scales along the coordinate axes on the phase plane portraits for the 1-st and the 2-nd oscillators are quite characteristic.

Non-identity of nonlinear dissipation results in specific form of the boundary of the main synchronization tongue, which looks like letter S.

Physical explanation. Oscillators are closely approximated by control parameters in area of small values of coupling parameter. Size of the limit cycle of the 2-nd oscillator is essentially greater than this one of the 1-st oscillator due to small relative nonlinear dissipation. So the 2-nd oscillator dominates over the 1-st one. Dissipative coupling causes damping of self-oscillations of the 2-nd oscillator with the transition across the line $\mu = \lambda_2$. Now the 1-st oscillator becomes the leading one. In region, where coupling parameter takes intermediate values, there is no leading oscillator. It is clearly seen from the phase portraits.

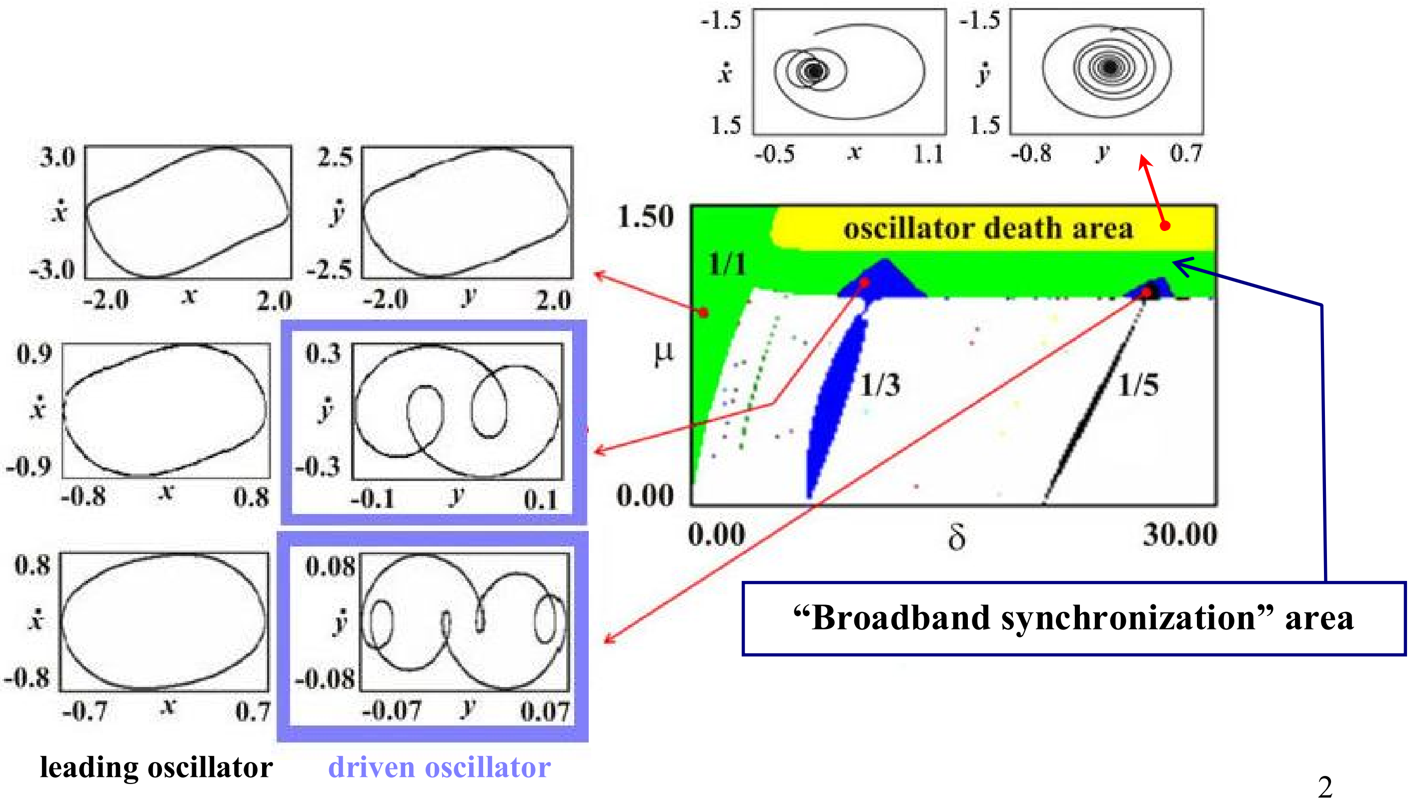
Case of identical oscillators $\lambda_1 = \lambda_2 = 1$, $\gamma = 1$



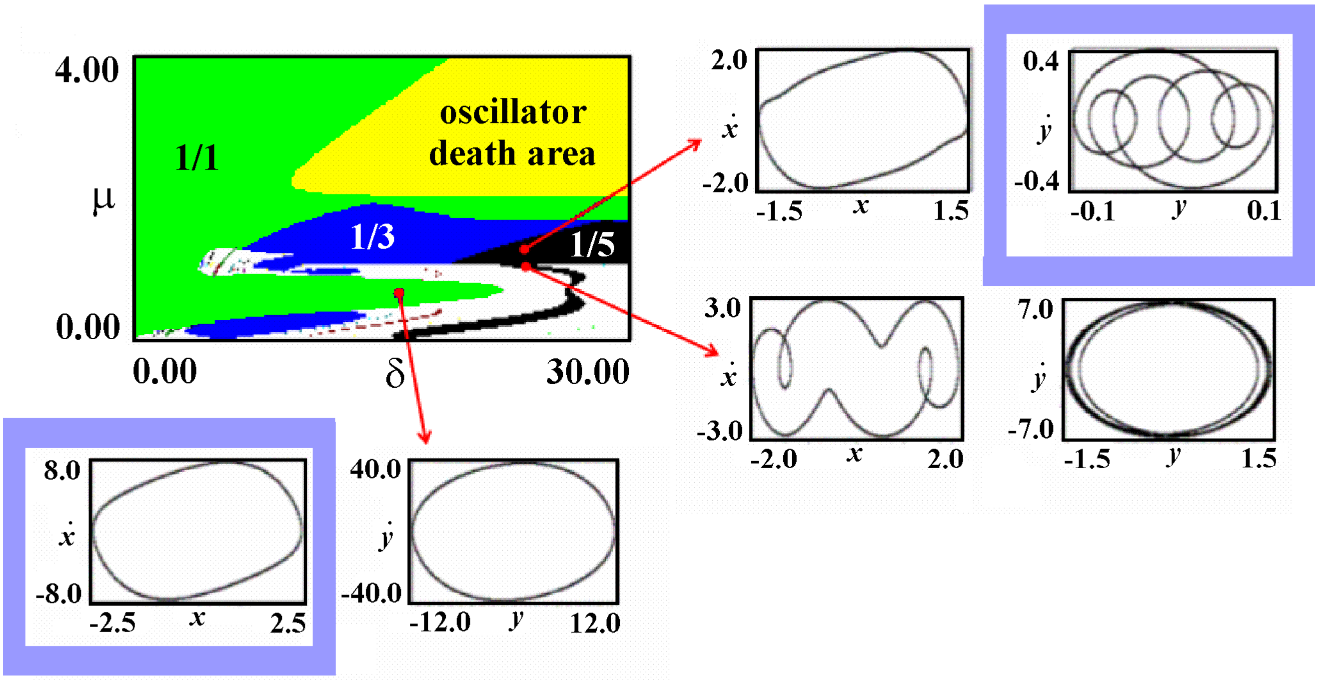
Case of non-identical oscillators

Non-identical parameters, controlling the Hopf bifurcation $\lambda_1 = 1.25$, $\lambda_2 = 1$,

Identical parameters of nonlinear dissipation $\gamma = 1$



Non-identical parameters of nonlinear dissipation $\gamma = 0.01$



driven oscillator leading oscillator

1.2. Analysis of the broadband synchronization by means of slow-flow equations

We define within the framework of quasiharmonic approximation:

$$x = \frac{1}{2}\left(ae^{it} + a^{*}e^{-it}\right), \quad y = \frac{1}{2}\left(be^{it} + b^{*}e^{-it}\right), \tag{2}$$

$$\frac{1}{2}\left(\dot{a}e^{it} + \dot{a}^{*}e^{-it}\right) = 0, \quad \frac{1}{2}\left(\dot{b}e^{it} + \dot{b}^{*}e^{-it}\right) = 0. \tag{3}$$

$$\tau = \frac{t}{2}, \; z = \frac{a}{2}, \; \omega = \frac{b}{2}, \tag{4}$$

The upper boundary of broadband synchronization area is the boundary of oscillator death area, so:

$$z \sim e^{i\varepsilon t}, \omega \sim e^{i\varepsilon t} \tag{5}$$

$$\Downarrow$$

$$\delta^2 = \frac{(\mu(\lambda_1 + \lambda_2) - \lambda_1\lambda_2)(\lambda_1 + \lambda_2 - 2\mu)^2}{(\lambda_1 - \mu)(\lambda_2 - \mu)}. \tag{6}$$

To obtain an expression for the lower boundary of broadband synchronization area we set

$$z(t) = R(t)e^{i\varphi_1}, \; \omega(t) = r(t)e^{i\varphi_2}, \; \psi = \varphi_2 - \varphi_1. \tag{7}$$

Then we obtain equations for amplitudes R, r of oscillators and their relative phase ψ:

$$\begin{aligned}
\frac{dR}{d\tau} &= R(\lambda_1-\mu)-R^3+\mu r\cos\psi,\\
\frac{dr}{d\tau} &= r(\lambda_2-\mu)-\gamma r^3+\mu R\cos\psi,\\
\frac{d\psi}{d\tau} &= \delta-\mu\left(\frac{r}{R}+\frac{R}{r}\right)\sin\psi.
\end{aligned} \tag{8}$$

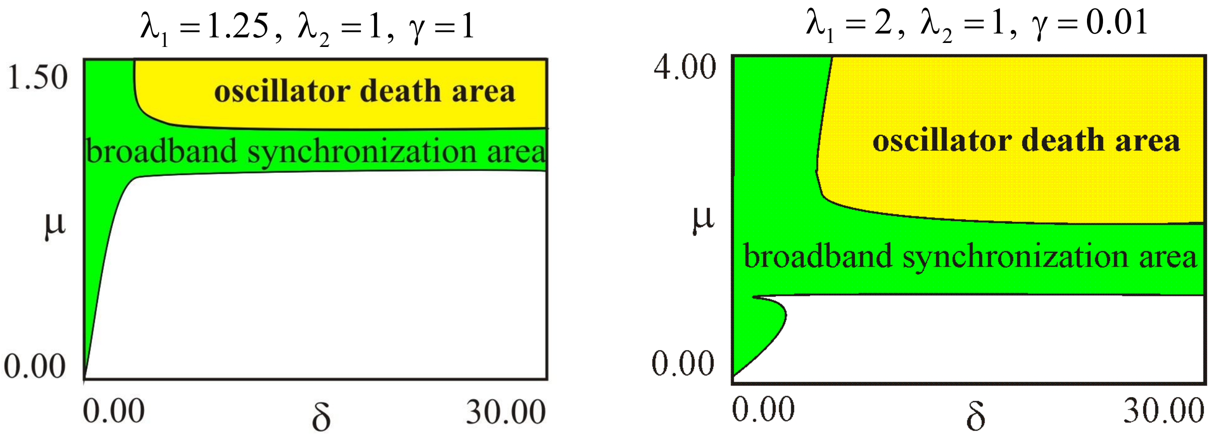
Steady amplitudes of limit cycles:

$$R\approx\sqrt{\lambda_1-\mu},\quad r\approx\sqrt{\frac{\lambda_2-\mu}{\gamma}}, \tag{9}$$

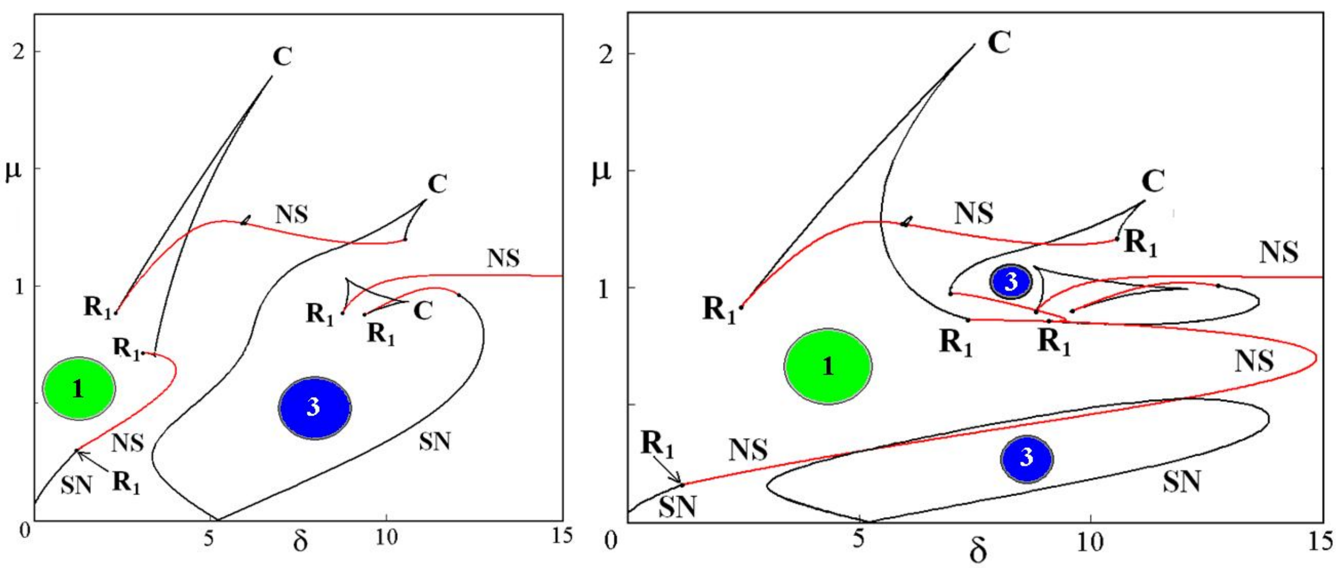
then the lower boundary of broadband synchronization area is

$$\boxed{\delta=\mu\sqrt{\frac{\lambda_2-\mu}{\gamma(\lambda_1-\mu)}}+\mu\sqrt{\frac{\gamma(\lambda_1-\mu)}{\lambda_2-\mu}}.} \tag{10}$$

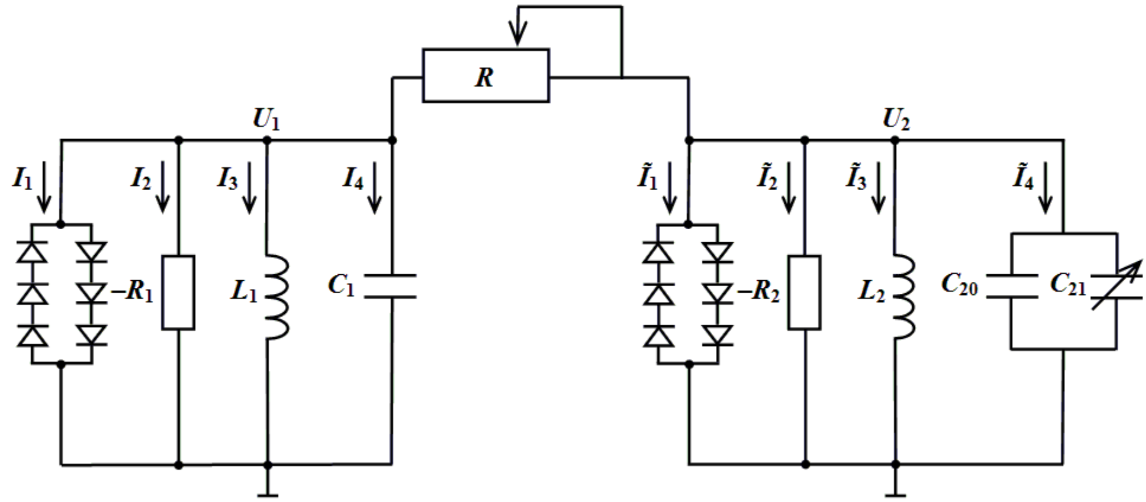
Analytic boundaries of the main synchronization area



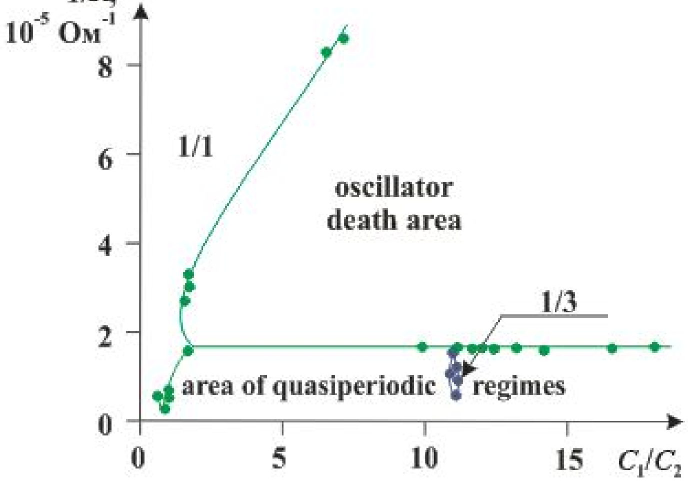
Bifurcation analysis



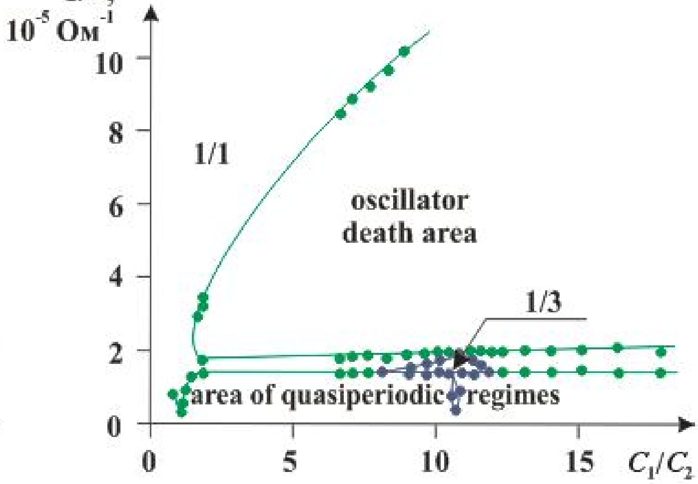
1.3. Experimental system
Non-identical parameters, controlling the Hopf bifurcation

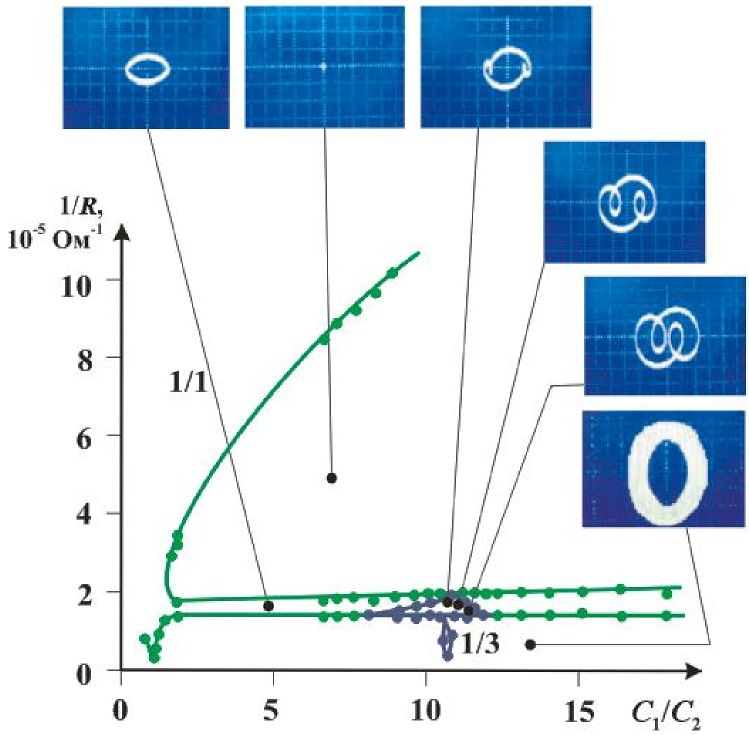


Case of identical oscillators



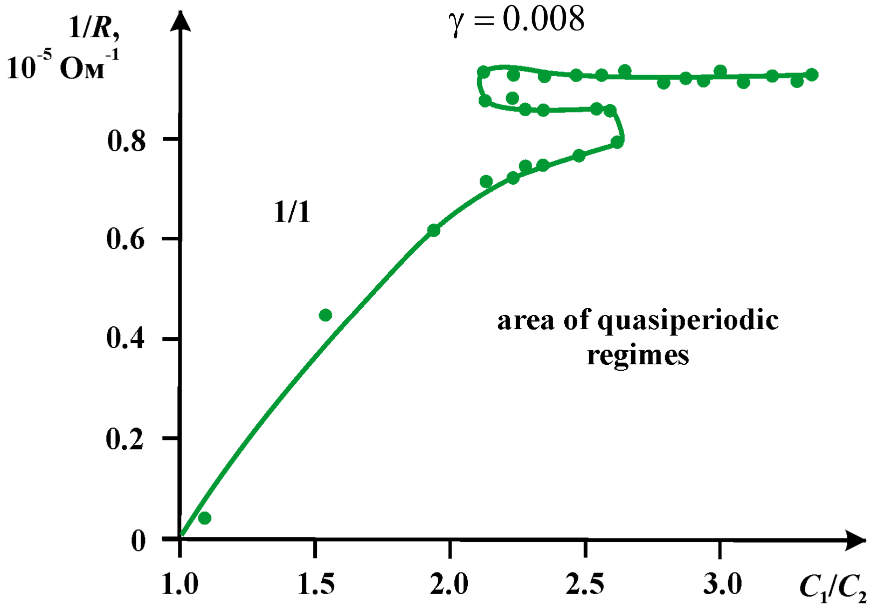
Case of non-identical oscillators





Non-identity in nonlinear dissipation

To provide different nonlinear dissipation in active oscillators we use different number of diodes in the active oscillators: 10 diodes in the 1-st generator and 2 diodes in the 2-nd generator.



2. External synchronization

Consider the system of two coupled van der Pol oscillators under external influence

$$\begin{aligned} &\ddot{x}-(\lambda_1-x^2)\dot{x}+\left(1-\Delta/2\right)x+\mu(\dot{x}-\dot{y})=B\sin\omega t,\\ &\ddot{y}-(\lambda_2-y^2)\dot{y}+\left(1+\Delta/2\right)y+\mu(\dot{y}-\dot{x})=0. \end{aligned} \quad (12)$$

λ_1 and λ_2 are parameters characterizing the excess above the threshold of the Andronov – Hopf bifurcation in autonomous oscillators; Δ is the frequency mismatch between the autonomous oscillators; μ is the coefficient of dissipative coupling; B is the amplitude of external influence; ω is the frequency of external influence.

2.1 Analysis by means of slow-flow equations

$$x = ae^{i\omega t} + a^* e^{-i\omega t}, \quad y = be^{i\omega t} + b^* e^{-\omega i t}, \quad (13)$$

$$\dot{a}e^{i\omega t} + \dot{a}^* e^{-i\omega t} = 0, \quad \dot{b}e^{i\omega t} + \dot{b}^* e^{-i\omega t} = 0, \quad (14)$$

$$a = \mathrm{R}e^{i\psi_1},\ b = re^{i\psi_2},\ \Omega = \omega - 1,\ \delta_1 = -\frac{\Delta}{4} - \Omega, \delta_2 = \frac{\Delta}{4} - \Omega,\ b = B/4. \quad (15)$$

Equations for amplitudes R, r of oscillators and their phases ψ_1, ψ_2:

$$\begin{aligned} &2\frac{dR}{dt} = R(\lambda_1-\mu) - R^3 + \mu r\cos(\psi_2-\psi_1) - 2b\cos\psi_1,\\ &2\frac{dr}{dt} = r(\lambda_2-\mu) - r^3 + \mu R\cos(\psi_1-\psi_2),\\ &\frac{d\psi_1}{dt} = \delta_1 + \frac{r}{2R}\mu\sin(\psi_2-\psi_1) + \frac{b}{R}\sin\psi_1,\\ &\frac{d\psi_2}{dt} = \delta_2 + \frac{R}{2r}\mu\sin(\psi_1-\psi_2). \end{aligned} \quad (16)$$

Steady amplitudes of limit cycles: $\quad R \approx \sqrt{\lambda_1 - \mu}, \quad r \approx \sqrt{\lambda_2 - \mu}\,.$ (17)

$$\Downarrow (16,17)$$

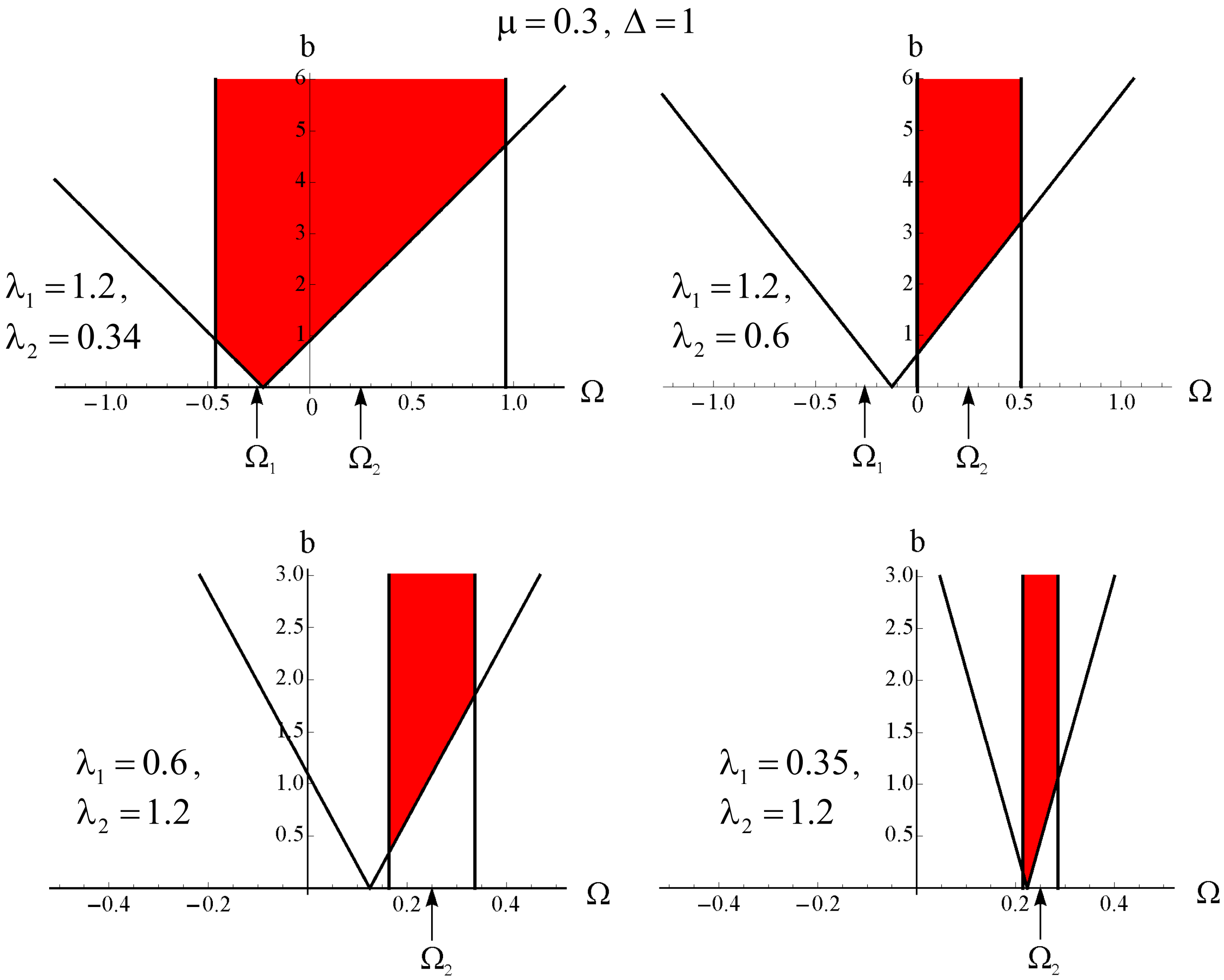
$$\frac{d\psi_1}{dt} = \delta_1 + \frac{\mu}{2}\sqrt{\frac{\lambda_2-\mu}{\lambda_1-\mu}}\sin(\psi_2-\psi_1) + \frac{b}{\sqrt{\lambda_1-\mu}}\sin\psi_1,$$
$$\frac{d\psi_2}{dt} = \delta_2 + \frac{\mu}{2}\sqrt{\frac{\lambda_1-\mu}{\lambda_2-\mu}}\sin(\psi_1-\psi_2) \qquad (18)$$

Phase locking condition $\frac{d\varphi_1}{dt} = \frac{d\varphi_2}{dt}$ leads to the equations for boundaries of the synchronization area:

(defines typical form of a corner) $\quad \Omega = \frac{\Delta}{4}\frac{\lambda_2-\lambda_1}{\lambda_1+\lambda_2-2\mu} \pm b\frac{\sqrt{\lambda_1-\mu}}{\lambda_1+\lambda_2-2\mu},$ (19)

(defines width of locking range) $\quad \Omega = \frac{\Delta}{4} \pm \frac{\mu}{2}\sqrt{\frac{\lambda_1-\mu}{\lambda_2-\mu}}.$ (20)

Analytic boundaries of the synchronization area (Red color)

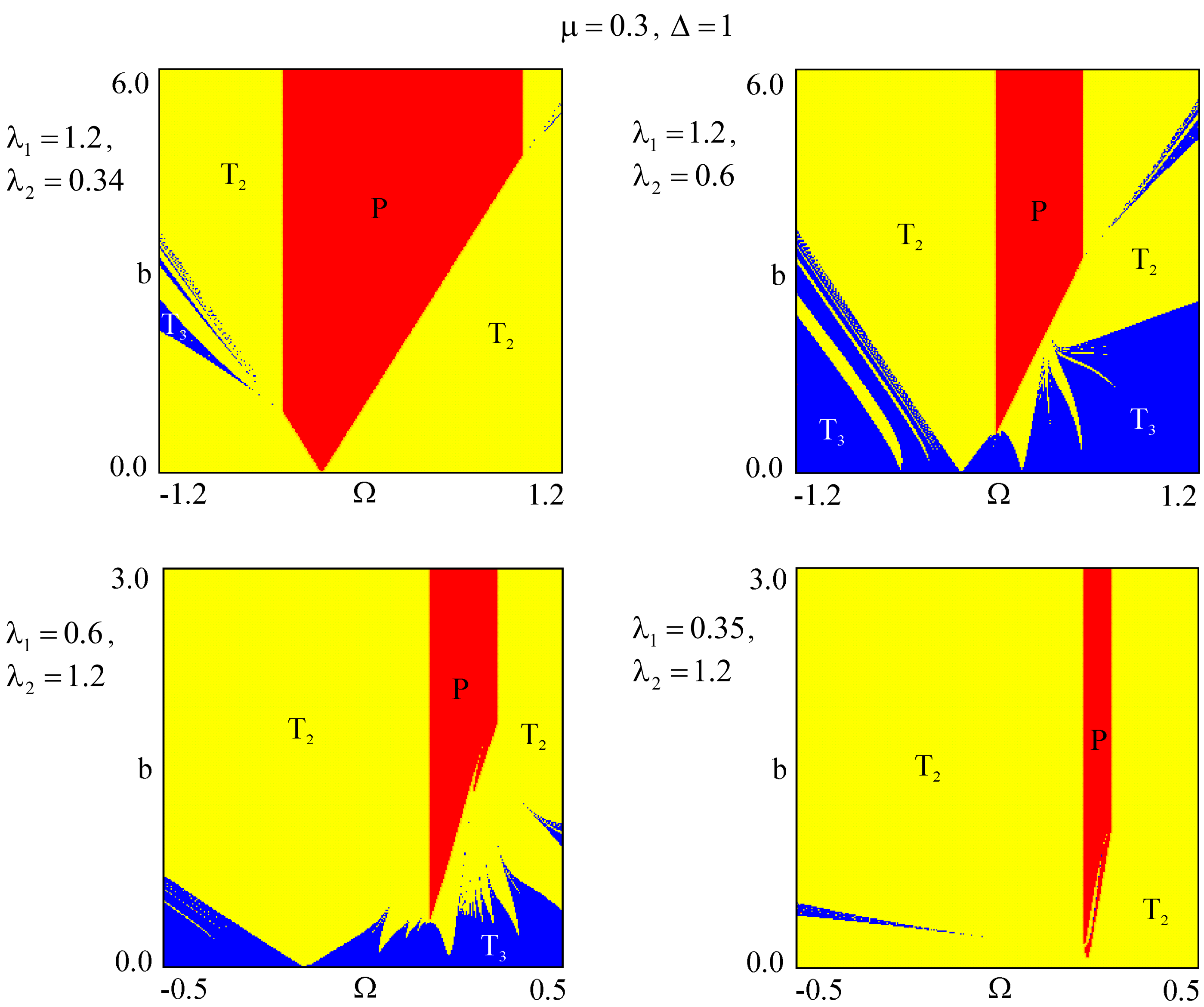


$\Omega_1 = -0.25,$
$\Omega_2 = 0.25$ } fundamental frequencies of the 1-st and the 2-nd oscillators

2.2 Parameter planes

For construction of the parameter plane we calculate two Liapunov exponents Λ_1, Λ_2 in each point of the parameter plane frequency Ω – amplitude b of external influence. Then we mark following regimes by different colors:

- **Red color P** – existence of stable fixed point (phase locking), $\Lambda_1 < 0, \Lambda_2 < 0$;
- **Yellow color T₂** – quasiperiodic regime, which corresponds to the double-frequency torus, $\Lambda_1 = 0, \Lambda_2 < 0$;
- **Blue color T₃** – quasiperiodic regime, which corresponds to the three-frequency torus, $\Lambda_1 = 0, \Lambda_2 = 0$.



Conclusions

- ✓ Possibility of a special synchronization regime on the parameter plane frequency mismatch δ – coupling value μ in an infinitively long band between oscillator death and quasiperiodic areas is shown for dissipatively coupled van der Pol oscillators, non-identical in values of parameters controlling the Hopf bifurcation. Width of this band depends on the difference of control parameters $\lambda_1 - \lambda_2$.
- ✓ Non-identity in nonlinear dissipation results in specific form of the boundary of the main synchronization tongue, which looks like letter S on the parameter plane frequency mismatch δ – coupling value μ.
- ✓ The phase locking area and quasiperiodic areas, which correspond to the double- and three-frequency tori, were found for dissipatively coupled van der Pol oscillators under external harmonic influence on the parameter plane frequency Ω – amplitude b of external influence.
- ✓ The results of numerical investigation are in close fit with the results of analytical and experimental investigation.

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