Characterizing nonlinear dynamics with covariant Lyapunov vectors

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Lyapunov Exponents

- Chaotic dynamics is characterized by exponential sensitivity to initial conditions:

$$\vec{x}_{t+1} = \vec{\mathbf{F}}(\vec{x}_t)$$

$$\left\| \delta\, \vec{x}_t \right\| \approx \left\| \delta\, \vec{x}_0 \right\| \cdot \exp\left[\lambda_1\, t \right] \qquad t >> 1$$

$$\frac{d}{dt} \vec{x}_t = \vec{\mathbf{F}}(\vec{x}_t)$$

- Tangent evolution of linearized perturbations is ruled by the *Jacobian:*

$$\mathbf{J}_t \;: \qquad \left[\mathbf{J}_t \right]_{\mu\nu} = \frac{\partial F_\mu(\vec{x}_t)}{\partial x_\nu}$$

$$\delta\, \vec{x}_t = \mathbf{M}(\vec{x}_0, t)\ \delta\, \vec{x}_0$$

$$\mathbf{M}(\vec{x}_0, t) = \left(\mathbf{J}_{t-1} \mathbf{J}_{t-2} \cdots \mathbf{J}_{t_0+1} \mathbf{J}_{t_0} \right)$$

$$\frac{d}{dt} \mathbf{M}(\vec{x}_0, t) = \mathbf{J}_t\, \mathbf{M}(\vec{x}_0, t) \qquad \mathbf{M}(\vec{x}_0, 0) = \mathbf{I}$$

Lyapunov Exponents

- The existence of a complete set of N LEs is granted by the Oseledets multiplicative theorem:

$$\Lambda_+(\vec{x}_0) = \lim_{t\to\infty}\left[\mathbf{M}^{\mathrm{T}}(\vec{x}_0,t)\mathbf{M}(\vec{x}_0,t)\right]^{1/(2t)}$$

$$\Lambda_+(\vec{x}_0)\vec{e}_+^{\,j}(\vec{x}_0) = \gamma_j\,\vec{e}_+^{\,j}(\vec{x}_0)$$

$$\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_N \qquad\qquad \lambda_j = \ln\gamma_j$$

- There exist a sequence of nested subspaces connected with these growth rates:

$$\mathbf{R}^N = \Gamma_{\vec{x}_0}^{(1)} \supset \Gamma_{\vec{x}_0}^{(2)} \supset \cdots \supset \Gamma_{\vec{x}_0}^{(N)} \qquad \lambda_j \quad \text{exp. growth rate of } \vec{u} \in \Gamma_{\vec{x}_0}^{(j)} \setminus \Gamma_{\vec{x}_0}^{(j+1)}$$

$$\dim\left(\Gamma_{\vec{x}_0}^{(j)}\right) = N - j + 1$$

• LEs quantify the growth of volumes in tangent space

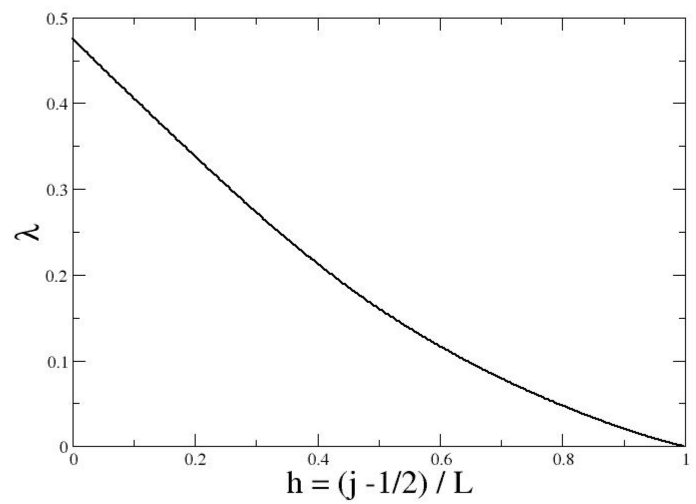
• Entropy production (Kolmogorov-Sinai entropy):

$$H_{KS} = \sum_{\lambda_i > 0} \lambda_i$$

• Attractor dimension in dissipative systems (Kaplan Yorke Formula)

$$D_{KY} = k + \frac{\sum_{i=1}^{k} \lambda_i}{|\lambda_{k+1}|}$$

• There exist a thermodynamic limit for Lyapunov spectra in spatially ext. systems:



Lyapunov Vectors ?

- After exponents (i.e. eigenvalues), people got interested in vectors (i.e. eigenvectors ?) to quantify stable and unstable directions in tangent space.

- Hierarchical decomposition of spatiotemporal chaos

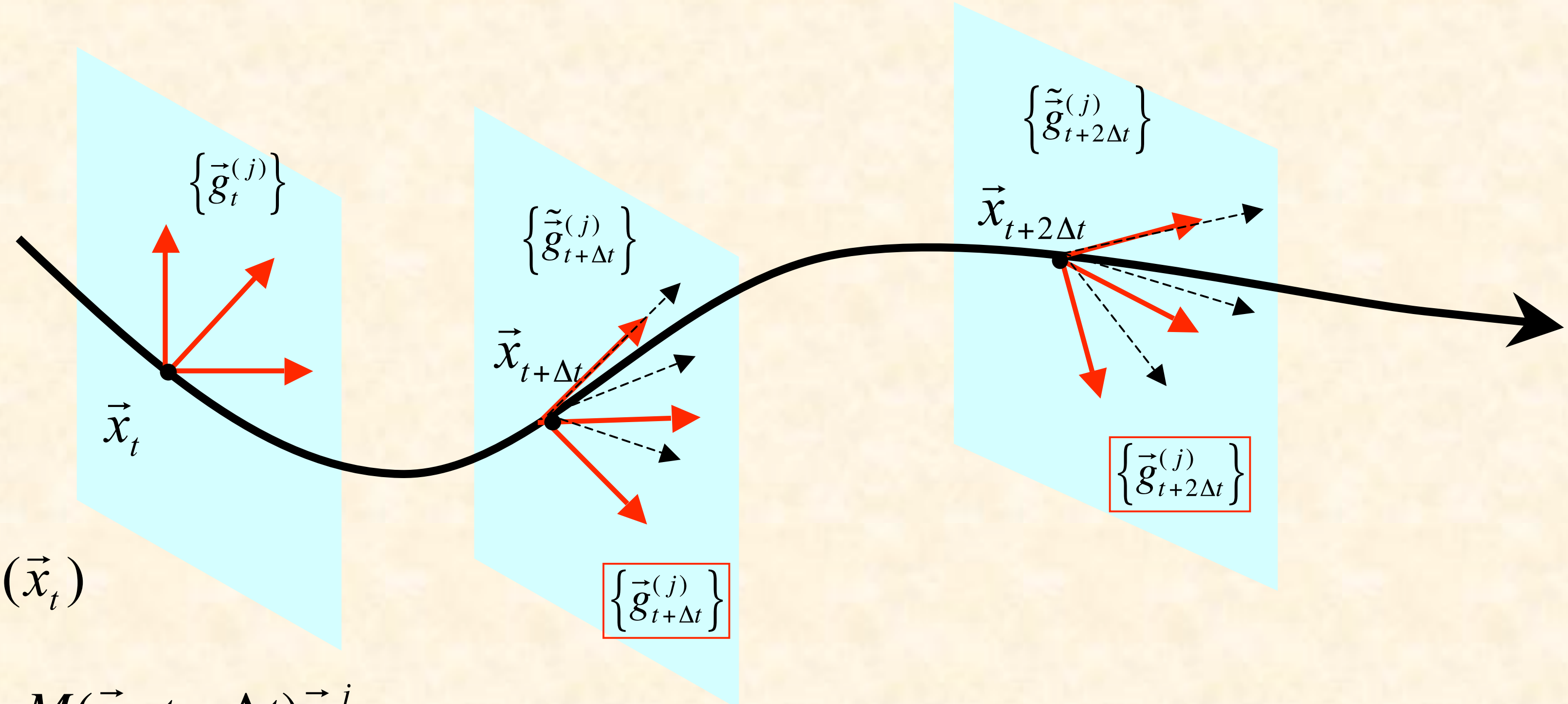
- Optimal forecast in nonlinear models (e.g. in geophysics)

- Study of "hydrodynamical modes" in near-zero exponents and vectors (access to transport properties ?)

But… which vectors ?

- i.e. bred vectors, singular vectors, Gram Schmidt vectors, covariant vectors…

Gram Schmidt vectors



$$\frac{d\vec{x}_t}{dt} = \vec{\mathbf{F}}(\vec{x}_t)$$

$$\tilde{\vec{g}}^{\,j}{}_{t+\Delta t} = M(\vec{x}_t, t+\Delta t)\vec{g}^{\,j}{}_t$$

Gram Schmidt vectors are obtained by GS orthogonalization (Benettin *et al.* 1980)

$$\tilde{\mathbf{G}}_t = \left(\tilde{g}_t^{\,1}\middle|\cdots\middle|\tilde{g}_t^{\,N}\right)$$

$$\mathbf{Q}_t = \left(g_t^{\,1}\middle|\cdots\middle|g_t^{\,N}\right)$$

Upper triangular

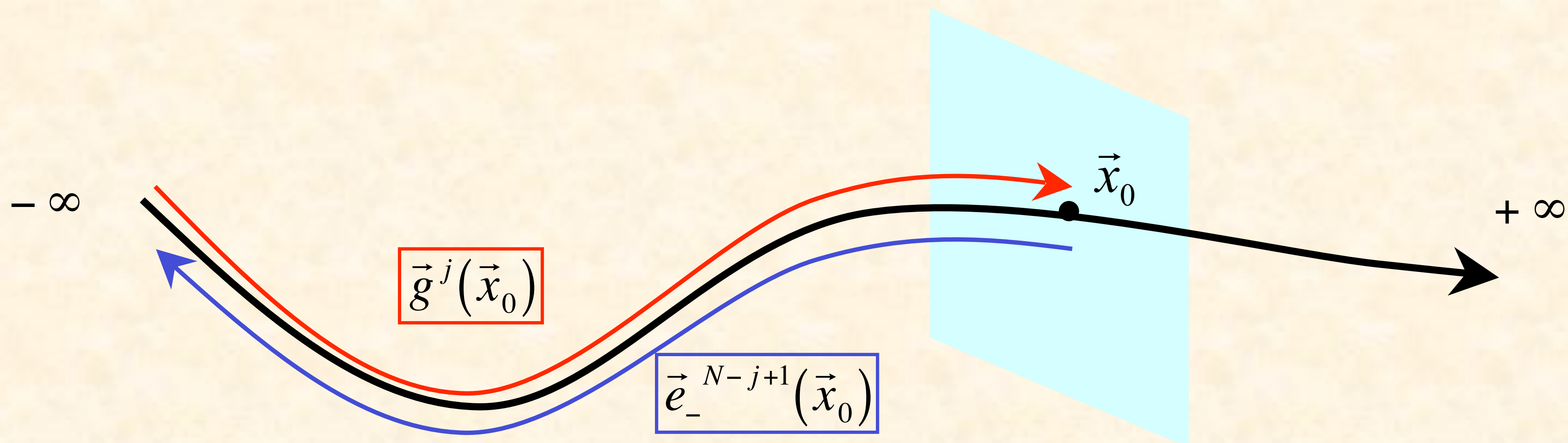
$$\tilde{\mathbf{G}}_{t+\Delta t} = \mathbf{Q}_{t+\Delta t}\mathbf{R}_{t,\Delta t}$$

- It can be shown that any orthonormal set of vectors eventually converge to a well defined basis *(Ershov and Potapov, 1998)*

- For time-invertible systems they coincide with the eigenvectors of the backward Oseledec matrix:

$$\vec{g}^{\,j} \rightarrow \vec{e}_{-}^{\,N-j+1}$$

$$\Lambda_{-}(x_0) = \lim_{t\to-\infty}\left[\mathbf{M}^{-1}(x_0,t)^T\,\mathbf{M}^{-1}(x_0,t)\right]^{1/2t}$$



But…

- They are orthogonal, while *stable* and *unstable* manifolds are generally not.

- Dynamical properties are "washed away" by orthonormalization, which is norm dependent, while LEs are not (for a wide class of norms).

- They are not invariant under time reversal, while LEs are (sign-wise):

$$\vec{g}_{+}^{\,j} \neq \vec{g}_{-}^{\,N-j+1} \qquad \lambda^{+}_{j} = -\lambda^{-}_{N-j+1}$$

- They are not covariant with dynamics and do not yield correct growth factors:

$$\mathbf{M}(\vec{x}_t, t+\Delta t)\,\vec{g}_t^{\,j} \neq \gamma_{t,\Delta t}^{(j)}\,\vec{g}_{t+\Delta t}^{\,j} \qquad \left\langle \ln \left\| \mathbf{M}(\vec{x}_t, t+\Delta t)\,\vec{g}_t^{\,j} \right\| \right\rangle \neq \lambda_j$$

$$\vec{g}_t^{\,j} \equiv \vec{g}^{\,j}(\vec{x}_t)$$

Covariant Lyapunov vectors $\boldsymbol{v}$

• Oseledets (1968) & Ruelle (1979) – Oseledets splitting

$$\vec{v}^{\,j} \quad \text{spans} \quad \mathbf{E}_{\vec{x}_0}^{(j)} = \underbrace{\boldsymbol{\Gamma}_{\vec{x}_0}^{(j)}}_{\dim\left[\Gamma_{\vec{x}_0}^{(j)}\right] = N-j+1} \cap \overbrace{\overline{\boldsymbol{\Gamma}}_{\vec{x}_0}^{(N-j+1)}}^{\dim\left[\overline{\Gamma}_{\vec{x}_0}^{(N-j+1)}\right] = j}$$

$$\boldsymbol{\Gamma}_{\vec{x}_0}^{(j)} = \mathbf{U}_+^{(J)}(\vec{x}_0) \oplus \cdots \oplus \mathbf{U}_+^{(N)}(\vec{x}_0)$$

$$\overline{\boldsymbol{\Gamma}}_{\vec{x}_0}^{(N-j+1)} = \mathbf{U}_-^{(1)}(\vec{x}_0) \oplus \cdots \oplus \mathbf{U}_-^{(j)}(\vec{x}_0)$$

$$\mathbf{U}_\pm^{(J)}(\vec{x}_0) \quad \text{eigenspaces of} \quad \Lambda_\pm(\vec{x}_0)$$

$$\dim\left[\boldsymbol{\Gamma}_{\vec{x}_0}^{(j)}\right] = N - j + 1 \qquad \dim\left[\overline{\boldsymbol{\Gamma}}_{\vec{x}_0}^{(j)}\right] = j$$

• They are covariant with dynamics and do yield correct growth factors (LEs):

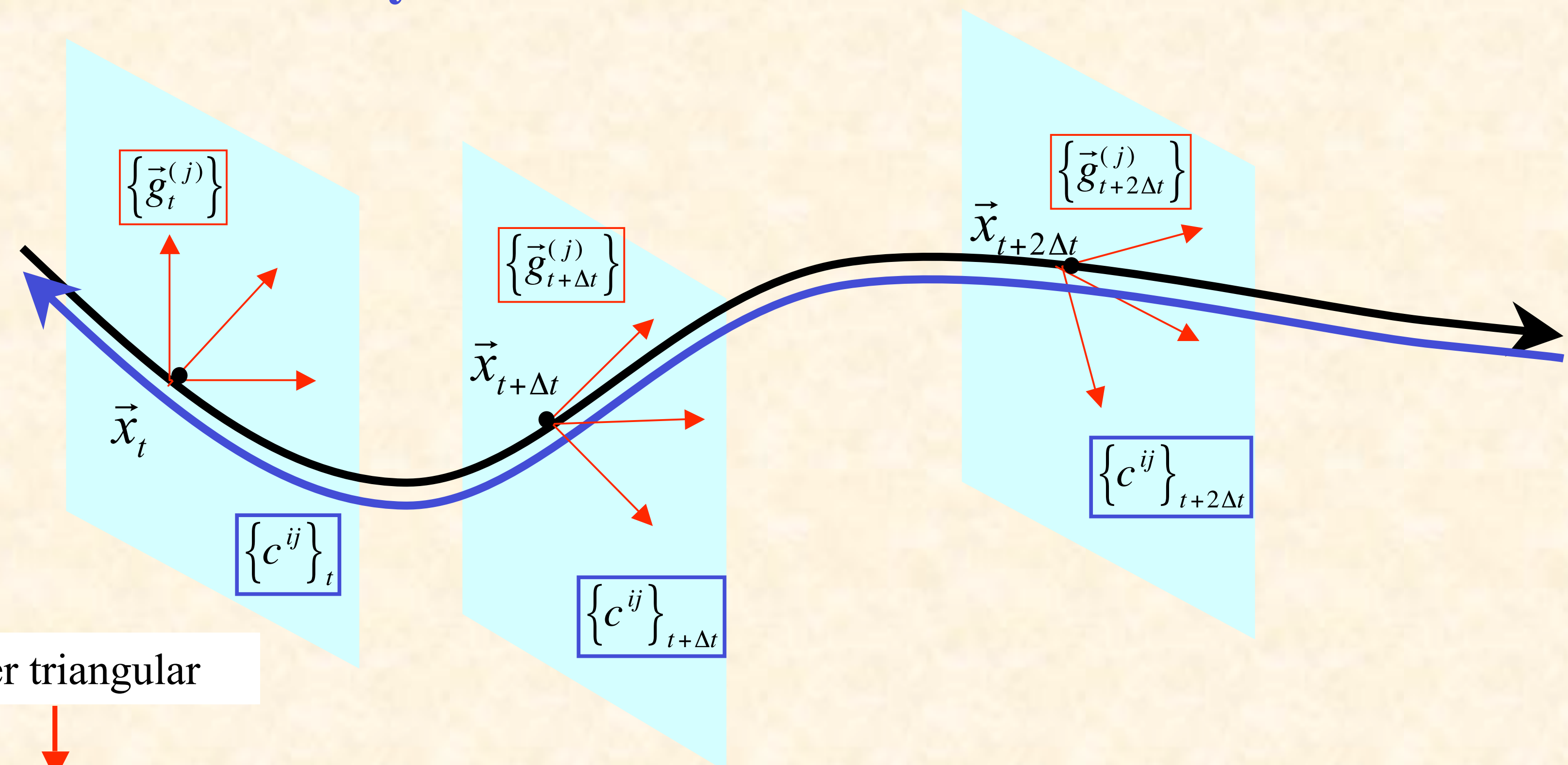
$$\mathbf{M}(\vec{x}_t, t+\Delta t)\vec{v}_t^{\,j} = \gamma_j \vec{v}_{t+\Delta t}^{\,j} \qquad \left\langle \ln \left\| \mathbf{M}(\vec{x}_t, t+\Delta t)\vec{v}_t^{\,j} \right\| \right\rangle = \lambda_j$$

After Ruelle

- Brown, Bryant & Abarbanel (1991) – Covariant vectors in time series data analysis
- Legras & Vautard; Trevisan & Pancotti (1996) – Covariant vectors in Lorenz 63
- Politi *et. al.* (1998) – Covariant vectors satisfy a node theorem for periodic orbits
- Wolfe & Samelson (2007) – Intersection algorithm, more efficient for $j \ll N$

Lack of a practical algorithm to compute them
No studies of ensemble properties in large systems

Computing covariant Lyapunov Vectors *v* by forward-backward iterations



Upper triangular

$$[\mathbf{C}_t]_{ij} = c_t^{ij} = \left(\vec{g}_t^{\,i} \cdot \vec{v}_t^{\,j}\right)$$

Consider vectors which are linear combinations of the first *j* Gram-Schmidt vectors *g*

$$\vec{v}_t^{\,j} = \sum_{i=1}^{j} c_t^{ij} \vec{g}_t^{\,i}$$

$$\sum_{i=1}^{j} \left[c_t^{ij}\right]^2 = 1$$

1. R evolves the coefficients C according to tangent dynamics

Diag matrix w. local growth factors

Covariant evolution means:

$$\mathbf{V}_{t+\Delta t}\,\Delta_{t,\Delta t} = \mathbf{M}_{t,\Delta t}\ \mathbf{V}_t \qquad \left(\mathbf{M}_{t,\Delta t} \equiv \mathbf{M}(\vec{x}_t, t+\Delta t)\right)$$

(*Expand CLV on GS basis*)

$$\left(v_t^1 \middle| v_t^2 \middle| \cdots \middle| v_t^N\right) \equiv \mathbf{V}_t = \mathbf{Q}_t\mathbf{C}_t$$

$$\mathbf{Q}_{t+\Delta t}\mathbf{C}_{t+\Delta t}\,\Delta_{t,\Delta t} = \mathbf{M}_{t,\Delta t}\ \mathbf{Q}_t\mathbf{C}_t$$

(*use QR decomposition*)

$$\mathbf{M}_{t,\Delta t}\mathbf{Q}_t = \tilde{\mathbf{G}}_{t+\Delta t} = \mathbf{Q}_{t+\Delta t}\mathbf{R}_{t,\Delta t}$$

$$\mathbf{Q}_{t+\Delta t}\mathbf{C}_{t+\Delta t}\,\Delta_{t,\Delta t} = \mathbf{Q}_{t+\Delta t}\mathbf{R}_{t,\Delta t}\mathbf{C}_t$$

one gets the evolution rule

$$\mathbf{C}_{t+\Delta t}\,\Delta_{t,\Delta t} = \mathbf{R}_{t,\Delta t}\mathbf{C}_t$$

2. Moving backwards insures convergence to the "right" covariant vectors

$$\mathbf{R}_{t,\Delta t}{}^{-1}\tilde{\mathbf{C}}_{t+\Delta t}\,\tilde{\Delta}_{t,\Delta t} \to \mathbf{C}_t$$

(*consider two different random initial conditions*)

$$\tilde{\mathbf{C}}_{t+\Delta t}\,\tilde{\Delta}_{t,\Delta t} = \mathbf{R}_{t,\Delta t}\tilde{\mathbf{C}}_t \qquad\qquad \tilde{\tilde{\mathbf{C}}}_{t+\Delta t}\,\tilde{\tilde{\Delta}}_{t,\Delta t} = \mathbf{R}_{t,\Delta t}\tilde{\tilde{\mathbf{C}}}_t$$

A. If C are *upper triangular with non-zero diagonal*, one can verify that

$$\tilde{\tilde{\Delta}}_{t,\Delta t}\,,\tilde{\Delta}_{t,\Delta t} \xrightarrow{\Delta t \to \pm\infty} \mathrm{diag}\left(e^{\pm\Delta t\,\lambda_1}, e^{\pm\Delta t\,\lambda_2}, \cdots, e^{\pm\Delta t\,\lambda_N}\right)$$

B. By simple manipulations

$$\mathbf{R}_{t,\Delta t} = \tilde{\mathbf{C}}_{t+\Delta t}\,\tilde{\Delta}_{t,\Delta t}\tilde{\mathbf{C}}_t^{-1} = \tilde{\tilde{\mathbf{C}}}_{t+\Delta t}\,\tilde{\tilde{\Delta}}_{t,\Delta t}\tilde{\tilde{\mathbf{C}}}_t^{-1}$$

$$\Rightarrow \left[\tilde{\tilde{\mathbf{C}}}_{t+\Delta t}^{-1}\tilde{\mathbf{C}}_{t+\Delta t}\right] = \tilde{\tilde{\Delta}}_{t,\Delta t}\left[\tilde{\tilde{\mathbf{C}}}_t^{-1}\tilde{\mathbf{C}}_t\right]\tilde{\Delta}_{t,\Delta t}^{-1}$$

(*by matrix components*)

$$\Rightarrow \left[\tilde{\tilde{\mathbf{C}}}_{t+\Delta t}^{-1}\tilde{\mathbf{C}}_{t+\Delta t}\right]_{\mu\nu} \xrightarrow{\Delta t \to \pm\infty} \exp\left[\Delta t\left(\lambda_\mu - \lambda_\nu\right)\right]\left[\tilde{\tilde{\mathbf{C}}}_t^{-1}\tilde{\mathbf{C}}_t\right]_{\mu\nu}$$

$$\Rightarrow \left[\tilde{\tilde{\mathbf{C}}}_{t+\Delta t}^{-1}\tilde{\mathbf{C}}_{t+\Delta t}\right]_{\mu\nu} \approx \begin{cases} 0 & \mu > \nu \\ \exp\left[\Delta t\left(\lambda_\mu - \lambda_\nu\right)\right] & \mu < \nu \qquad \lambda_\mu - \lambda_\nu > 0 \\ \phi_\mu & \mu = \nu \end{cases}$$

If we follow the reversed dynamics

$$\tilde{\tilde{\mathbf{C}}}_{t+\Delta t}^{-1}\tilde{\mathbf{C}}_{t+\Delta t} \xrightarrow{\Delta t \to -\infty} \Phi \qquad (\textit{diagonal matrix})$$

$$\Rightarrow \tilde{\mathbf{C}}_{t+\Delta t} \xrightarrow{\Delta t \to -\infty} \tilde{\tilde{\mathbf{C}}}_{t+\Delta t}\Phi$$

All random initial conditions converge to the same ones, apart a prefactor

Thus this reversed dynamics converges to covariant vectors for almost any initial condition

Covariant Lyapunov Vectors properties

• They coincide with *stable* and *unstable* manifolds

• They are invariant under time reversal.

$$\vec{v}_+^{\,j} = \vec{v}_-^{\,N-j+1} \qquad\qquad \lambda^+_j = -\lambda^-_{N-j+1}$$

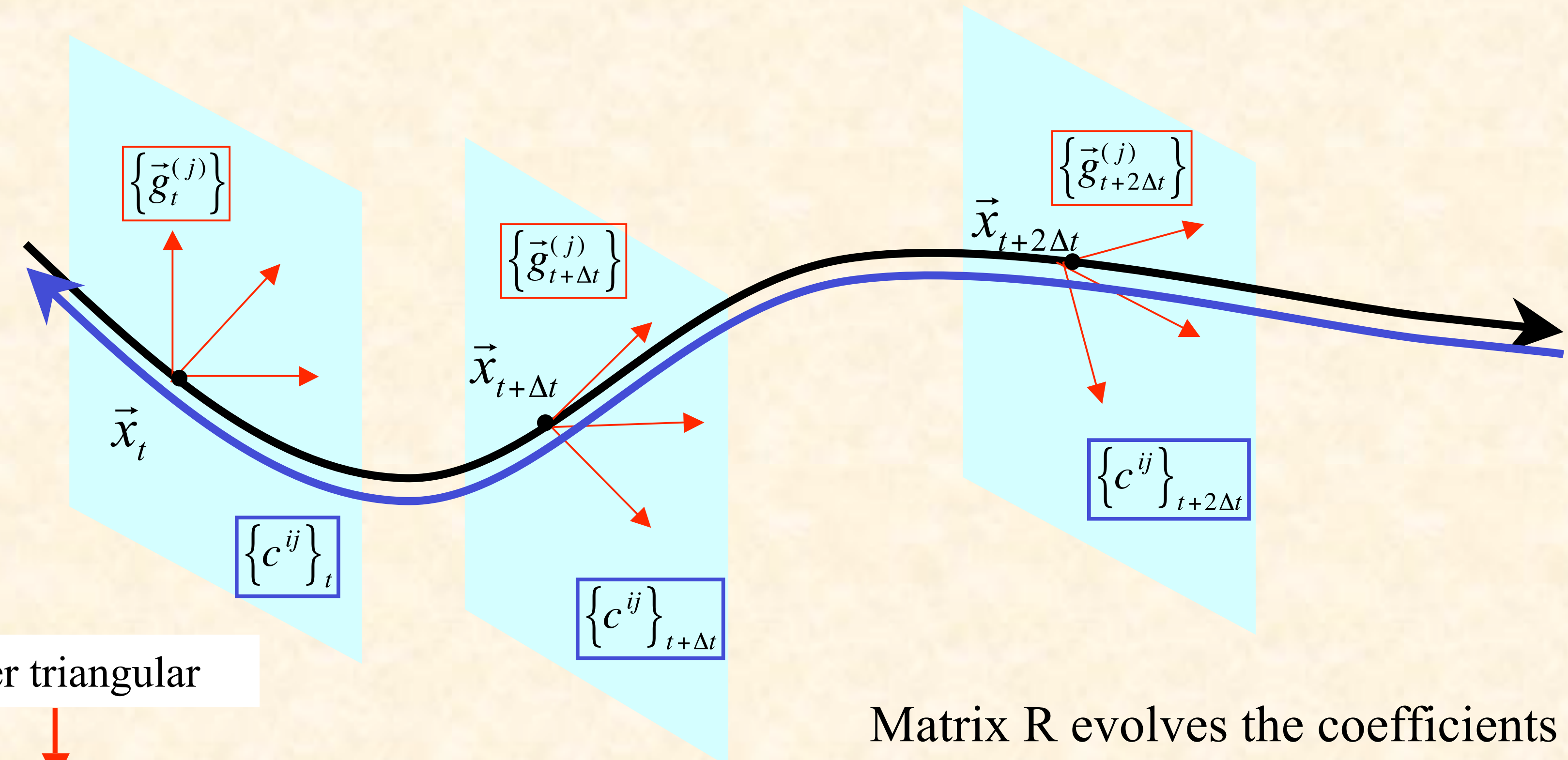
• They are covariant with dynamics and do yield correct growth factors (LEs):

$$\mathbf{M}_{t,\Delta t}\vec{v}_t^{\,j} = \gamma^{(j)}_{t,\Delta t}\,\vec{v}_{t+\Delta t}^{\,j} \qquad\qquad \left\langle \ln\left\| \mathbf{M}_{t,\Delta t}\vec{v}_t^{\,j} \right\| \right\rangle_t = \lambda_j$$

• They are norm independent and, for time reversible systems, coincide with the Oseledec splitting (*Ruelle 1979*)

• They can be computed for non time reversible systems too by following backward a stored forward trajectory

The stable algorithm for covariant Lyapunov Vectors



Upper triangular

$$\left[\mathbf{C}_t\right]_{ij} = c_t^{ij} = \left(\vec{g}_t^{\,i} \cdot \vec{v}_t^{\,j}\right)$$

$$\vec{v}_t^{\,j} = \sum_{i=1}^{j} c_t^{ij} \vec{g}_t^{\,i}$$

Matrix R evolves the coefficients C according to tangent dynamics

$$\left(\mathbf{R}_{t,\Delta t}\right)^{-1} \mathbf{C}_{t+\Delta t} \Delta_{t,\Delta t} = \mathbf{C}_t$$

This linearized evoluytion is convergent in the time reversed (linearized) dynamics

A Simple recipe

• Start from a random initial condition.

• Run a forward transient to obtain convergence of GS vectors

• Continue your phase space trajectory continuously storing the QR decomposition of tangent space.

• Run a final backward transient only storing the R matrices from QR

• Generate a random upper triangular matrix C

• Evolve C backward by inverting R matrices along the backward transient

• Convergence to CLV coefficients is ruled by difference between nearest LEs

• Once backward transient has been done and CLV coefficients are converged, continue to move backward along trajectories. CLV can be recovered as V=QC

• Some further tricks to ease memory storage in RAM are possible

Some applications

- Measure angles between CLV or linear combinations of CLV:
  numerical measures of hyperbolicity violations.

- Study the so called Lyapunov Hydrodynamic modes in Hamiltonian systems…

- Data assimilation algorithms ?

- Study the localization of modes associated to LE: hierarchical decomposition of ST chaos ?

- Tangent space decomposition may reveal the effective degrees of freedom large disspative systems

- Analyze collective behavior in large dynamical system

FG, P. Poggi, A. Turchi, H. Chaté, R. Livi, and A. Politi, *PRL* **99**, 130601 (2007).

K.A. Takeuchi, FG, H. Chaté, *PRL* **103**, 154103 (2009).

Collective behavior

$$x_i^{t+1} = (1-K) f(x_i^t) + \frac{K}{N} \sum_j f(x_j^t) + \eta_i^t \qquad X^t = \frac{1}{N} \sum_j x_j^t$$

- Collective, apparently low dimensional behavior of some global, mean field variable(s), possibly varying on time scales much larger that the ones of the individual elements. Individual oscillators stay unlocked.

Microscopic chaos

- (infinitely) many DOFs
- disordered, chaotic behavior

Collective behavior

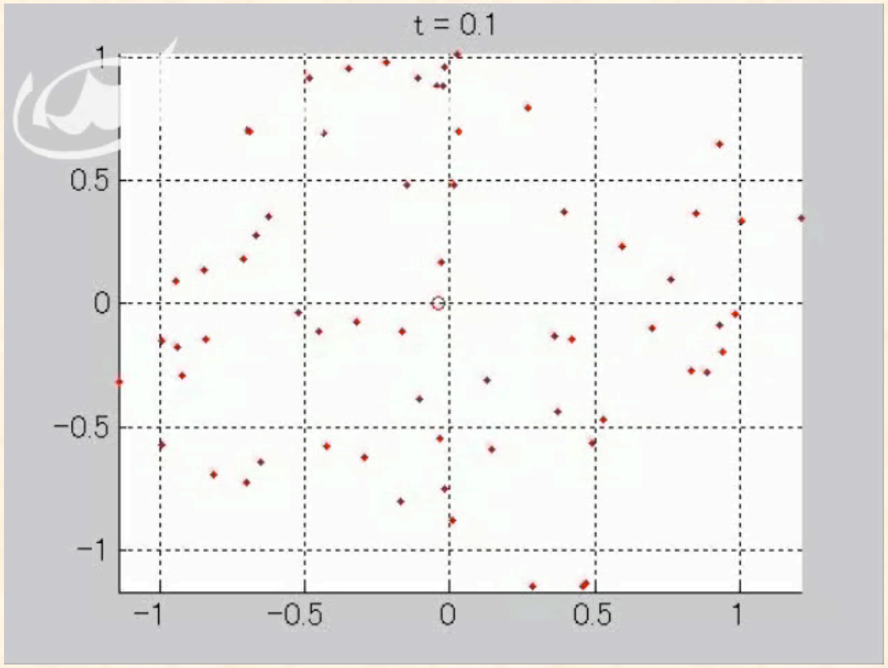
- fewer DOFs (possibly finite)
- various time-dependent behavior

Lyapunov analysis?

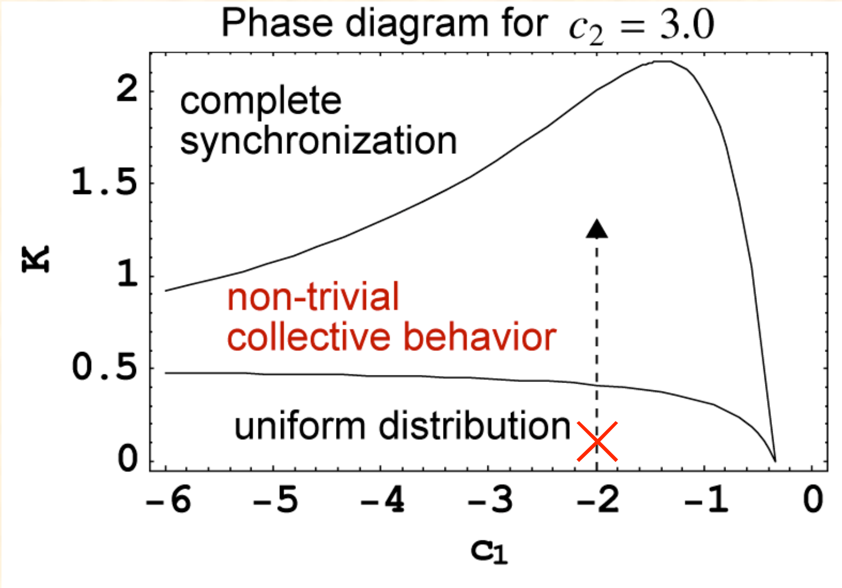
A model system: Globally coupled limit cycle oscillators

Landau Stuart oscillators *Kuramoto & Nakagawa (1994, 1995)*

$$\dot{W}_j = W_j - (1 + ic_2)|W_j|^2 W_j + K(1 + ic_1)(\overline{W} - W_j) \qquad \overline{W} = \frac{1}{N}\sum_{j=1}^{N} W_j$$



$K = 0.1$



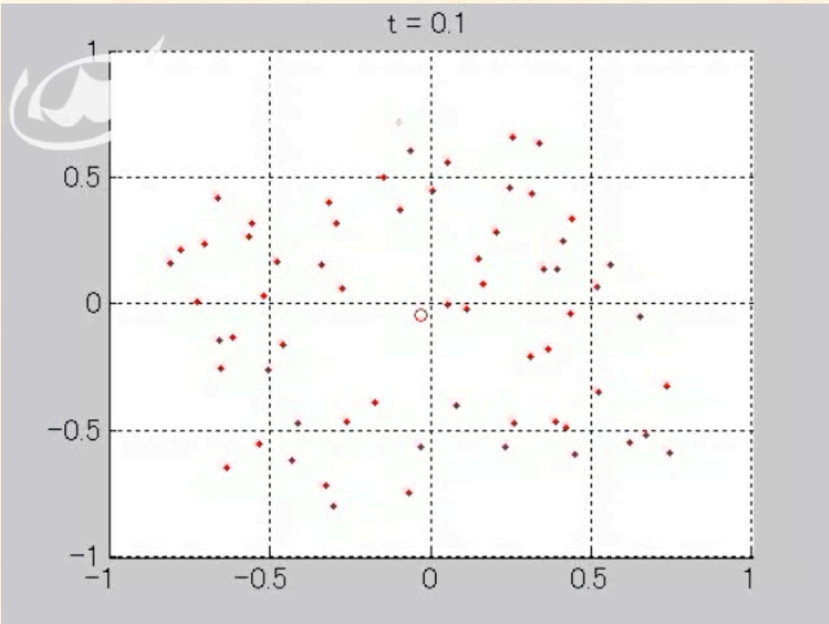
● : individual oscillators

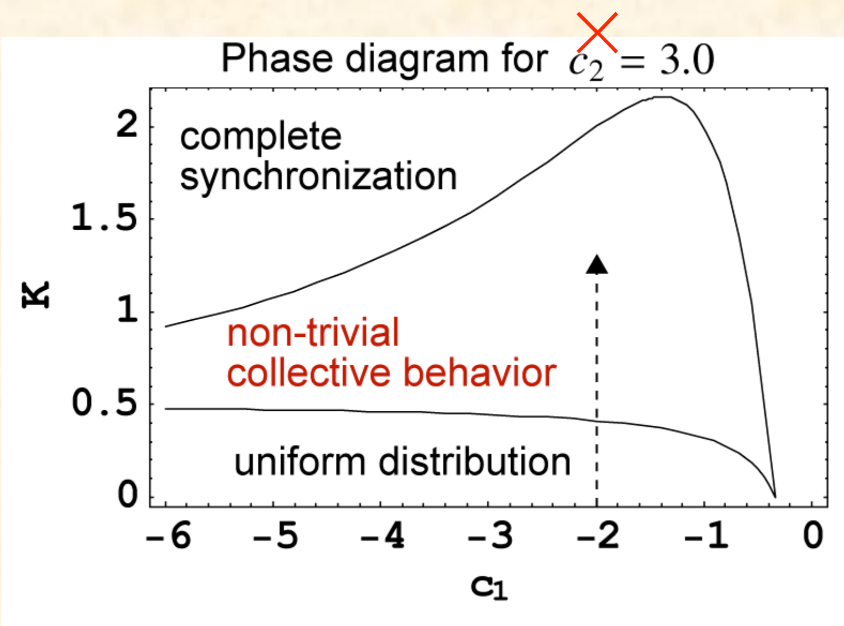
○ : collective dynamics

A model system: Globally coupled limit cycle oscillators

Ginzburg Landau oscillators *Kuramoto & Nakagawa (1994, 1995)*

$$\dot{W}_j = W_j - (1 + ic_2)\left|W_j\right|^2 W_j + K(1 + ic_1)\left(\overline{W} - W_j\right) \qquad \overline{W} = \frac{1}{N}\sum_{j=1}^{N} W_j$$



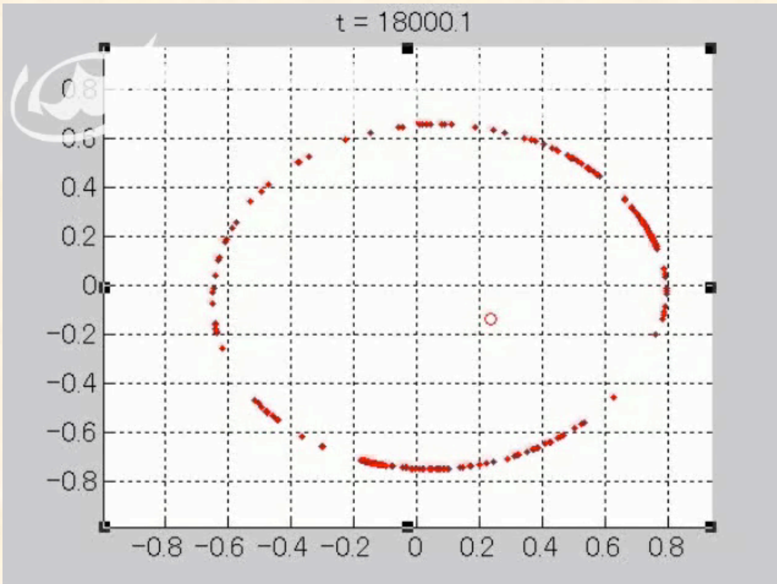


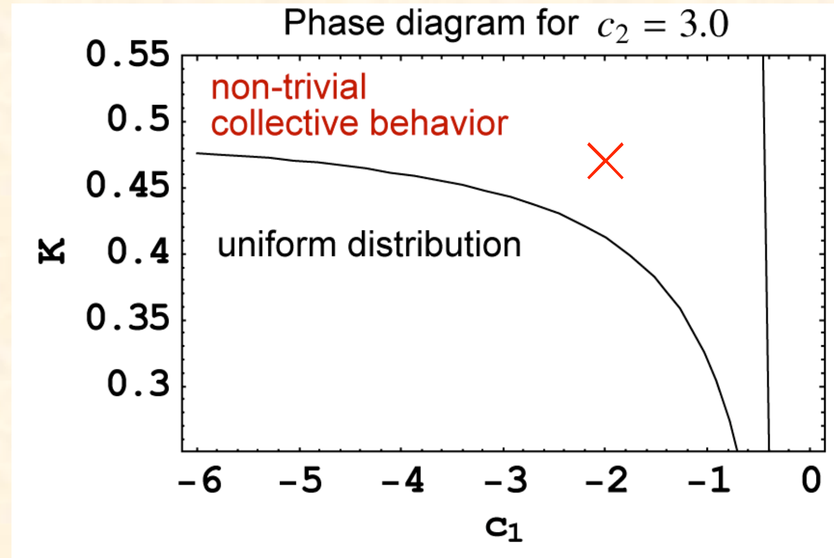
● : individual oscillators

○ : collective dynamics

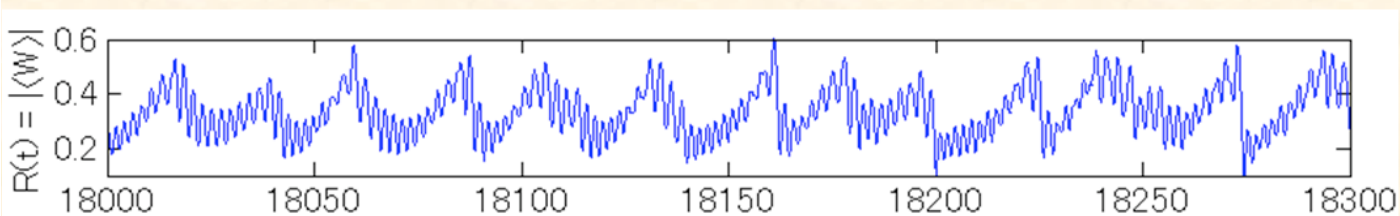
$K = 2.5$

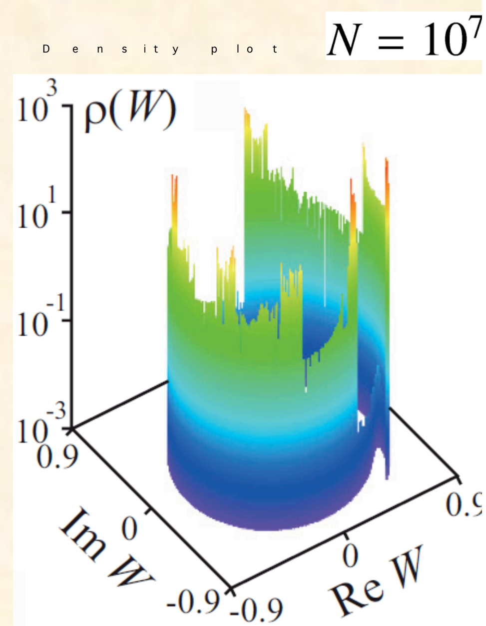
Intermediate couplung: nontrivial collective begavior





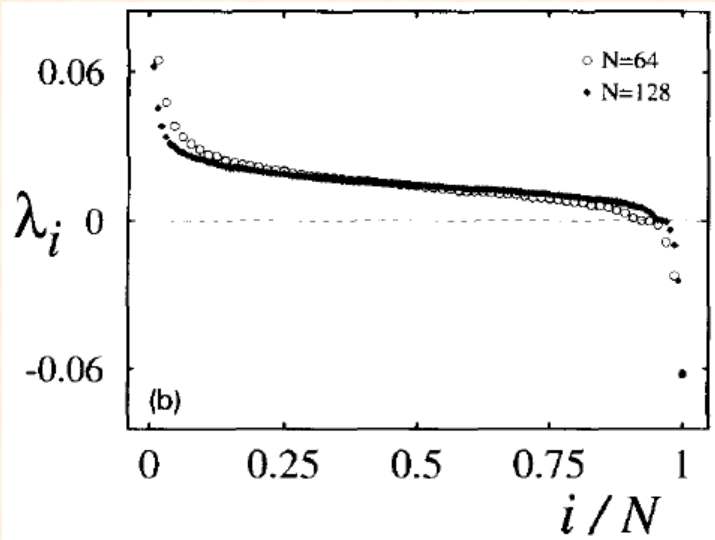
$K = 0.470$

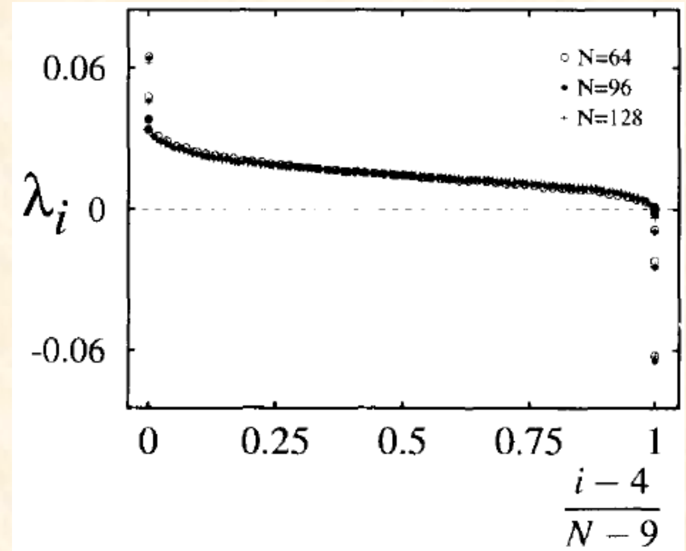




individual oscillators: chaotic

collective dynamics: weakly chaotic





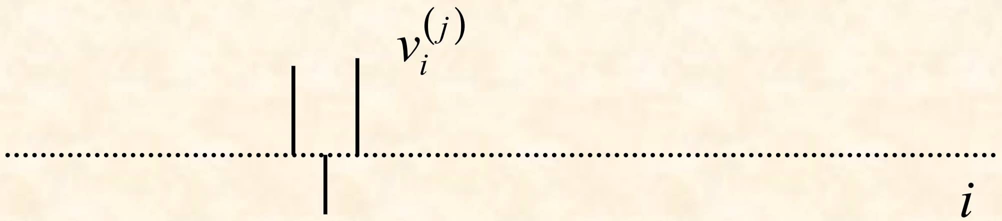
•Can one detect collective modes by infinitesimal Lyapunov analysis?

- Extensive LE, continuous part of the Lyapunov spectrum corresponding to microscopic dynamics

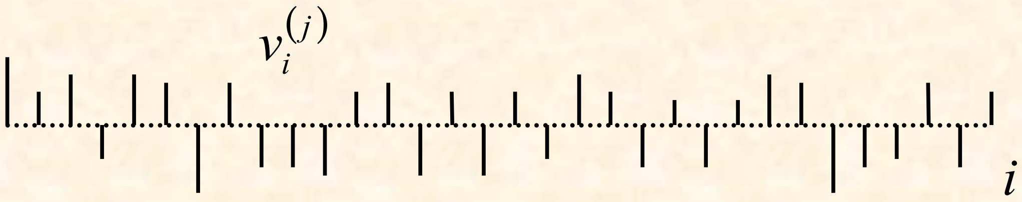
- Non extensive LE, discrete part of the Lyapunov spectrum corresponding to coherent modes

- Does it exist a well defined thermodynamic limit for Lyapunov spectra in globally coupled systems (i.e. *extensivity*)?

Conjecture: CLV are a tool to characterize collective modes



- Localized, extensive covariant Lyapunov vectors corresponding to microscopic dynamics



- Delocalized, nonextensive covariant Lyapunov vectors corresponding to collective modes

•Localization properties of vector j can be characterized by the inverse participation ratio

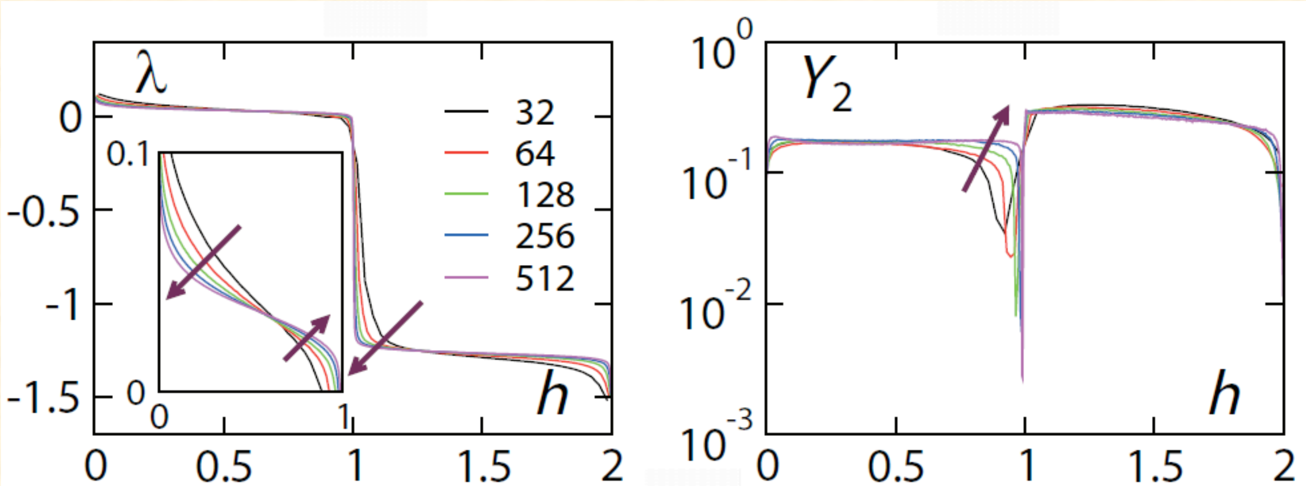
$$Y_2(j) = \left\langle \sum_i \left[v^{(j)} \right]_i^4 \right\rangle$$

• Localized: nonvanishing Y_2

$$Y_2(j) \approx 1/\ell + L^{-\gamma}$$

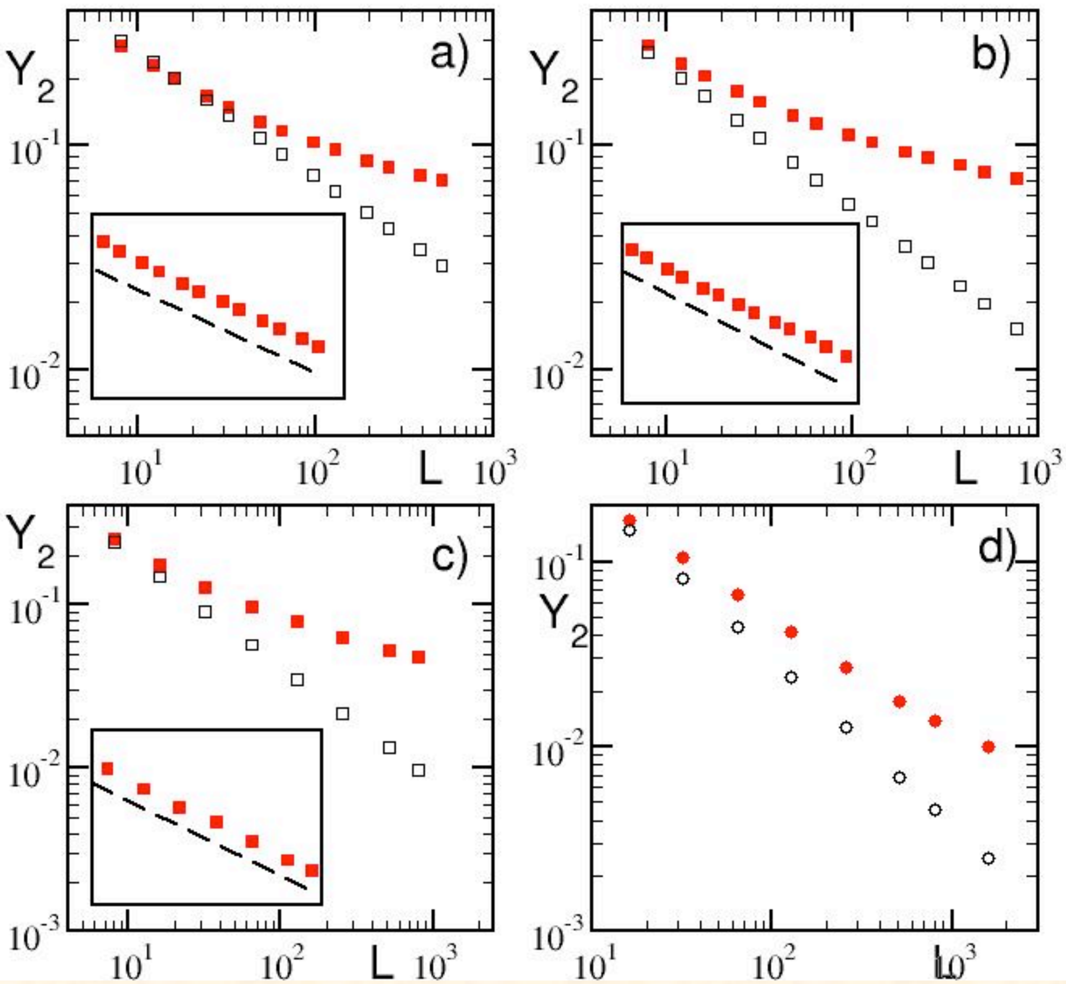
• Delocalized: vanishing Y_2

$$Y_2(j) \approx 1/L$$



$h \equiv (i - 0.5)/N$: rescaled index

Localization in spatially extended systems – Numerical results



CLV

GSV

$$h = i/L = 0.2$$

a) CML of Tent maps

$$x^i_{t+1} = (1-2\varepsilon) f\left(x^i_t\right) + \varepsilon\left[f\left(x^{i+1}_t\right) + f\left(x^{i-1}_t\right)\right]$$

$$f(x) = \begin{cases} a\,x & 0 \le x < 1/a \\ \frac{a}{1-a}(x-1) & 1 \ge x \ge 1/a \end{cases}$$

b) Symplectic Maps

$$p^i_{t+1} = p^i_t + \mu\left[g\left(q^{i+1}_t - q^i_t\right) - g\left(q^i_t - q^{i-i}_t\right)\right]$$

$$q^i_{t+1} = q^i_t + p^i_{t+1} \qquad g(x) = \frac{1}{2\pi}\sin(2\pi x)$$

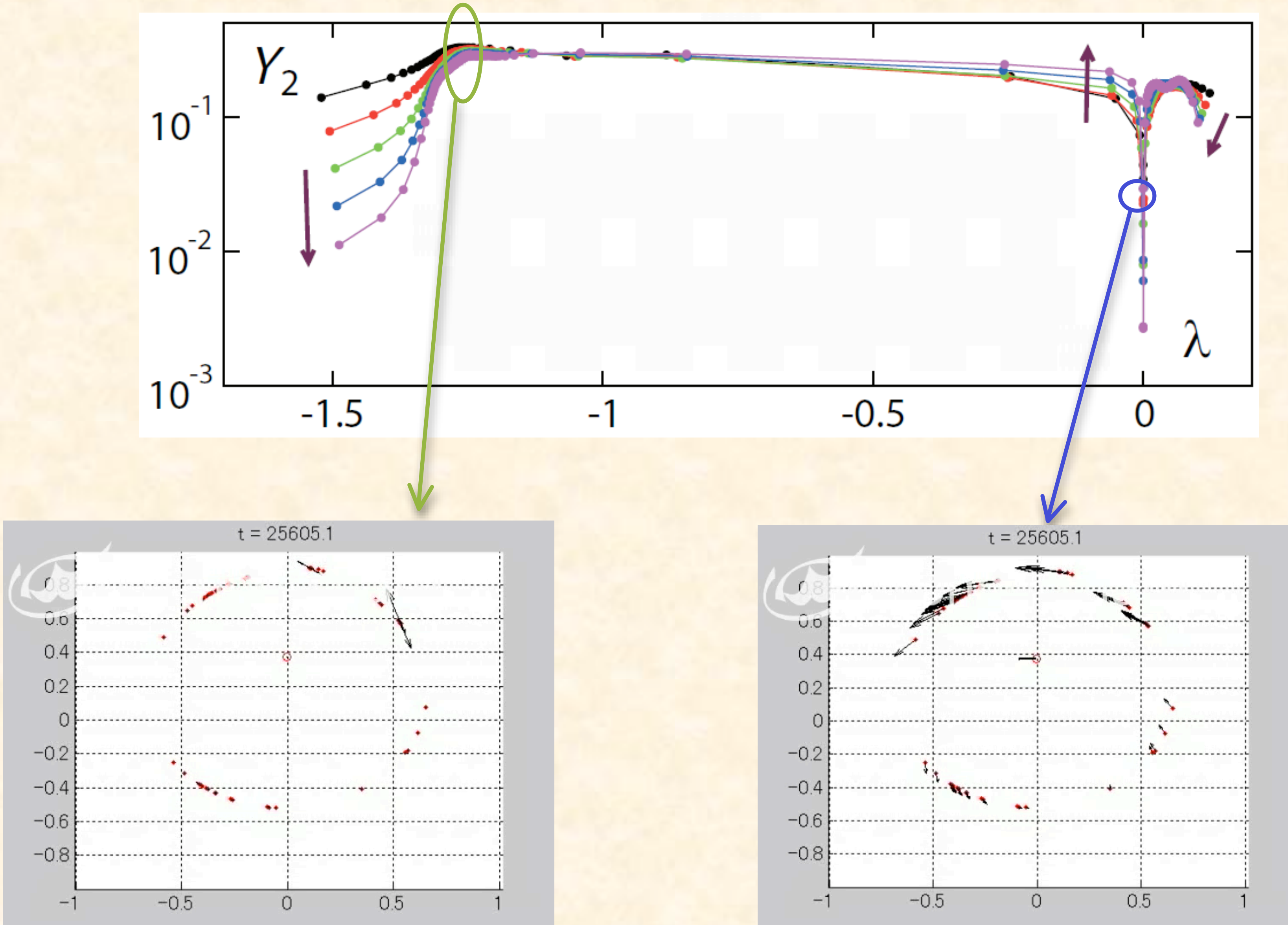
Continuous time Hamiltonian systems

$$\ddot{q}_i = F\left(q_{i+1} - q_i\right) - F\left(q_i - q_{i-1}\right)$$

c) Rotators $\quad F(x) = \sin(x)$

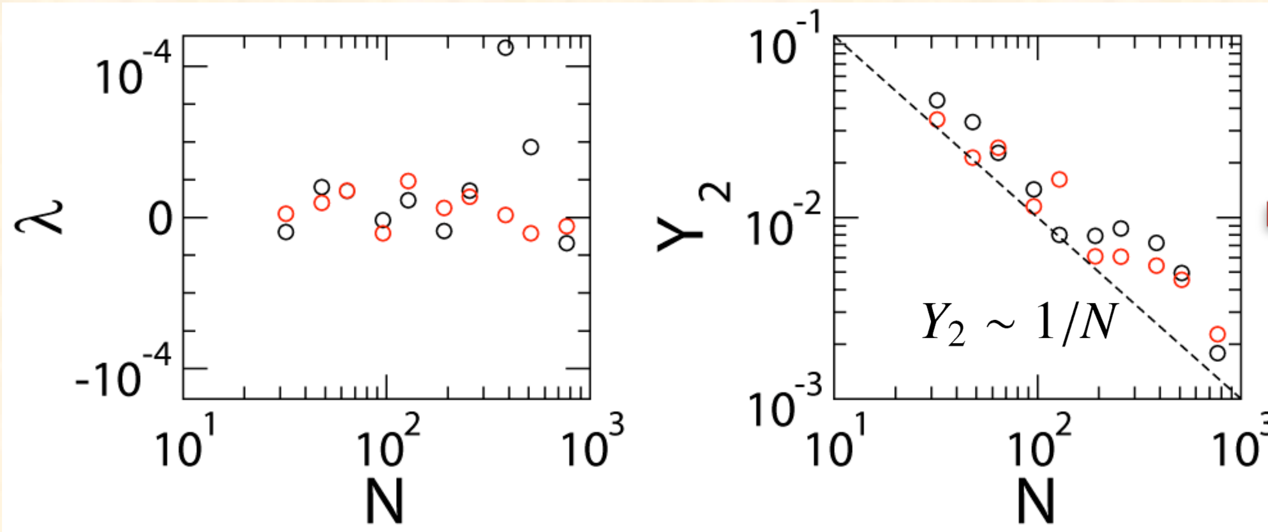
d) FPU $\quad F(x) = x + x^3$

Parametric plot of Y_2 vs λ



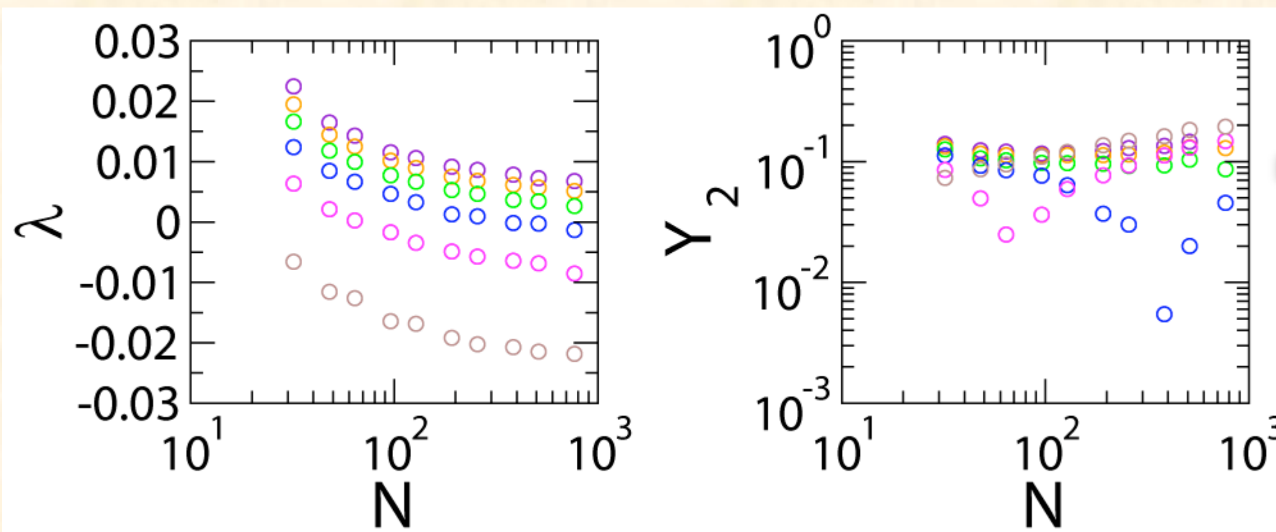
Zero-modes

two numerical zeros



collective!

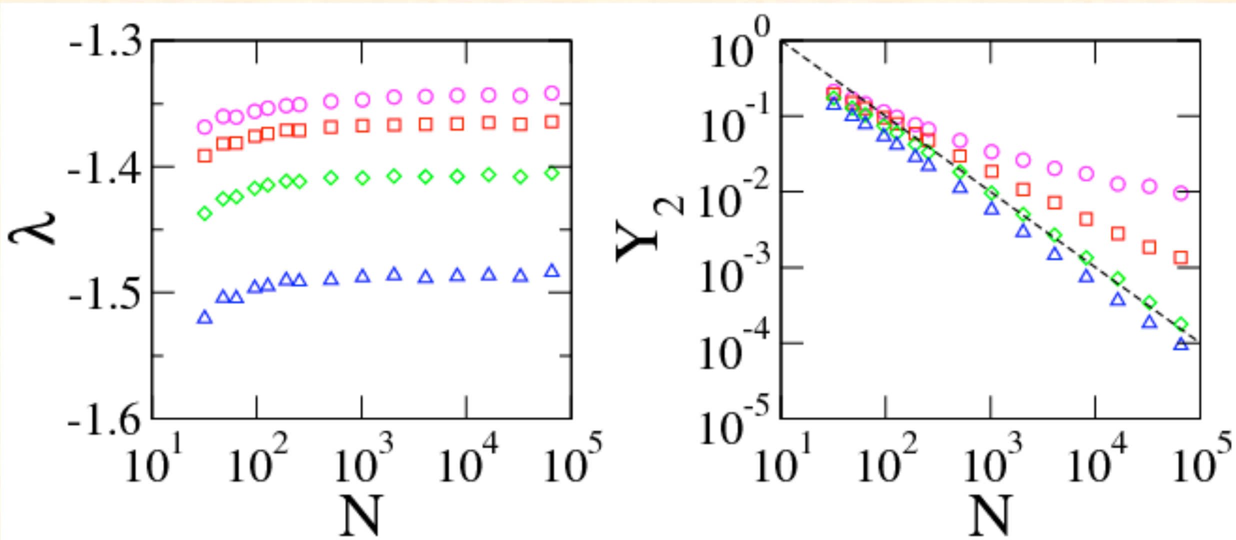
others around zero



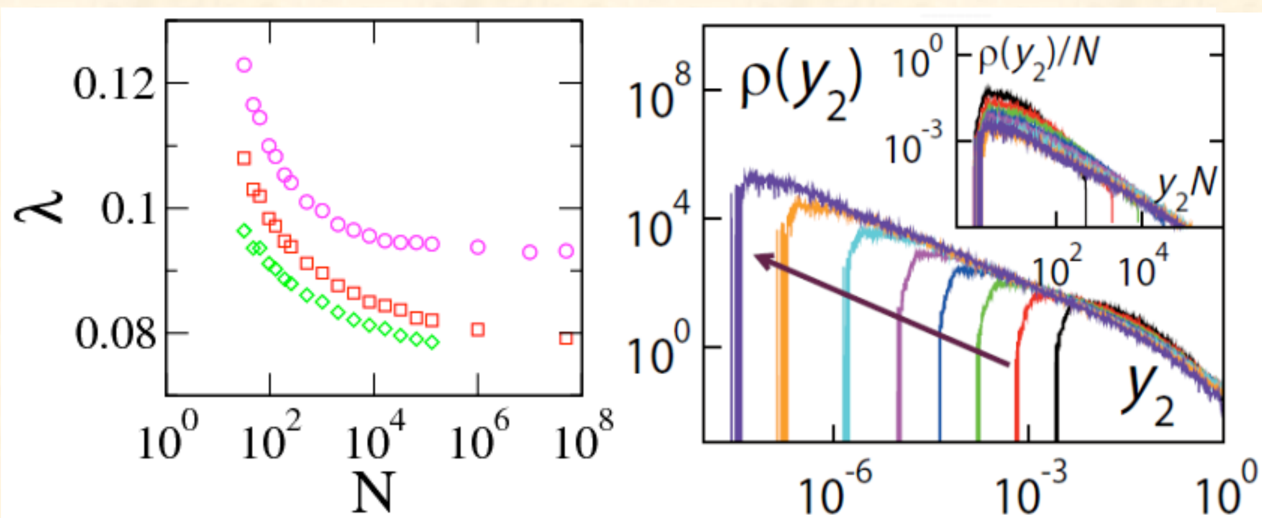
microscopic

Most Positive & Negative Modes

most negative modes



most positive modes



1 positive collective mode!

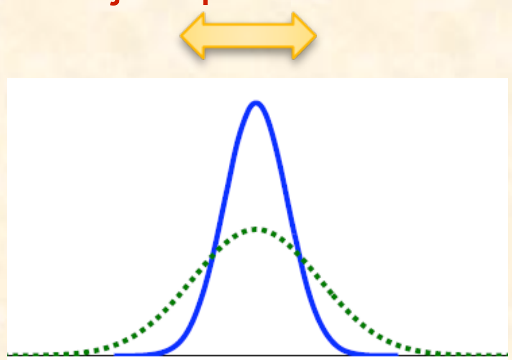
On collective CLV structure

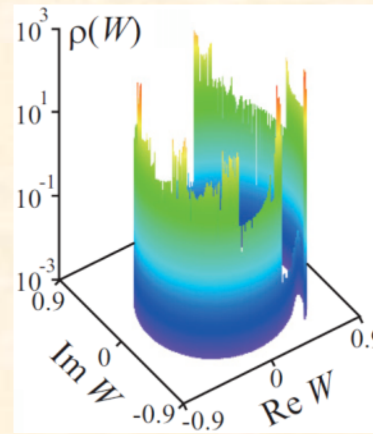
λ > 0 collective

moves peaks entirely

λ < 0 collective

adjust peak width





λ = 0 collective

Degenerate: global change in phases
and traslation along the trajectory

Relation to Perron-Frobenius description

We can “directly” look at the thermodynamic limit through evolution of distribution function via PF equation.

Globally coupled logistic maps with bounded smooth noise (Karumaswamy dist.)

$$x_i^{t+1} = (1-K)f(x_i^t) + K\langle f(x)\rangle + \xi_i^t$$

$$f(x) = 1 - ax^2, a = 1.57$$

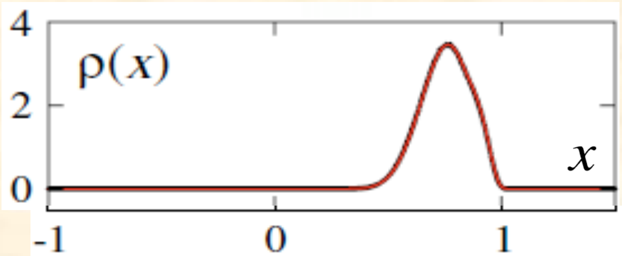
$$\rho^{t+1}(x) = \int \rho_{\text{noise}}(F^t(y) - x)\rho^t(y)\mathrm{d}y$$

with $F^t(y) = (1-K)f(y) + K\int f(z)\rho^t(z)\mathrm{d}z$

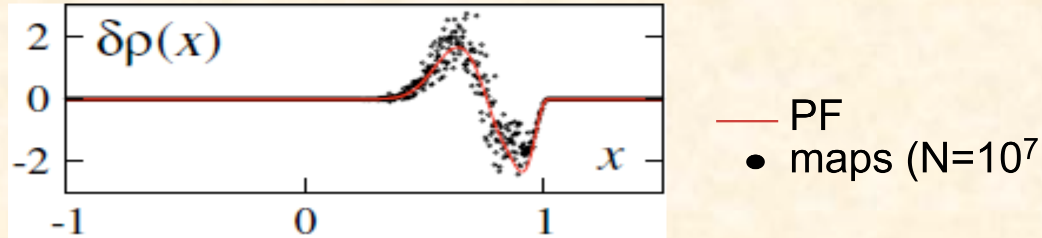
$$\rho_N(\xi) = 15\chi^2(1-\chi^3)^4$$

$$\chi = (\xi/\sigma + 1)/2 \in [0,1]$$

⇨ We compare Lyapunov modes from PF dynamics and delocalized Lyapunov modes from maps

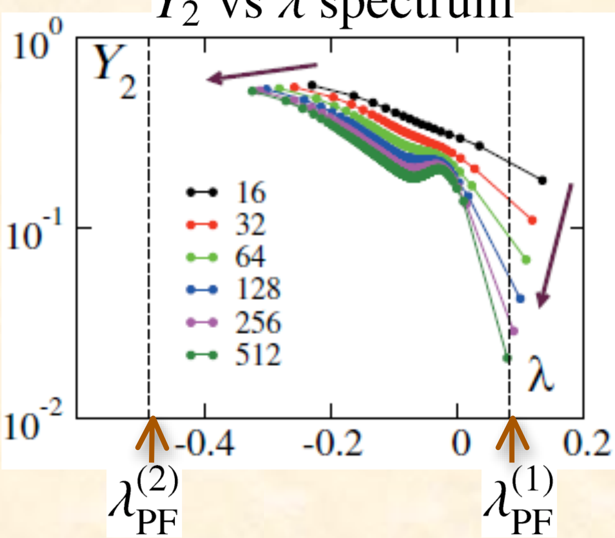


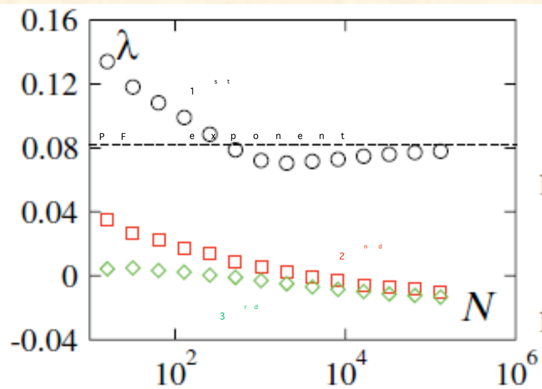
profile of 1st vector

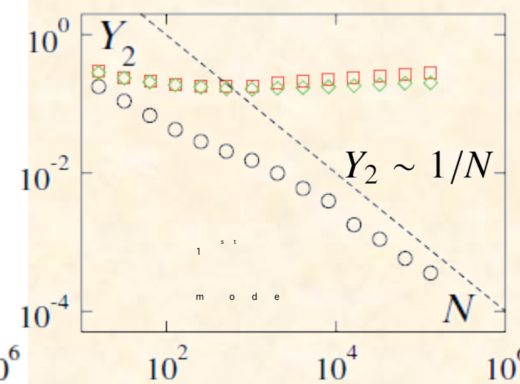


$K = 0.28 \qquad \sigma = 0.1$

Y_2 vs λ spectrum







Quantitative correspondence
between
PF mode & delocalized collective mode!

Conclusions

- Covariant Lyapunov Vectors are the right vectorial quantities to analyze spatiotemporal dynamics.

- They are covariant with dynamics, invariant under time reversal, norm independent and allow to compute LEs by ensamble averages

- For time reversible systems they coincide with Oseledec splitting

- CLVs yield drastically different behavior with respect to GSV (where orthonormalization induced "noise" distrupt dynamical properties) for what concerns spatially extended systems.

- They can be used to detect and analize collective modes in globally coupled systems, thus they can be used to analyze/discriminate different time and length scales in spatiotemporal chaotic systems.

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Yang, K. A. Takeuchi, FG, H. Chaté, and G. Radons, *PRL* **102** 074102 (2009).