Hyperbolic chaos in extended systems constructed of elements with hyperbolic dynamics

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Exploring Complex Dynamics in High–Dimensional Chaotic Systems:
From Weather Forecasting to Oceanic Flows
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What is hyperbolic chaos?

- Each trajectory on the attractor at each point has well defined stable and unstable manifolds. No tangencies between them.
- The “simplest” type of chaos, because permits rigorous mathematical study.
- Structurally stable chaos, because its properties remain unaltered within a wide range of parameters values.

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High dimensional hyperbolicity

Hyperbolic chaos in low–dimensional systems is studied well.

The interesting question is to study a hyperbolic chaos in spatially distributed system with a large number of degrees of freedom.

We take identical oscillators with hyperbolic chaos and construct a chain introducing diffusive coupling:

- the chaos is hyperbolic when the coupling is absent or very small so that the oscillators are independent or almost independent;
- the chaos is hyperbolic when the coupling is very strong so that the oscillators are fully synchronized;
- the subject is to understand what happens between these two limiting cases.

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System of two coupled amplitude equations

Motivation: amplitude equations for two coupled non autonomous van der Pol oscillators with hyperbolic chaos
[S. P. Kuznetsov, Phys. Rev. Lett. 95 (2005) 144101]

$$\dot{a} = Aa\cos(2\pi t/T) - (1+\mathrm{i}c)|a|^2 a - \mathrm{i}\epsilon\, b$$
$$\dot{b} = -Ab\cos(2\pi t/T) - (1+\mathrm{i}c)|b|^2 b - \mathrm{i}\epsilon\, a^2$$

- complex variables $a \equiv a(t)$ and $b \equiv b(t)$;
- external force with amplitude A and period T;
- rotation with frequency c if no force and interaction;
- skew coupling with parameter ϵ.

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Detecting hyperbolicity using Covariant Lyapunov Vectors

1. Compute CLVs at some point of the attractor.
2. Vectors corresponding to positive Lyapunov exponents span expanding subspace of the tangent space, and those with negative exponents span the contracting subspace.
3. Mutual orientation of two subspaces is characterized by the principal angles[1]. If the first principal angle θ_1 vanishes then the contracting and expanding subspaces have a tangency. The tangency means the violation of hyperbolicity.
4. Store a lot of θ_1 at different points of the attractor and compute their distribution $P(\theta_1)$. Attractor is hyperbolic if this distribution is clearly separated from the origin.

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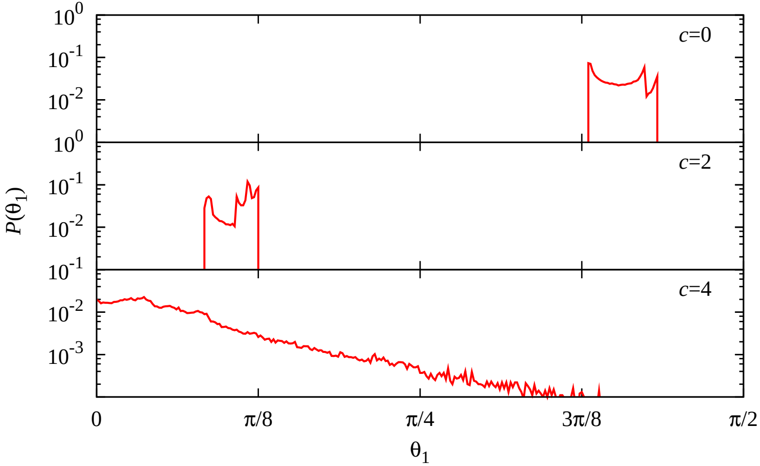
Distributions of the first principal angle θ_1

Hyperbolicity at $c = 0$, and $c = 2$. Non-hyperbolic chaos at $c = 4$.

Show equations



Equations of the chain

Chain of N oscillators with hyperbolic chaos:

$$\dot{a}_n = A\cos(2\pi t/T)a_n - (1+\mathrm{i}c)|a_n|^2 a_n - \mathrm{i}\epsilon\, b_n + \kappa(a_n)/h^2,$$
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- diffusive coupling: $\kappa(z_n) = z_{n-1} - 2z_n + z_{n+1}$ $(n = 1, \dots N-2)$;
- h controls the strength of the coupling;
- no-flux b.c.: $\kappa(z_0) = 2(z_1 - z_0)$, $\kappa(z_{N-1}) = 2(z_{N-2} - z_{N-1})$.

We want to know when this chain demonstrates hyperbolic chaos.

Control parameters:

- h controls the coupling (small h = high coupling);
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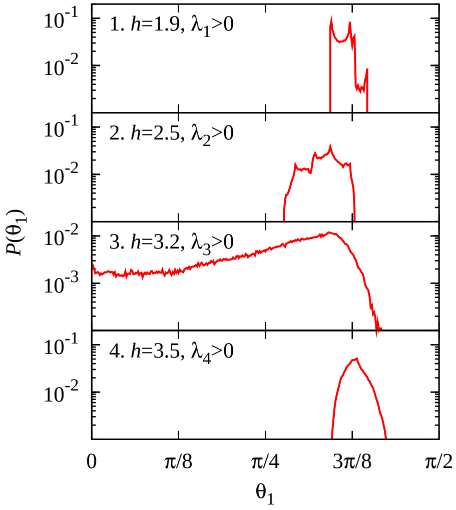
Scenario at $c = 0$, $N = 4$

When h grows, i.e., the coupling gets smaller:

1. $\lambda_1 > 0$. Full chaotic synchronization. Hyperbolic chaos.
2. $\lambda_2 > 0$. Chaos is still hyperbolic.
3. $\lambda_3 > 0$. Destruction of hyperbolicity.
4. $\lambda_4 > 0$. Hyperbolicity is recovered.



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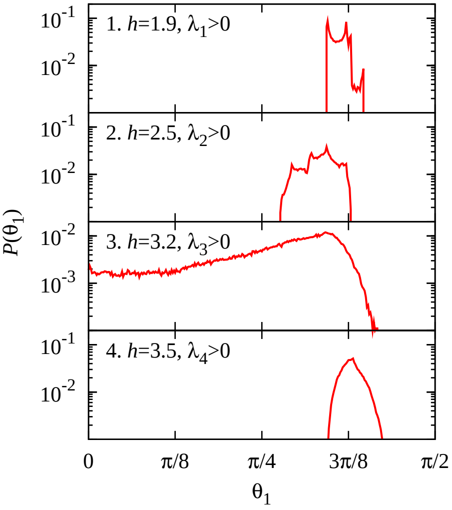

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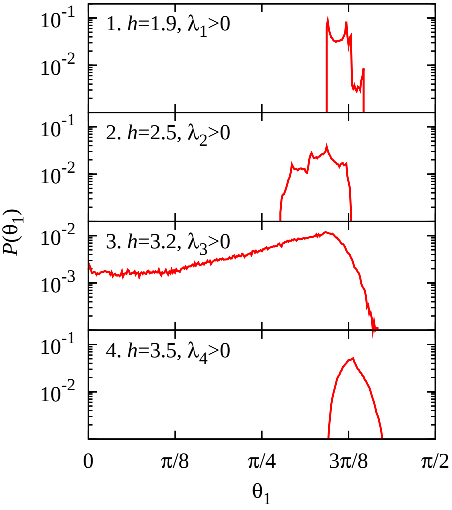

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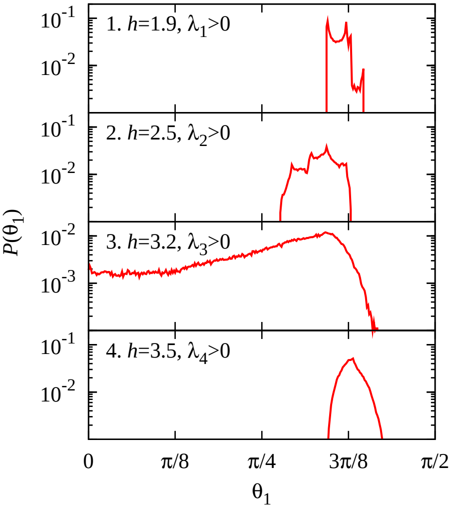
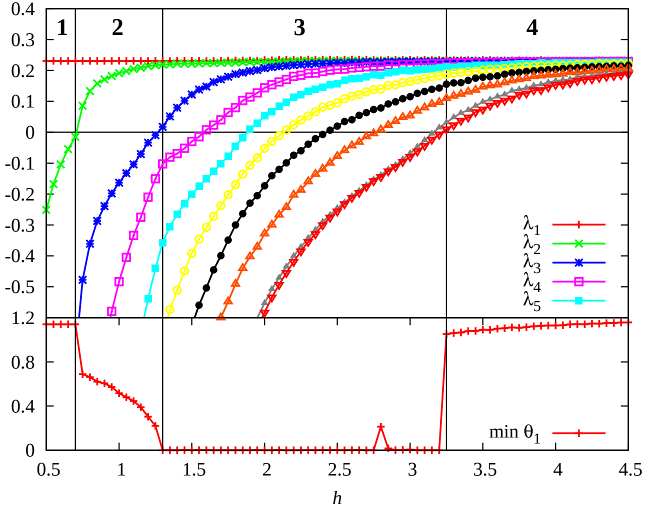


Lyapunov exponents and min. angle θ_1 at $c = 0$, $N = 10$



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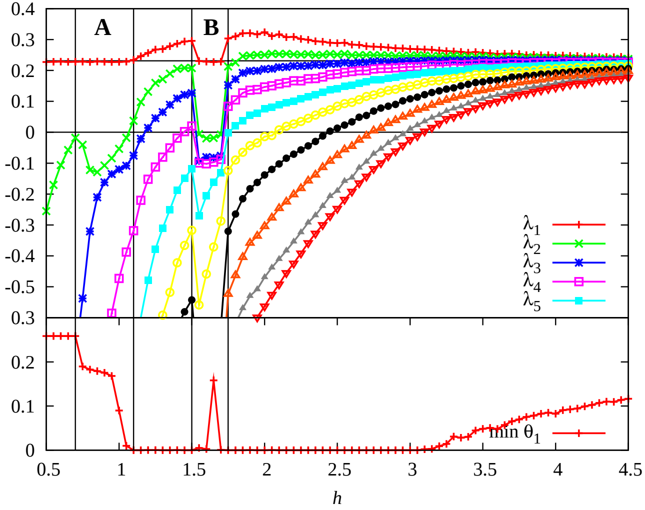
Lyapunov exponents and min. angle θ_1 at $c = 2$, $N = 10$



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What have we learned from these pictures?

Hyperbolicity at strong coupling exists:

1. When oscillators are fully synchronized and there is 1 positive Lyapunov exponent.
2. When there are 2 positive Lyapunov exponents at $c < 1$ (domain **2**).
3. When full synchronization is absent, but the chain has 1 positive Lyapunov exponent at $c > 1$ (domains **A** and **B**).

The chaos is also hyperbolic when the coupling is small so that each oscillator has its own positive Lyapunov exponent.

As we shall see below, the chaos at strong coupling is hyperbolic when the chain is synchronized somehow.

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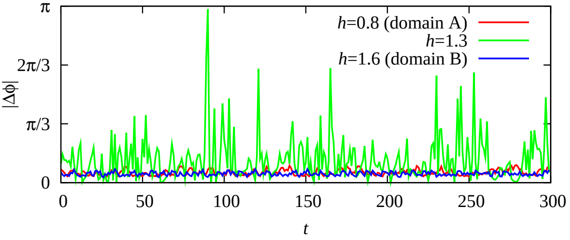
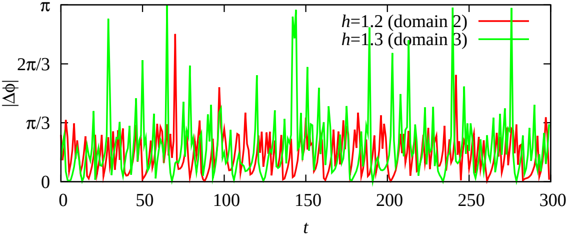
Difference between two complex phases

$$\Delta\phi = \arg(a_{N/2}) - \arg(a_{N/2+1})$$

- $c = 2$, doms. A,B
  Strong synchronization of complex phases

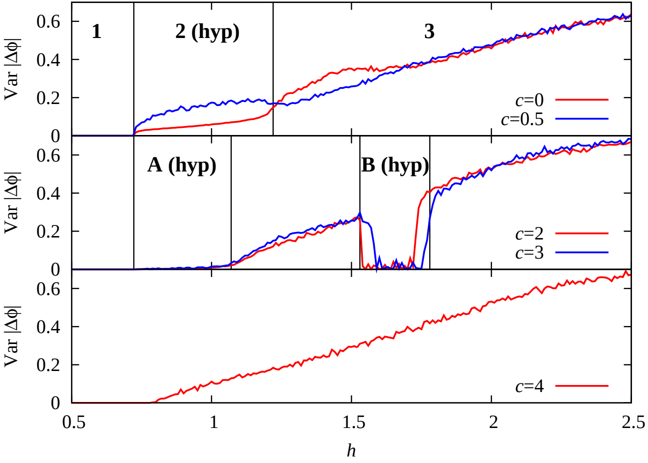


- $c = 0$, dom. 2
  Weak synchronization of complex phases



Variance of $|\Delta\phi|$ against h

Chaos is hyperbolic when oscillators are synchronized!

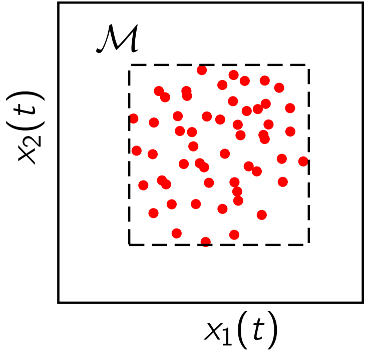


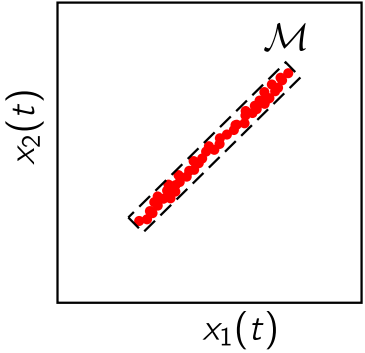
How can we detect synchronization in a chain of oscillators?

Synchronization results in the reduction of the effective dimension of the manifold $\mathcal{M}$ containing the attractor.

no synchronization, $\mathcal{M} \subset \mathbb{R}^2$

synchronization, $\mathcal{M}$ has one essential dimension





The idea of Principal Components Analysis

- Collect a set of points $\vec{x}(1)$, $\vec{x}(2)$, ..., where $\vec{x}(t) = \{x_1(t), \dots, x_N(t)\}$.
- Compute the covariance matrix

$$C = \begin{pmatrix} \mathrm{cov}(x_1, x_1) & \dots & \mathrm{cov}(x_1, x_N) \\ \vdots & \ddots & \vdots \\ \mathrm{cov}(x_N, x_1) & \dots & \mathrm{cov}(x_N, x_N) \end{pmatrix}.$$

- Compute eigenvalues and eigenvectors of C and sort them by descending of eigenvalues. Eigenvectors are orthogonal to each other.

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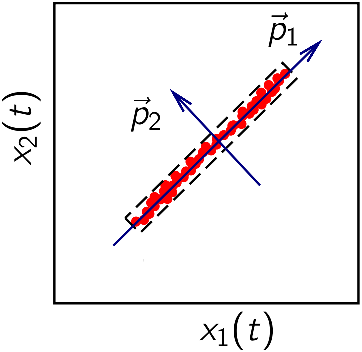
- The first eigenvector $\vec{p}_1$ goes along the most extended direction of the cloud, $\vec{p}_2$ gives the second important dimension and so on.
- Eigenvalues μ_i characterize the sizes of the cloud in these directions.



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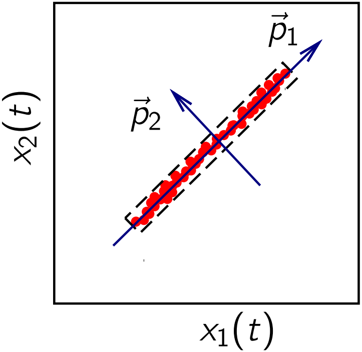

Local PCA for multidimensional attractor

The supporting manifold $\mathcal{M}$ may not be flat. The PCA should be applied locally.

- Take a point on the attractor.
- Collect a cloud of attractor points that fall in a small vicinity of the first point.
- Construct the covariance matrix and find its sorted eigenvalues μ_i.
- Repeat this for many points of the attractor and average the eigenvalues.

We expect that in presence of the synchronization several first averaged eigenvalues are much higher than all others.

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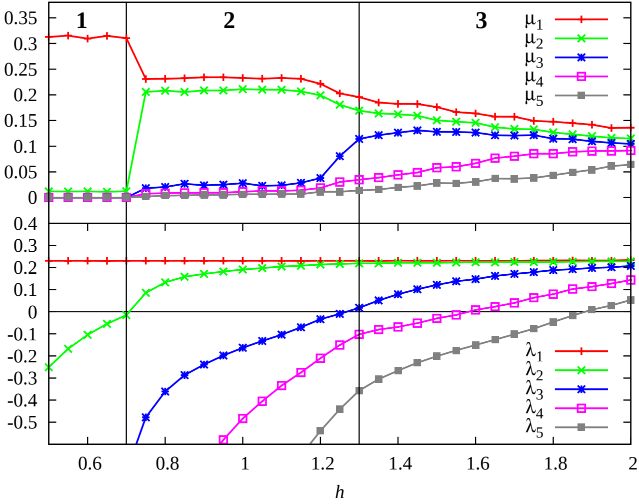
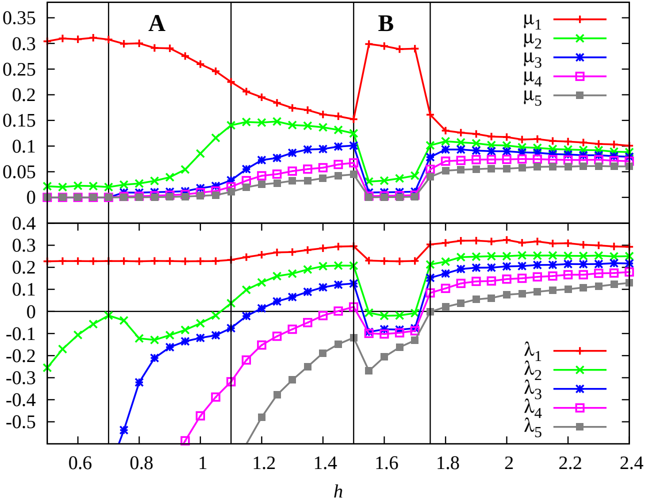
Averaged PCA eigenvalues and Lyapunov exponents, $c = 0$



Averaged PCA eigenvalues and Lyapunov exponents, $c = 2$



What have we learned now?

For the considered chain we observe:

1. Chaos at strong coupling (small h) is hyperbolic when the chain is synchronized.
2. The synchronization manifests itself as a small effective dimension of the supporting manifold $\mathcal{M}$.
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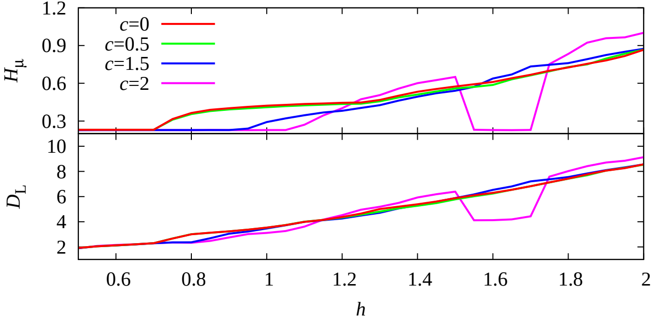
Kaplan–Yorke dimension and Kolmogorov–Sinai entropy

At strong coupling (small h) we observe:

- $H_\mu(h)$ (upper estimate of KS-entropy as a sum of positive Lyap. exponents) has a break when hyperbolicity disappears.
- Hyperbolic chaos exists at small values of KY-dimension D_{KY}.



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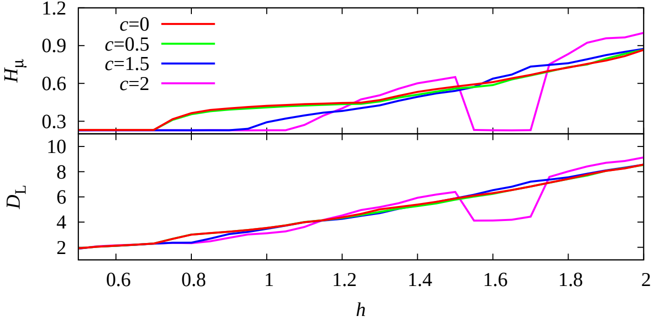

Results: Hyperbolic chaos in the chain

When the coupling is strong, the chaos is hyperbolic if the chain is synchronized:

- full chaotic synchronization;
- weak synchronization of complex phases at 2 positive Lyapunov exponents, and 2 essential dimensions of the supporting manifold;
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Otherwise, hyperbolicity exists when subsystems are almost independent so that each oscillator has its own unstable direction.

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