Approaches to design of realizable systems with structurally stable chaotic attractors

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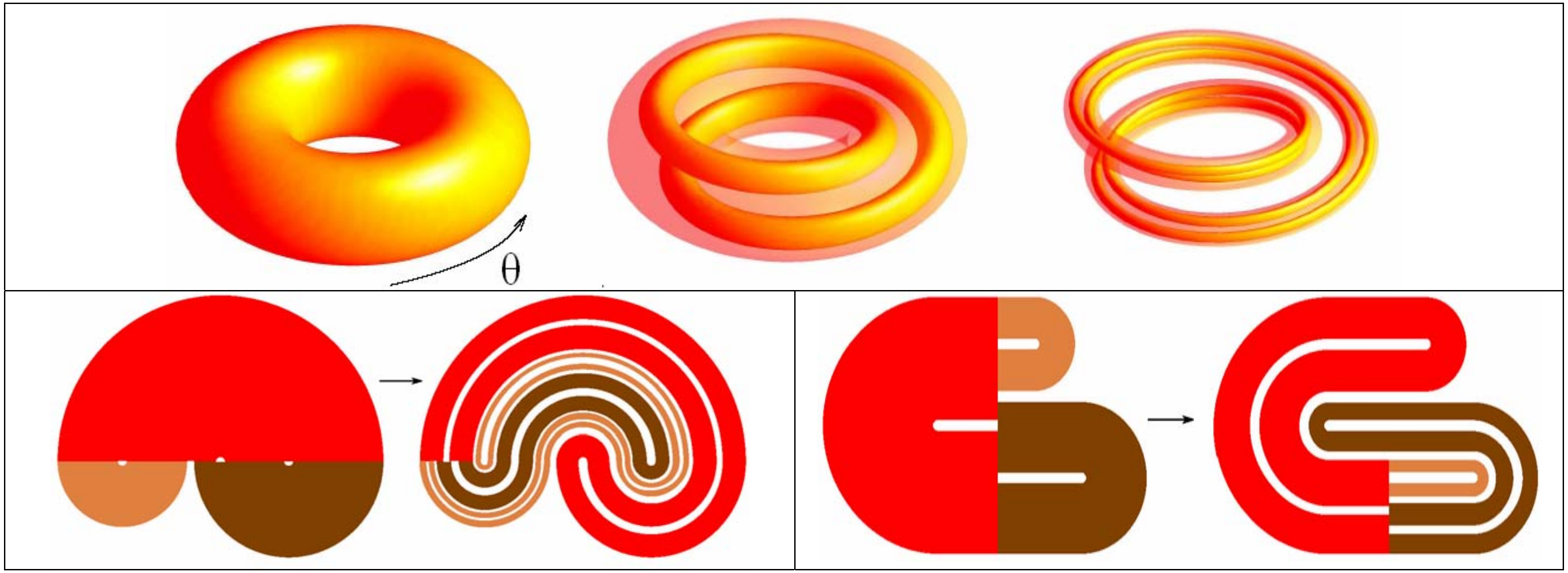
Uniformly hyperbolic attractors were introduced about 40 years ago, due to Smale, Anosov, Alekseev, Williams, Sinai, Ruelle, Newhouse and others. Traditional examples of uniformly hyperbolic attractors are discrete-time geometric models, like Smale-Williams attractor or Plykin attractor.



Initially, it was expected that they will be adequate for many real situations of chaotic behavior, like hydrodynamic turbulence etc. After time passed, it became clear that the early hyperbolic theory is too narrow to include majority of chaotic systems interesting for applications. So, efforts of matematicians were redirected on generalizations of the theory appropriate to wider classes of systems. E.g. concepts were elaborated of nonuniformly hyperbolic atteactors, partially hyperbolic systems, quasi-hyperbolic or singular hyperbolic attractors, quasiattractors, etc.

Largely forgotten remains the question: Is it possible, nevertheless, to find real-world systems, or to design realizable models in physics and technology, with uniformly hyperbolic attractors?

1) Some physical systems allow natural description in discretized time, and it is interesting to consider a possibility for uniformly hyperbolic attractors to appear in the respective maps. In addition to the geometric constructions, we need to have discrete-time models represented explicitly to compute practically significant characteristics of chaos.

2) Next, we have to turn to the continuous time case, as much more relevant in physics and technology; appropriate models should be designed and represented explicitly by differential equations.

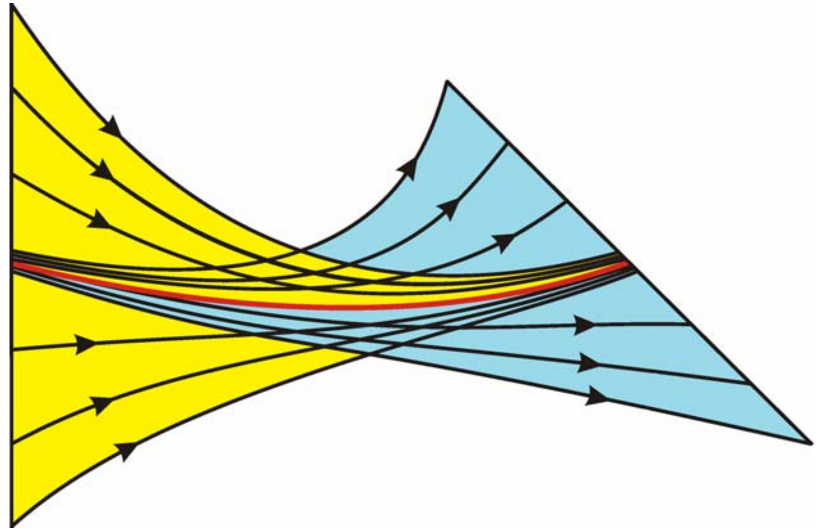
3) It is desirable to implement the uniformly hyperbolic attractors with combination of structural elements conventional in the oscillation theory and applications, like oscillators, couplings, and feedback loops.

4) The designed models has to be created, as real operating devices, e.g. in electronics, mechanics, nonlinear optics. Finally, applications for these devices have to be elaborated, and advantages over possible alternatives sustained.

The problem seems to be of fundamental significance. An analogy may be noticed with such a historical precedent as establishing correspondence between self-oscillations and limit cycles. Like the limit cycles in the previous century, the hyperbolic attractors should find their place, as mathematical images of real nonlinear phenomena. Especially, because of structural stability: as objects, insensitive to variations in underlaying equations, the hyperbolic attrators are surely obliged to present in real-world nonlinear phenomena!

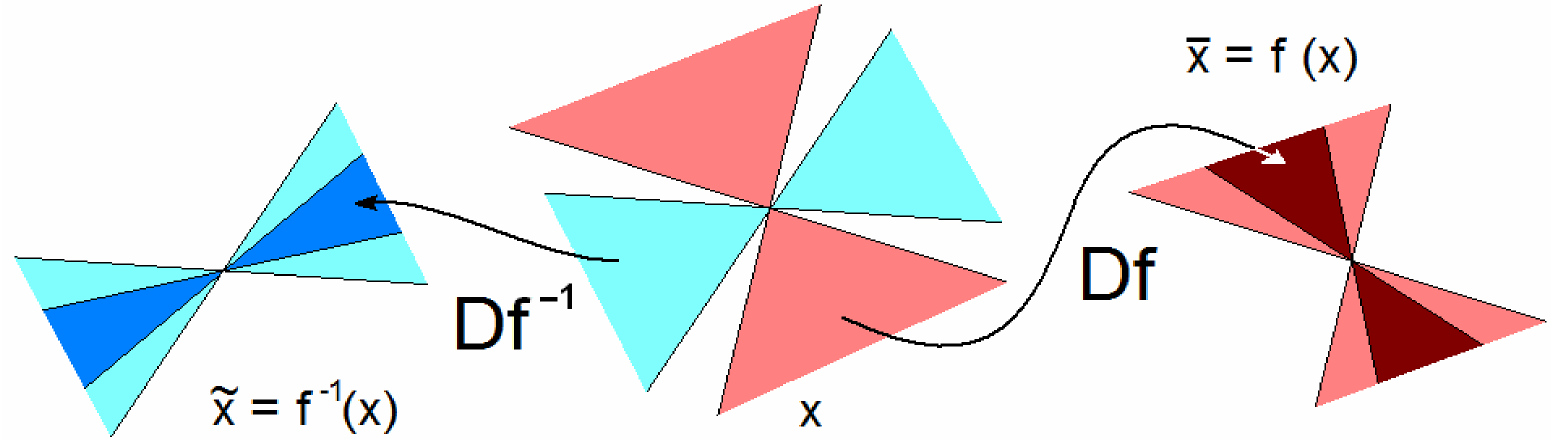
The present talk is aimed on a review of results obtained in the outlined direction by our research team is Saratov (in collaboration with prof. A.Pikovsky in Potsdam).

Uniformly hyperbolic attractor is an attracting object in phase space of a dissipative dynamical system that consists exclusively of **saddle trajectories.**



Their stable and unstable manifolds have the same dimension for all trajectories on the attractor; they should not be touching; only crossings at nonzero angles are allowed.

The hyperbolic nature of attractors can be verified with the cone criterion. The picture below illustrates this for a discrete-time system (iterated map).

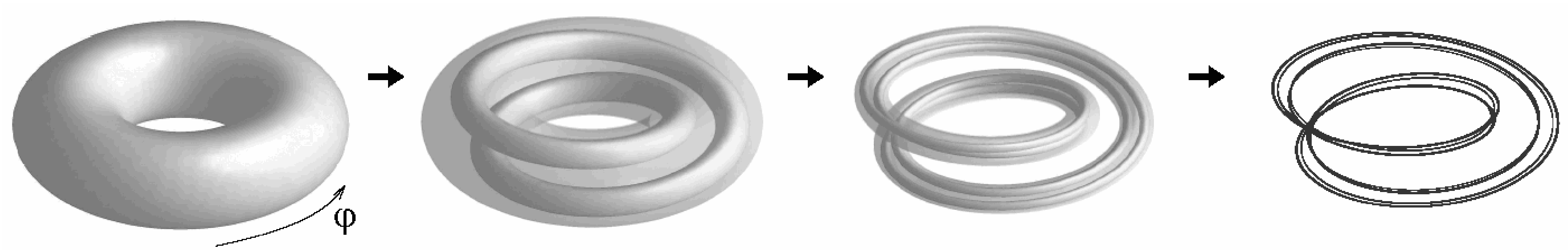


Cones of expanding and contracting infinitesimal perturbation vectors must exist at every point of a region containing the attractor, smoothly depending on the position. An image of expanding cone must be placed inside expanding cone at the image point, and preimage of contracting cone must be placed inside contracting cone at the preimage point.

For flows the same considerations are applicable in terms of Poincaré map.

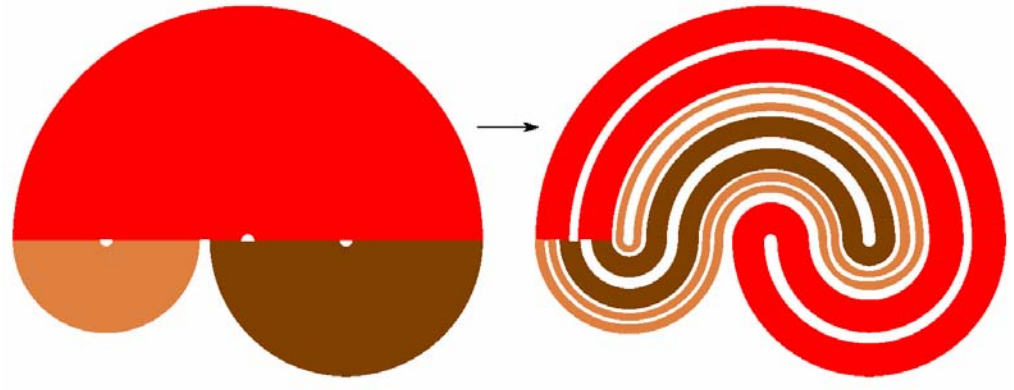
The geometric constructions of hyperbolic attractors

Smale-Williams attractor

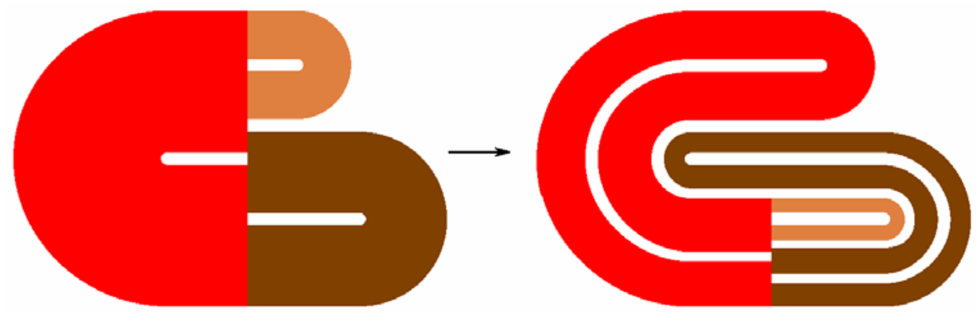


Plykin attractors

Original Plykin attractor

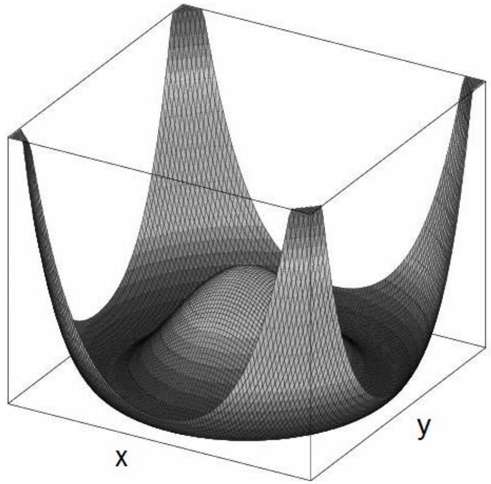


Plykin-type attractor



Smale-Williams attractors in a simple mechanical setup

Consider a particle on a plane (x, y) with friction force proportional to instant velocity, and in potential field $U(x,y) = -\frac{1}{2}(x^2+y^2)+\frac{1}{4}(x^2+y^2)^2$, with minimum on the unit circle. With period T, the particle undergoes kicks, accepting momentum $\mathbf{P}(x,y)$, depending on the instant position (x,y). To define the field $\mathbf{P}$, consider a set of particles placed initially on the unit circle $x=\cos\varphi,\ y=\sin\varphi$. After a kick, each particle accepts some momentum, and, in absence of the potential field, it would stop at $x'=x+P_x(x,y),\ y'=y+P_y(x,y)$. Require the particles to come just at the unit circle, forming a twofold loop, we set $x'=\cos\varphi',\ y'=\sin\varphi'$, $\varphi'=2\varphi$. It is the case if we specify the field components as

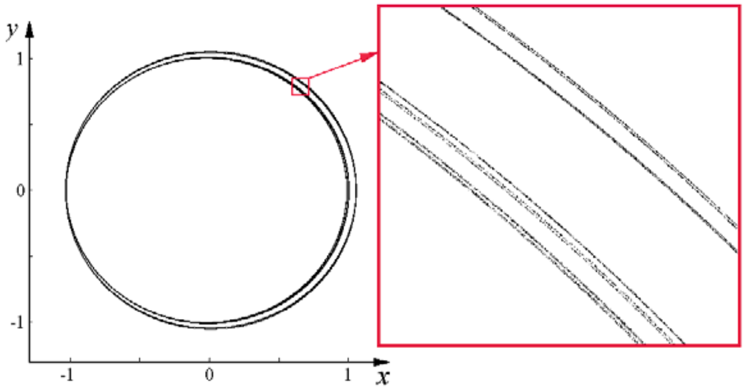


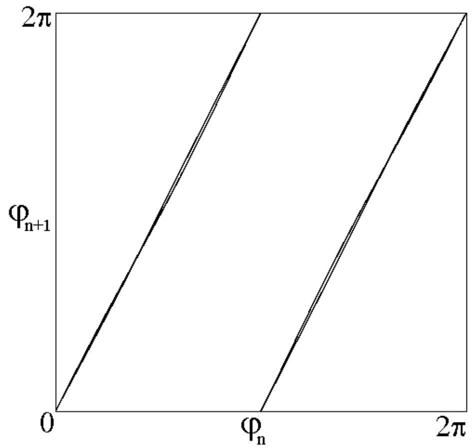
$$P_x(x,y)=x'-x=\cos 2\varphi-\cos\varphi=2x^2-x-1,\quad P_y(x,y)=y'-y=\sin 2\varphi-\sin\varphi=2xy-y.$$

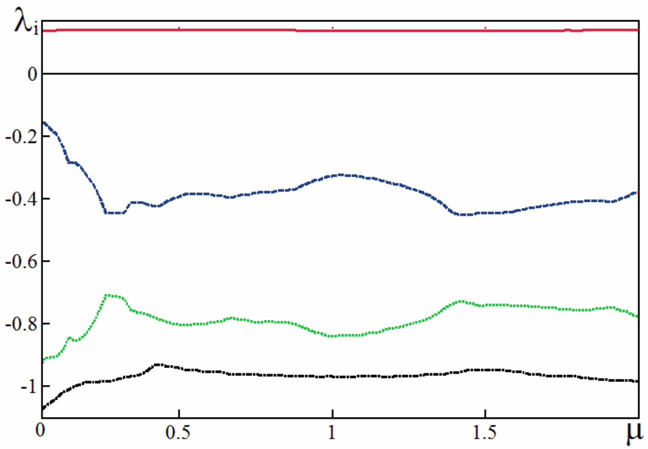
With unit mass and unit friction coefficient, introducing μ as intencity of the potential field, we get

$$\ddot{x}+\dot{x}-\mu x(1-x^2-y^2)=(2x^2-x-1)\sum\delta(t-nT),\quad \ddot{y}+\dot{y}-\mu y(1-x^2-y^2)=(2xy-y)\sum\delta(t-nT)$$

μ=0.44, T=5







The largest Lyapunov exponent: $\lambda_1 \approx T^{-1}\ln 2$. **Dimensions at** μ=0.44: D_{KY}=1,328, D_{GH}=1,325.

Design of hyperbolic attractors by means of evolution in successive stages

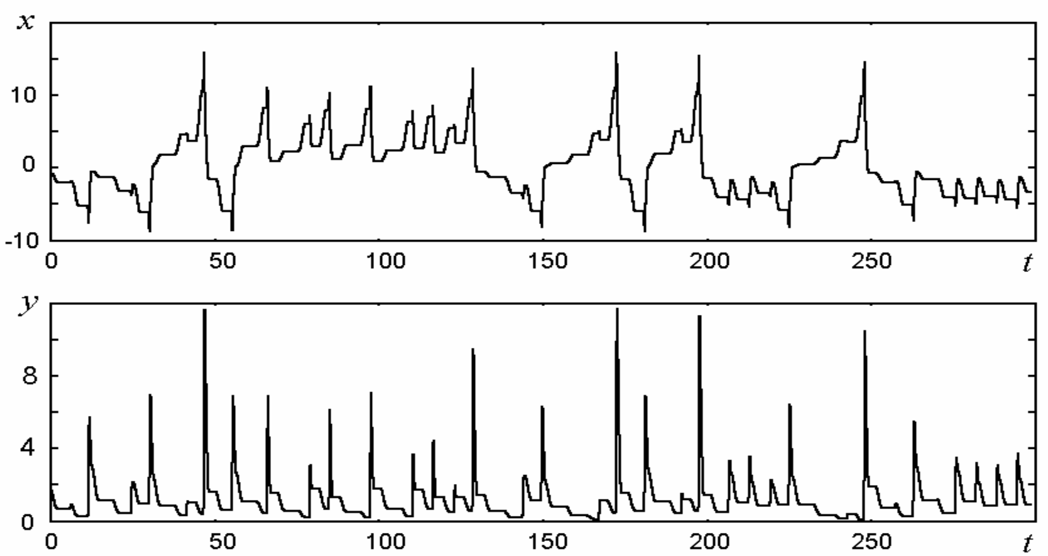
Suspending Plykin-type attractor: the Hunt model

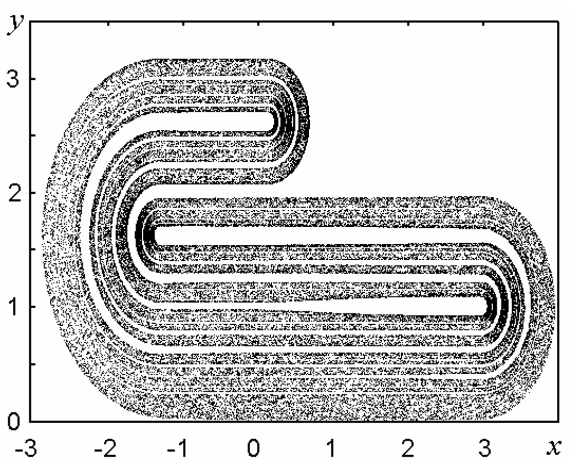
Hunt model is a non-autonomous system

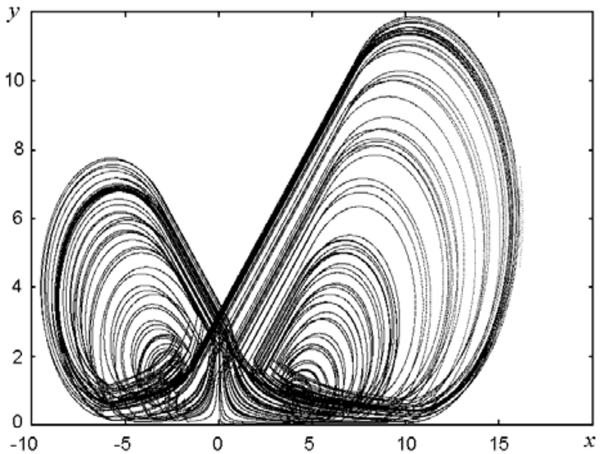
$$dx/dt = f_*(x, y, t),$$
$$dy/dt = g_*(x, y, t),$$

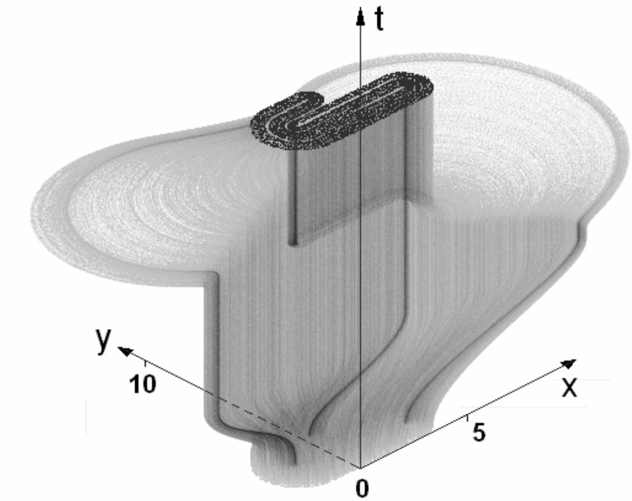
where the functions f_* and g_* are continuous and differentiable, of period 2π in time.

[Hunt T.J, PhD Thesis, 2000; Aidarova Yu.S. and Kuznetsov S.P., arXiv:0901.2727]









Lyapunov exponents: $\Lambda_1 = 0.9625$, $\Lambda_2 = -1.213$**, and attractor dimension** $D_{KY} \approx 1 + \Lambda_1 / |\Lambda_2| = 1.79$

Plykin-type attractor in a map generated by a flow on a sphere

[S.P. Kuznetsov, CNSNS, 14, 2009, 3487]

I. Flow down along circles of latitude, $0 \le t < 1$**,**

$$\dot{x} = -\varepsilon x y^2,\;\; \dot{y} = \varepsilon x^2 y,\;\; \dot{z} = 0.$$

II. Differential rotation around *z*-axis, $1 \le t < 2$**,**

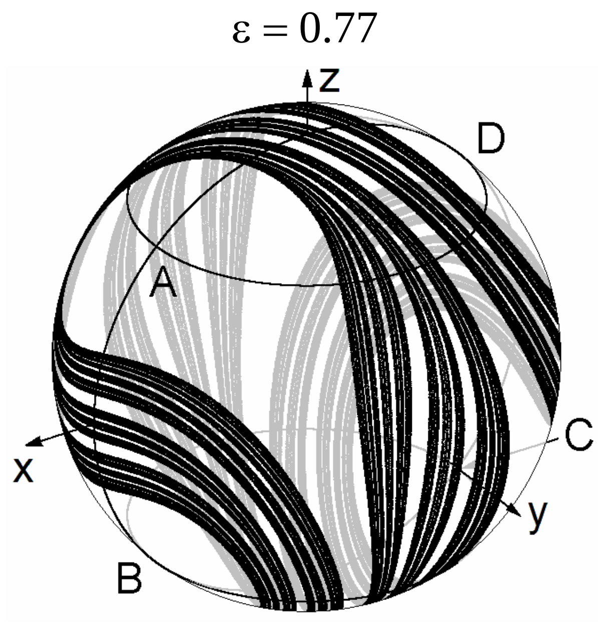
$$\dot{x} = \pi(z/\sqrt{2} + 1/2) y,\;\; \dot{y} = -\pi(z/\sqrt{2} + 1/2) x,\;\; \dot{z} = 0.$$

III. Flow down to the equator, $2 \le t < 3$**,**

$$\dot{x} = 0,\;\; \dot{y} = \varepsilon y z^2,\;\; \dot{z} = -\varepsilon y^2 z.$$

IV. Differential rotation around *x*-axis, $3 \le t < 4$**,**

$$\dot{x} = 0,\;\; \dot{y} = -\pi(x/\sqrt{2} + 1/2) z,\;\; \dot{z} = \pi(x/\sqrt{2} + 1/2) y.$$



Equivalently, it may be written as a unified set of equations:

$$\begin{aligned}
\dot{x} &= -f_1(t)\varepsilon x y^2 + f_2(t)\pi(z/\sqrt{2} + 1/2) y \\
\dot{y} &= f_1(t)\varepsilon x^2 y - f_2(t)\pi(z/\sqrt{2} + 1/2) x + f_3(t)\varepsilon y z^2 - f_4(t)\pi(x/\sqrt{2} + 1/2) z \\
\dot{z} &= -f_3(t)\varepsilon y^2 z + f_4(t)\pi(x/\sqrt{2} + 1/2) y
\end{aligned}$$

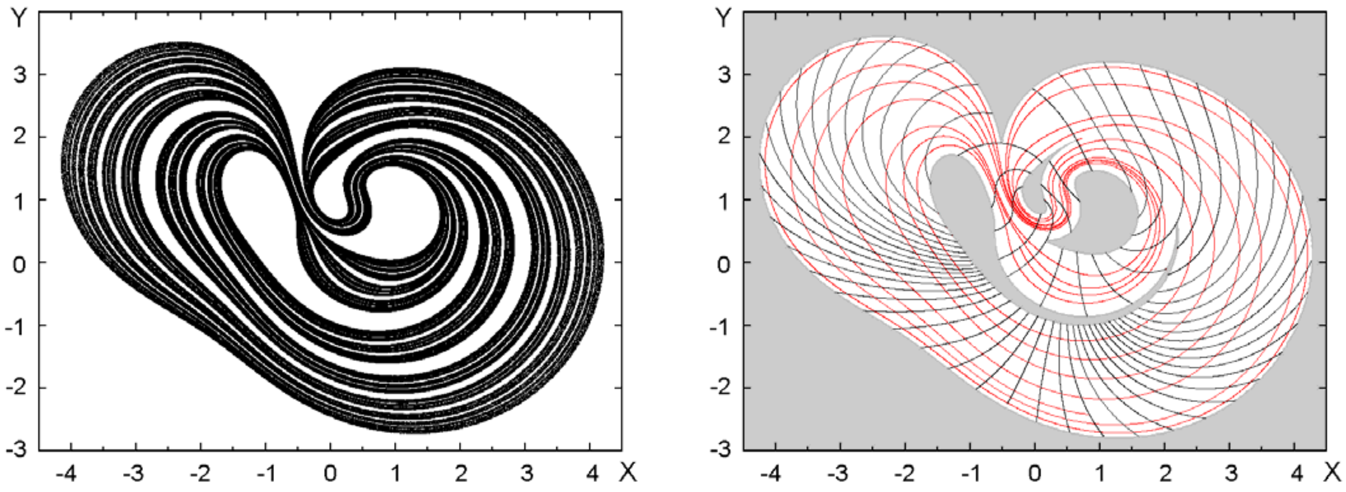
The Poincaré map

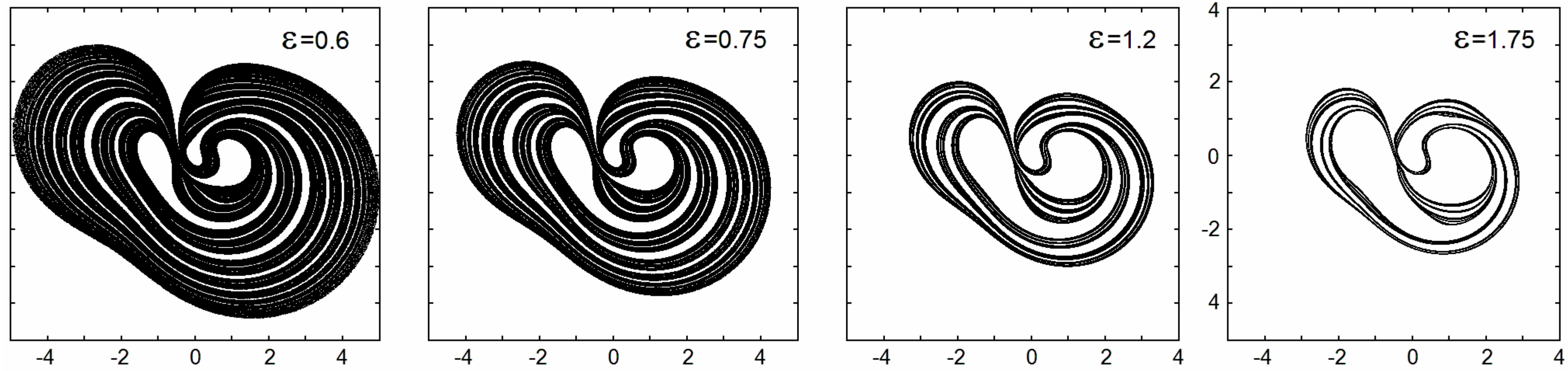
$$\mathbf{x}_{n+1} = \mathbf{f}_+(\mathbf{f}_-(\mathbf{x}_n)),\qquad \mathbf{f}_\pm(\mathbf{x}) = \begin{pmatrix} \pm z \\ \sqrt{x^2+y^2}\,\dfrac{y e^{\frac{\varepsilon}{2}(x^2+y^2)} \cos\frac{\pi}{2}(z\sqrt{2}+1) \pm x e^{-\frac{\varepsilon}{2}(x^2+y^2)} \sin\frac{\pi}{2}(z\sqrt{2}+1)}{\sqrt{x^2 e^{-\varepsilon(x^2+y^2)} + y^2 e^{\varepsilon(x^2+y^2)}}} \\ \sqrt{x^2+y^2}\,\dfrac{y e^{\frac{\varepsilon}{2}(x^2+y^2)} \sin\frac{\pi}{2}(z\sqrt{2}+1) \mp x e^{-\frac{\varepsilon}{2}(x^2+y^2)} \cos\frac{\pi}{2}(z\sqrt{2}+1)}{\sqrt{x^2 e^{-\varepsilon(x^2+y^2)} + y^2 e^{\varepsilon(x^2+y^2)}}} \end{pmatrix}$$

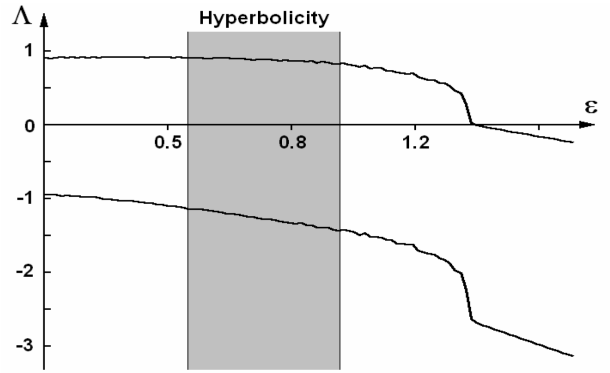
Lyapunov exponents: $\Lambda_1 = 0.959$**,** $\Lambda_2 = -1.141$**, and attractor dimension** $D_{KY} \approx 1 + \Lambda_1/|\Lambda_2| = 1.841$**.**

Plykin-type attractor on a plane

The variable change: $W = X + iY = \dfrac{x - z + iy\sqrt{2}}{x + z + \sqrt{2}}$







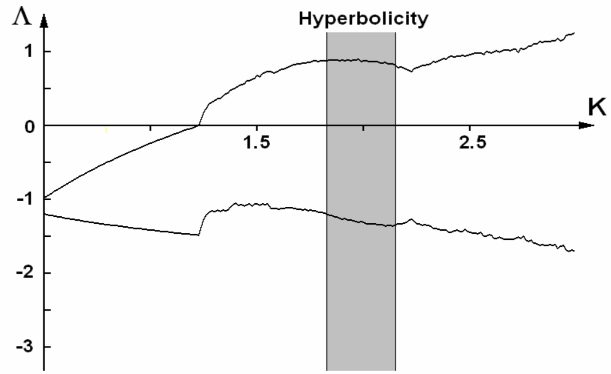
A version with smooth coefficients

$$\dot{X} = -2\varepsilon Y^2\Omega_1\left(\cos(\tfrac{\pi}{4}\cos\tfrac{\pi}{2}t) - X\sin(\tfrac{\pi}{4}\cos\tfrac{\pi}{2}t)\right)$$
$$+ KY\Omega_2\left(\cos(\tfrac{\pi}{4}\sin\tfrac{\pi}{2}t) - X\sin(\tfrac{\pi}{4}\sin\tfrac{\pi}{2}t)\right)\sin\tfrac{\pi t}{2},$$
$$\dot{Y} = 2\varepsilon Y\Omega_1\left(X\cos(\tfrac{\pi}{4}\cos\tfrac{\pi}{2}t) + \tfrac{1}{2}(1 - X^2 + Y^2)\sin(\tfrac{\pi}{4}\cos\tfrac{\pi}{2}t)\right)$$
$$- K\Omega_2\left(X\cos(\tfrac{\pi}{4}\sin\tfrac{\pi}{2}t) + \tfrac{1}{2}(1 - X^2 + Y^2)\sin(\tfrac{\pi}{4}\sin\tfrac{\pi}{2}t)\right)\sin\tfrac{\pi t}{2},$$

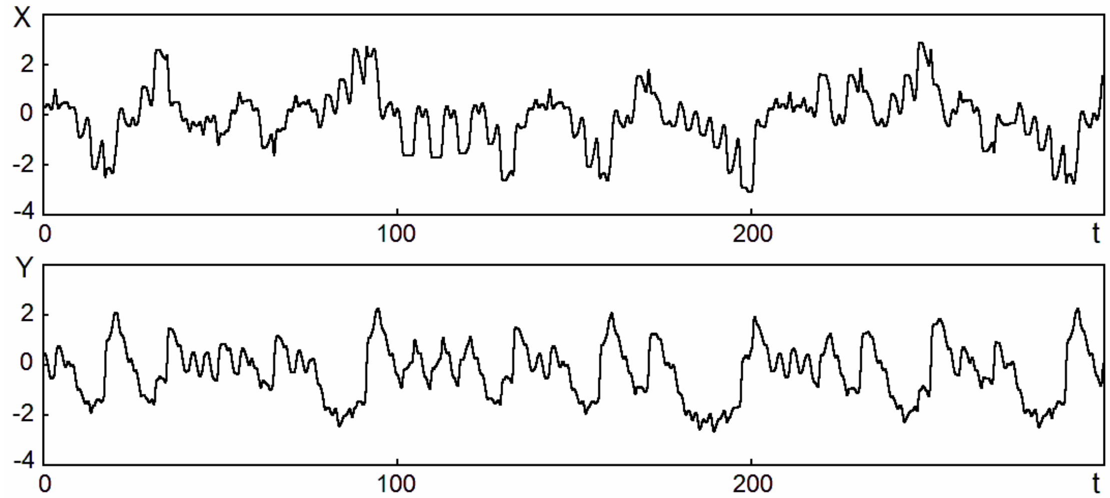
where

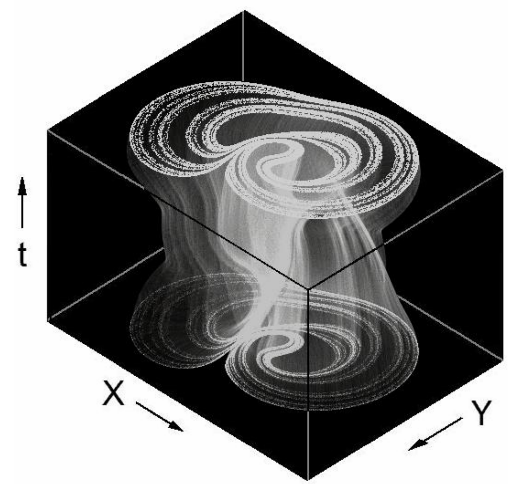
$$\Omega_1 = \frac{2X\cos(\tfrac{\pi}{4}\cos\tfrac{\pi}{2}t) + (1 - X^2 - Y^2)\sin(\tfrac{\pi}{4}\cos\tfrac{\pi}{2}t)}{\left(1 + X^2 + Y^2\right)^2},$$

$$\Omega_2 = \frac{-2X\sin(\tfrac{\pi}{4}\sin\tfrac{\pi}{2}t) + (1 - X^2 - Y^2)\cos(\tfrac{\pi}{4}\sin\tfrac{\pi}{2}t)}{1 + X^2 + Y^2} + \frac{1}{\sqrt{2}}.$$



At ε=0.72 and K=1.9 the Lyapunov exponents are $\lambda_1 \approx 0.221$ and $\lambda_2 \approx -0.315$, or for the Poincaré map $\Lambda_1 = \lambda_1 T \approx 0.884$, $\Lambda_2 = \lambda_2 T \approx -1.260$, and the Kaplan-Yorke dimension is $D_{KY} \approx 1.70$.





Flow model with the Smale-Williams attractor

[S.P.Kuznetsov, AND (Saratov), 17, 2009, No 4, 5-34; in Russian]

d=2, a=1.5, b=1.1, μ=3, d_2=2

I. Differential rotation around *x*-axis, $0 \le t < 1$**,**

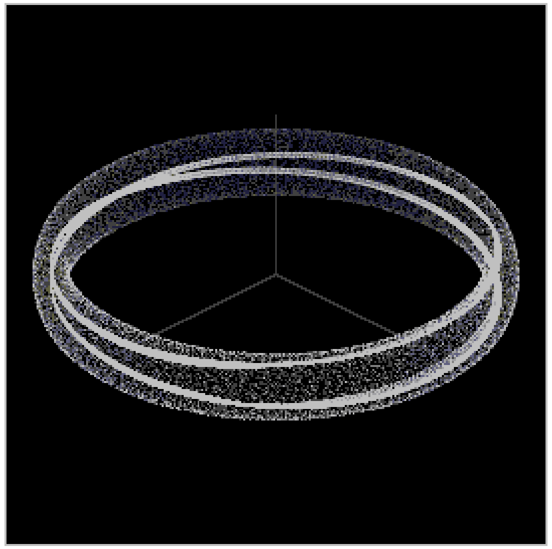
$$\dot{x} = 0,\ \ \dot{y} = -\tfrac{\pi}{2} xz,\ \ \dot{z} = \tfrac{\pi}{2} xy\,.$$

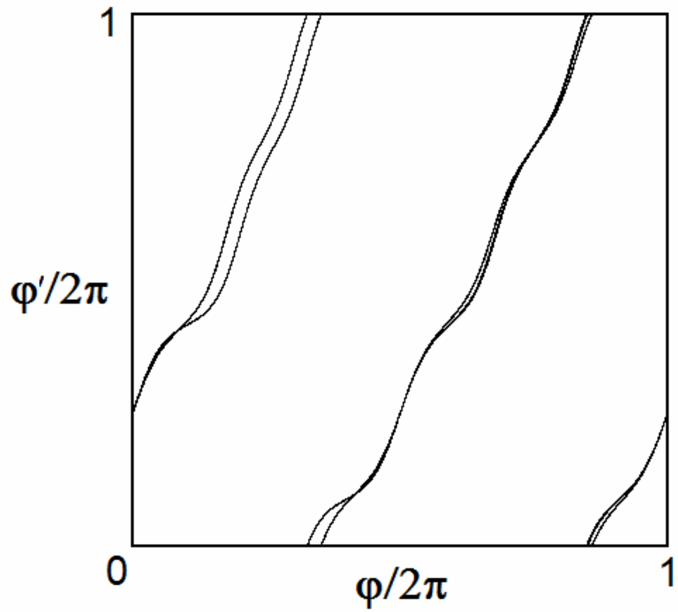
II. Nonunifirm displacement with compression in *y*-direction, $1 \le t < 2$

$$\dot{x} = -\tfrac{\pi}{2} z,\ \ \dot{y} = -d(y - ax^2 - az^2 + b),\ \ \dot{z} = \tfrac{\pi}{2} x\,.$$

III. Compression towards a unit circle in the plane *z*=0, $2 \le t < 3$

$$\dot{x} = \mu x(1 - x^2 - y^2),\ \ \dot{y} = \mu y(1 - x^2 - y^2),\ \ \dot{z} = -d_2 z$$





Equivalently, it may be written as a unified set of equations:

$$\dot{x} = -f_2(x)\tfrac{\pi}{2} z + f_3(x)\mu x(1 - x^2 - y^2),$$

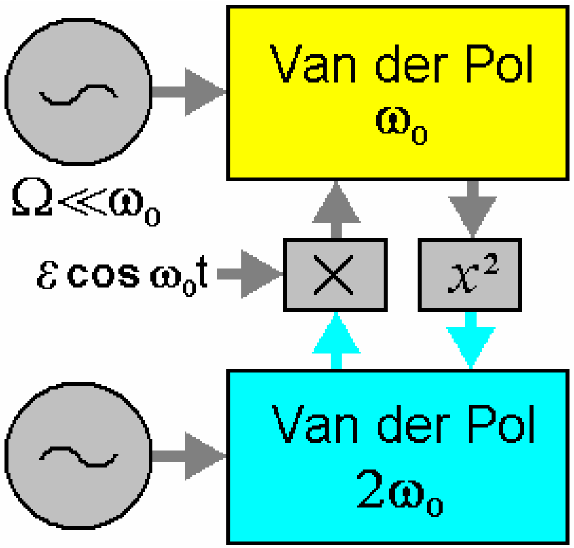
$$\dot{y} = -f_1(x)\tfrac{\pi}{2} xz - f_2(x) d(y - ax^2 - az^2 + b) + f_3(x)\mu y(1 - x^2 - y^2),$$

$$\dot{z} = f_1(x)\tfrac{\pi}{2} xy + f_2(x)\tfrac{\pi}{2} x - f_3(x) d_2 z$$

Design of hyperbolic attractors by means of phase manipulation in alternately excited oscillators

Smale-Williams attractor in coupled non-autonomous van der Pol oscillators

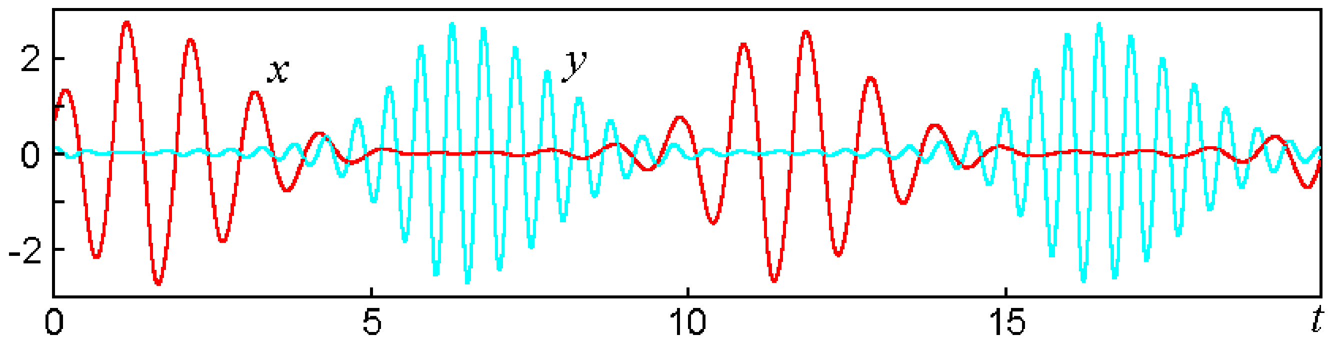
[S.P.Kuznetsov, Phys. Rev. Lett., 95, 2005, 144101]

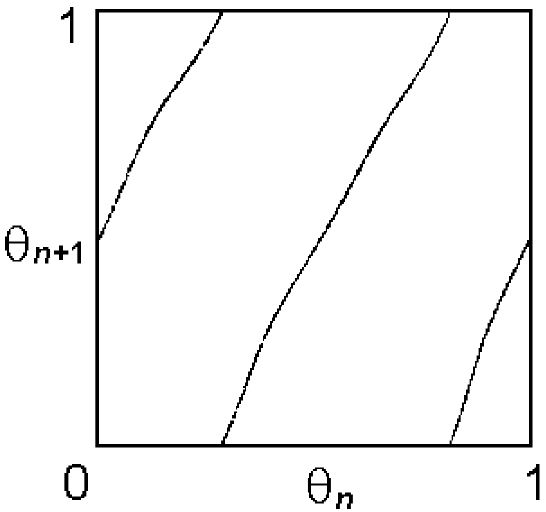


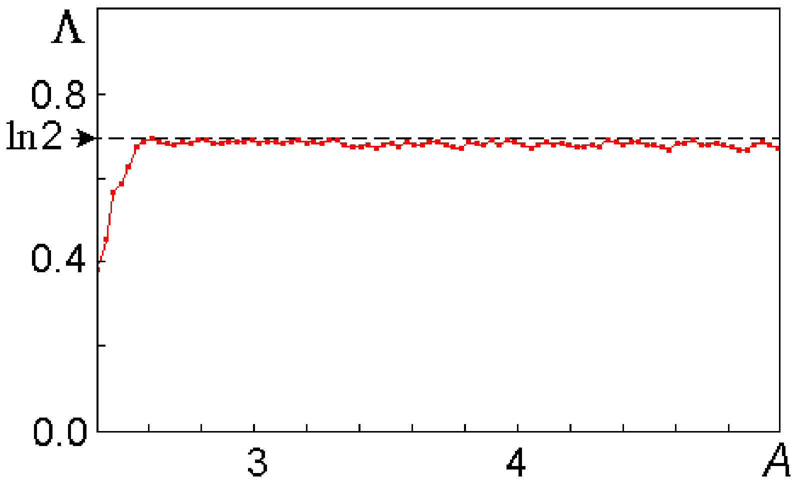
$$\ddot{x} - (A\cos\Omega t - x^2)\dot{x} + \omega_0^2 x = \varepsilon y \cos\omega_0 t$$
$$\ddot{y} - (-A\cos\Omega t - y^2)\dot{y} + 4\omega_0^2 y = \varepsilon x^2$$

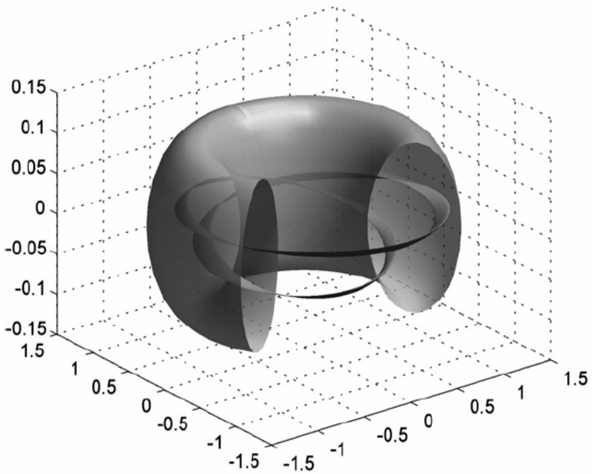
$$\theta_{n+1} = 2\theta_n \pmod{2\pi}$$

$\omega_0 = 2\pi$, $T=10$, $A=3$, $\varepsilon=0.5$





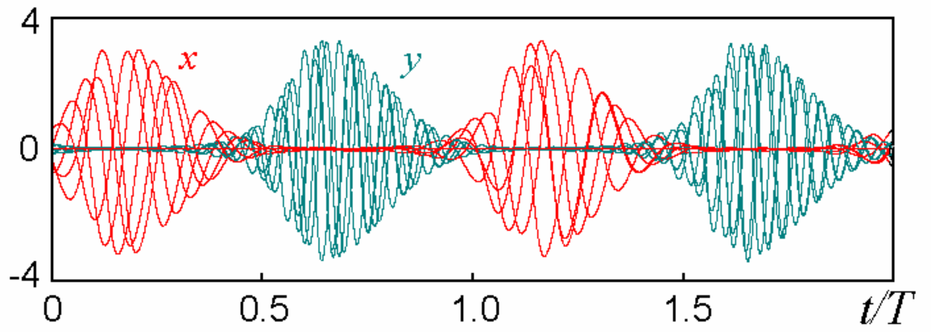




Parameters: $T=2\pi/\Omega=6$, $A=5$, $\varepsilon=0.5$

$$x_0 = x/0.812,\ x_1 = (u-0.438x)/0.721,\ x_2 = y+0.042x-0.226u,\ x_3 = v+0.218x-0.029u+0.118y,\ u=\omega_0^{-1}\dot{x},\ v=\omega_0^{-1}\dot{y}$$

Dynamics in time: chaos in phase

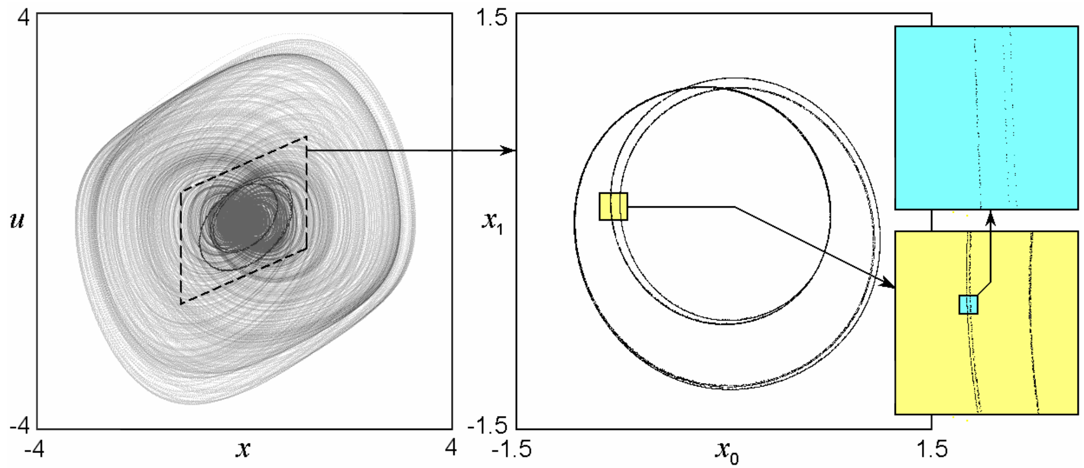


Lyapunov exponents:

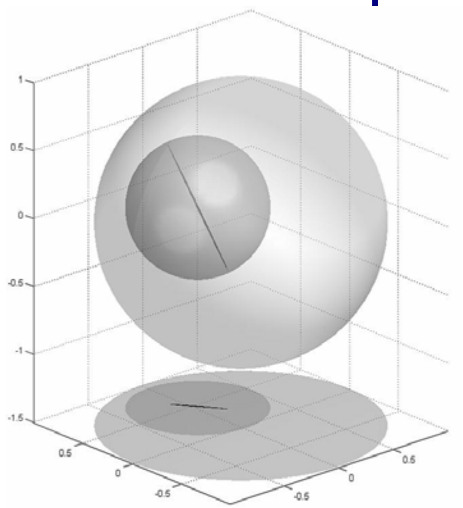
$$\Lambda_1 = 0.6832,\ \Lambda_2 = -2.602,\ \Lambda_3 = -4.605,\ \Lambda_4 = -6.538$$

Dimensions: D_{GP}=1.252, D_{KY}=1.263

Attractor: Cantor structure

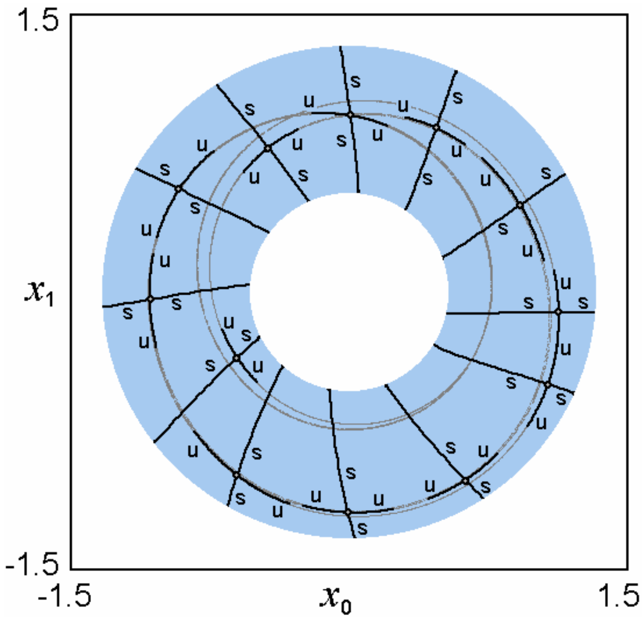


Cone criterion verification (at the most worse point)

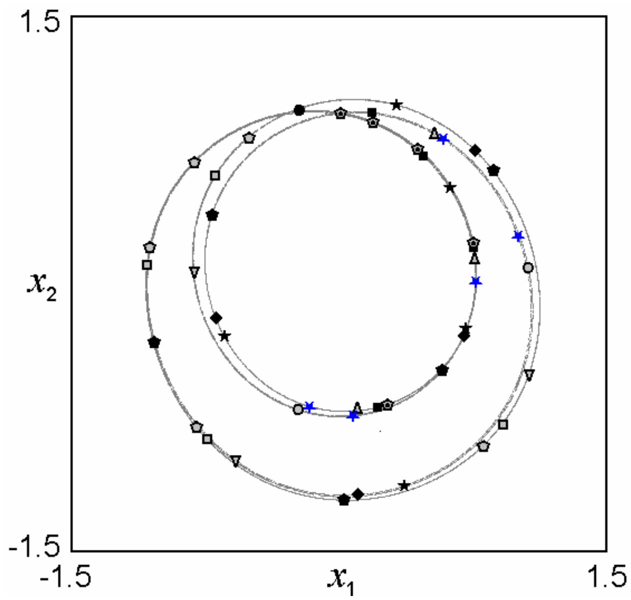


Kuznetsov&Sataev, Phys.Lett. A365, 2007, 97

Stable and unstable manifolds

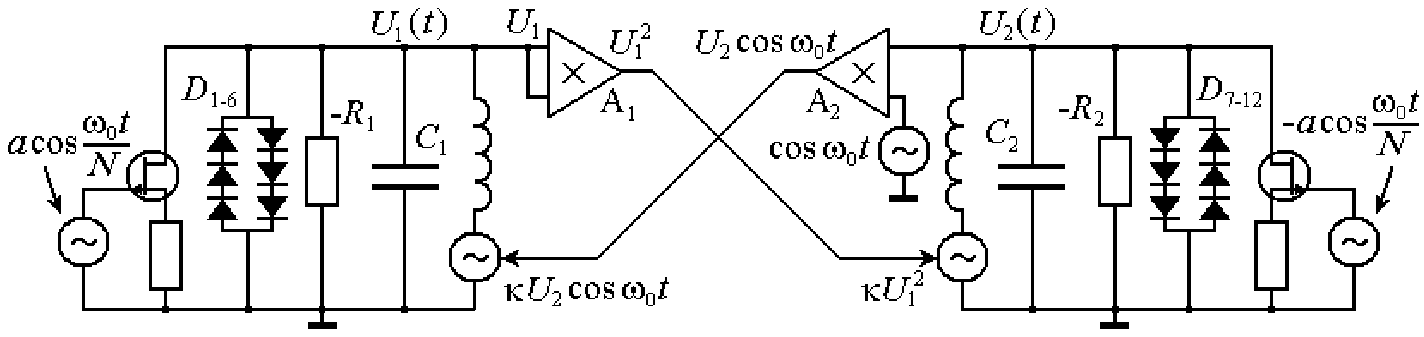


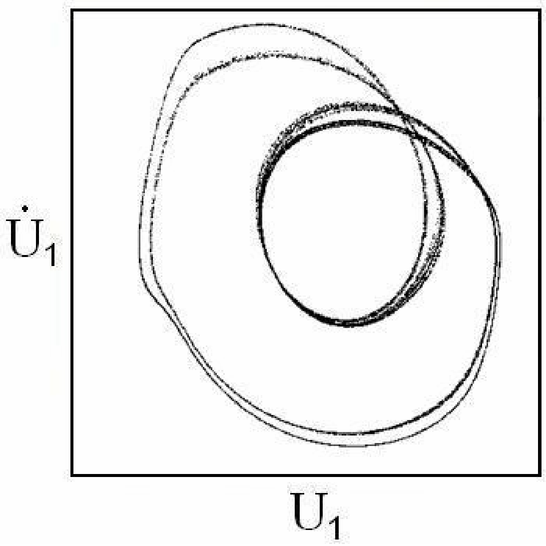
UPO in attractor

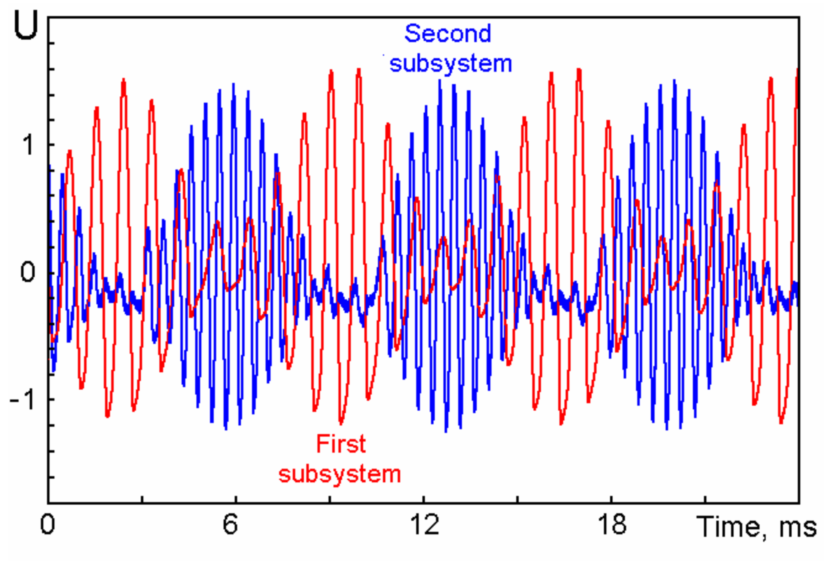


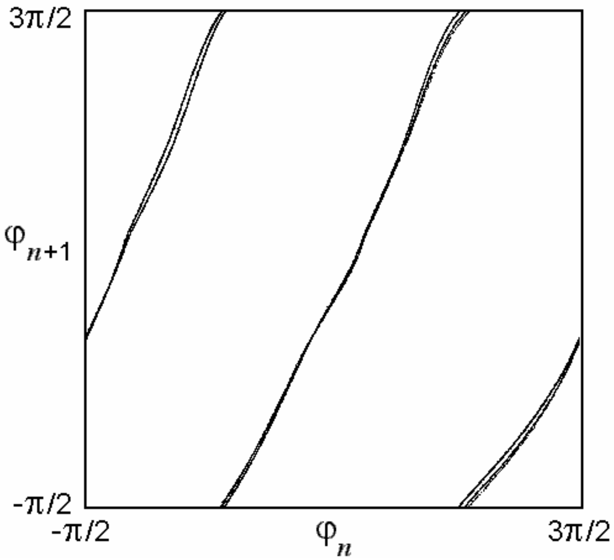
Electronic experiment

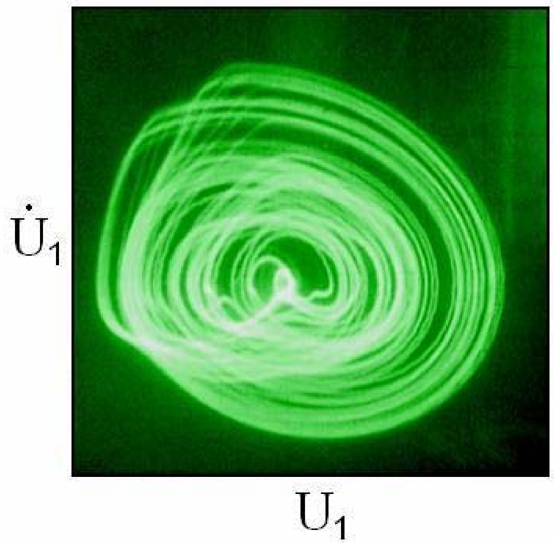
[S.P.Kuznetsov, E.P.Seleznev, JETP 102, 2006, 355]











Autonomous coupled oscillators with non-resonant excitation transfer

Smale-Williams attractor in minimal dimension [S.P.Kuznetsov, A.Pikovsky, Physica D232, 2007, 87]

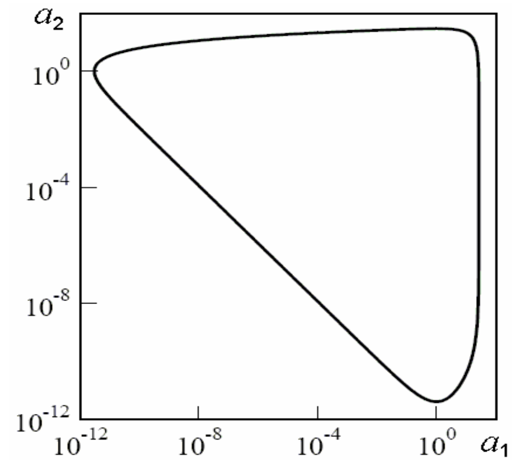
(1) Start with a version of the predator-pray model

$$\dot a_1 = 2(1 - a_2 + \tfrac{1}{2} a_1 - \tfrac{1}{50} a_1^2) a_1,\ \ \dot a_2 = 2(a_1 - 1) a_2$$

(2) Introduce “oscillatory” variables x and y to have $a_1 = x_1^2 + y_1^2$ and

$a_2 = x_2^2 + y_2^2$ satisfying these equations.

(3) Introduce additional coupling to ensure transfer of excitation from one oscillator to the second and back, with doubling of the phase of the oscillations in the course of the transmission.

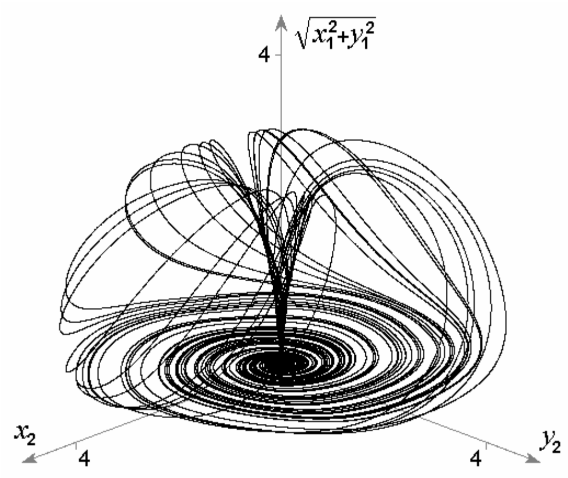


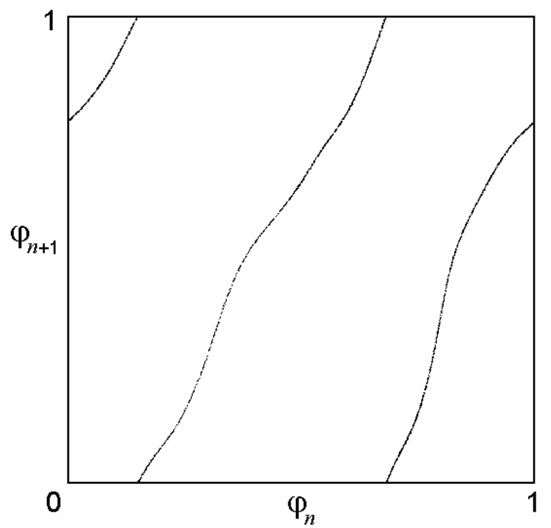
$$\dot x_1 = \omega_0 y_1 + [1 - (x_2^2 + y_2^2) + \tfrac{1}{2}(x_1^2 + y_1^2) - \tfrac{1}{50}(x_1^2 + y_1^2)^4] x_1 + \varepsilon x_2 y_2, \quad \dot x_2 = \omega_0 y_2 + (x_1^2 + y_1^2 - 1) x_2 + \varepsilon x_1,$$

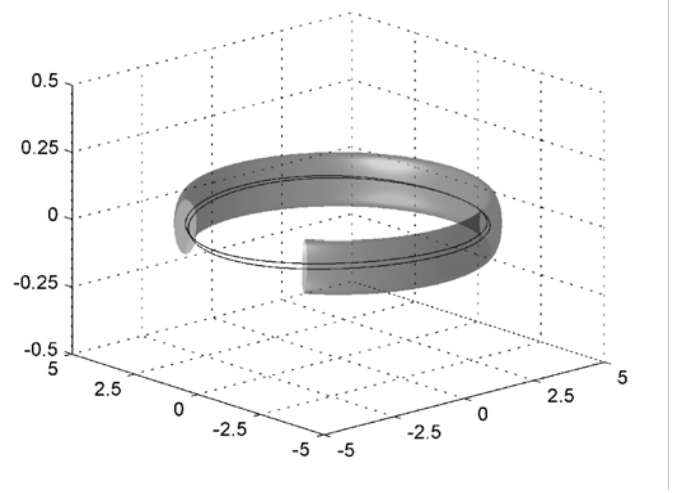
$$\dot y_1 = -\omega_0 x_1 + [1 - (x_2^2 + y_2^2) + \tfrac{1}{2}(x_1^2 + y_1^2) - \tfrac{1}{50}(x_1^2 + y_1^2)^4] y_1, \quad \dot y_2 = -\omega_0 x_2 + (x_1^2 + y_1^2 - 1) y_2$$

$\omega_0 = 2\pi$, $\varepsilon = 0.3$

Poincaré section: $S = x_1^2 + y_1^2 - x_2^2 - y_2^2 = 0$







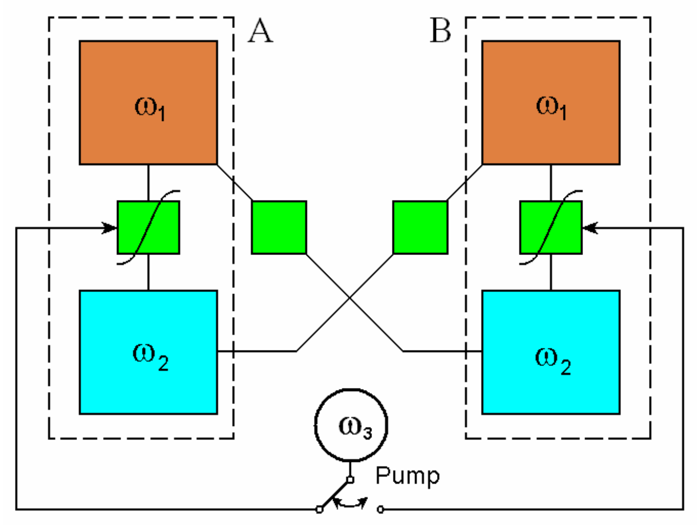
Parametric generator of chaos

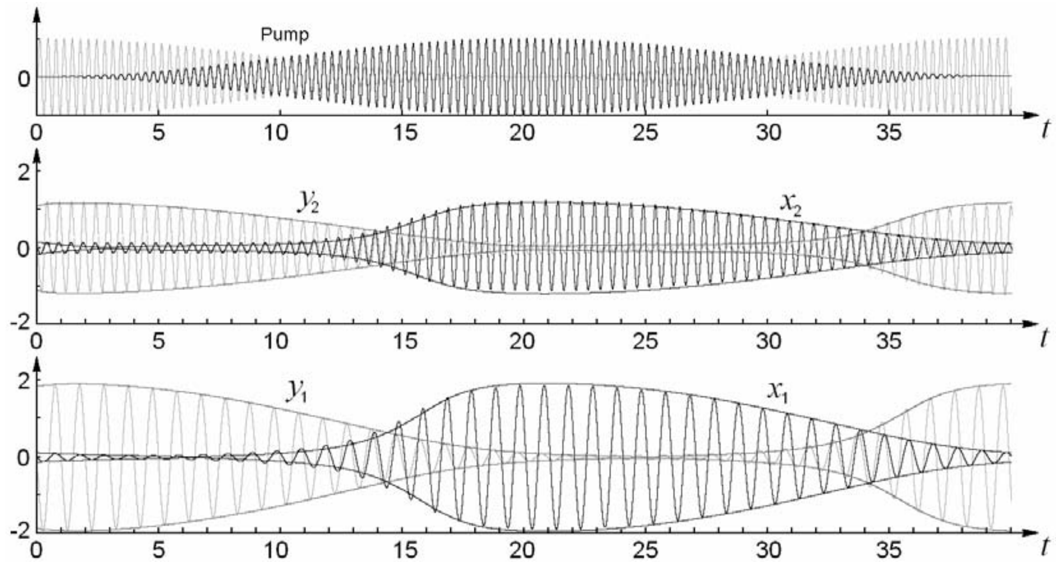
[S.P.Kuznetsov, JETP 106, 2008, No. 2, 380]

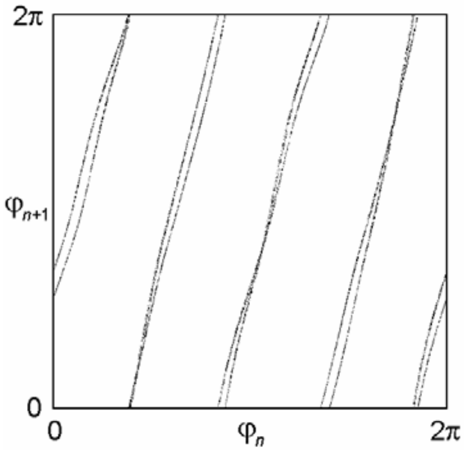
$$\ddot{x}_1 + \omega_1^2 x_1 = \kappa x_2 f(t)\sin\omega_3 t + 2\varepsilon x_1 y_2 - \alpha_1 \dot{x}_1 - \beta_1 \dot{x}_1^3, \quad \ddot{y}_1 + \omega_1^2 y_1 = \kappa y_2 g(t)\sin\omega_3 t + 2\varepsilon y_1 x_2 - \alpha_1 \dot{y}_1 - \beta_1 \dot{y}_1^3,$$

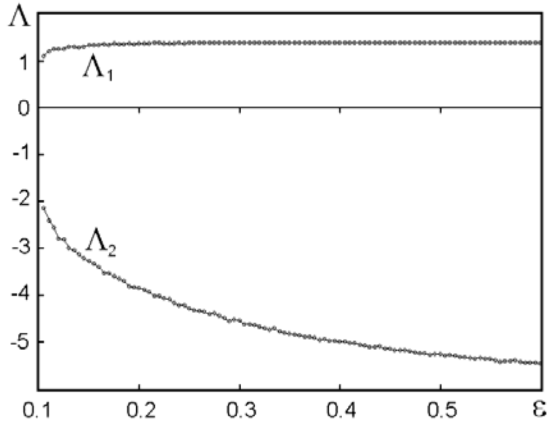
$$\ddot{x}_2 + \omega_2^2 x_2 = \kappa x_1 f(t)\sin\omega_3 t + \varepsilon y_1^2 - \alpha_2 \dot{x}_2 - \beta_2 \dot{x}_2^3, \quad \ddot{y}_2 + \omega_2^2 y_2 = \kappa y_1 g(t)\sin\omega_3 t + \varepsilon x_1^2 - \alpha_2 \dot{y}_2 - \beta_2 \dot{y}_2^3$$

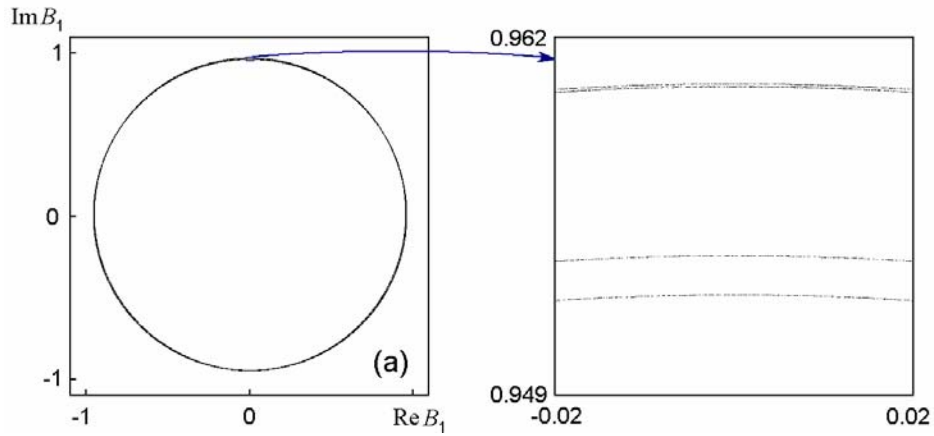
$$f(t) = \sin^2(\pi t/T),\ g(t) = \cos^2(\pi t/T),\ \omega_1=2\pi,\ \omega_2=4\pi,\ \omega_3=6\pi,\ T=40,\ \varepsilon=0.5,\ \kappa=35,\ a=0.6,\ b=0.01$$











Application of the phase manipulating principle in time-delayed systems

[S.P. Kuznetsov and V.I Ponomarenko, Tech. Phys. Lett., **34**, 2008, 771–773]

[S.P. Kuznetsov and A.S. Pikovsky, Europhysics Letters, **84**, 2008, 10013]

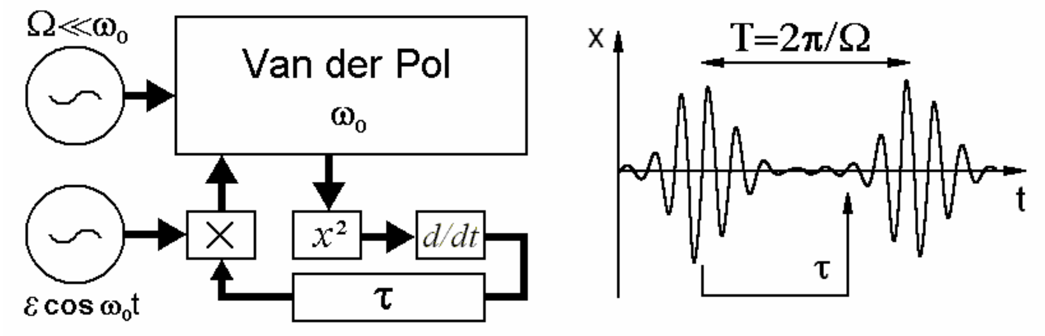
An appropriate class for implementation of the phase manipolation principle is represented by **systems with delayed feedback**.

In such systems it is possible to arrange generation of the hyperbolic chaos with a **single active element.**

On the other hand, the mathematical description is more difficult than that for low-dimensional models, because of **infinite dimension of the state space.** Indeed, to determine an instantaneous state one needs to specify not a finite set of variables, but a fragment of a signal on a time interval of the delay time.

A self-oscillatory element with slow periodic variation of bifurcation parameter, and with auxiliary signal

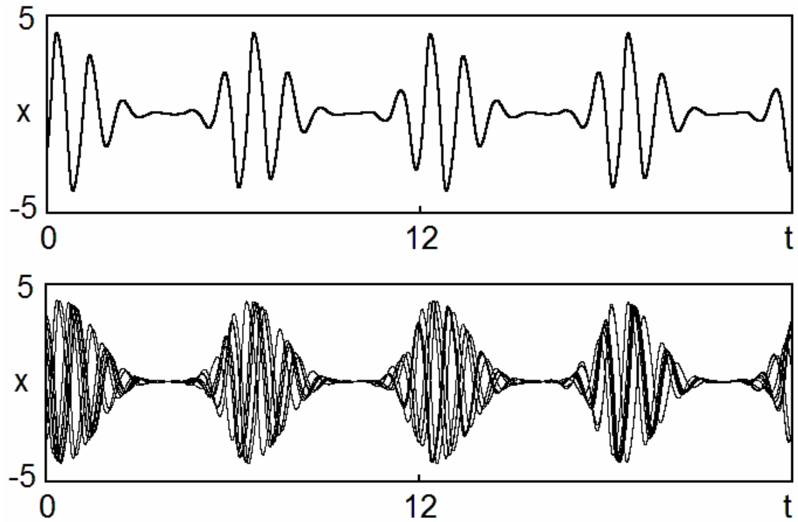
[S.P. Kuznetsov and V.I Ponomarenko, Tech. Phys. Lett., 34, 2008, 771–773]

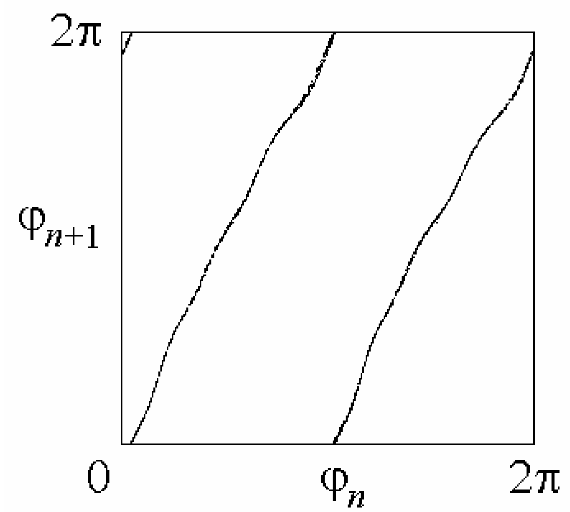


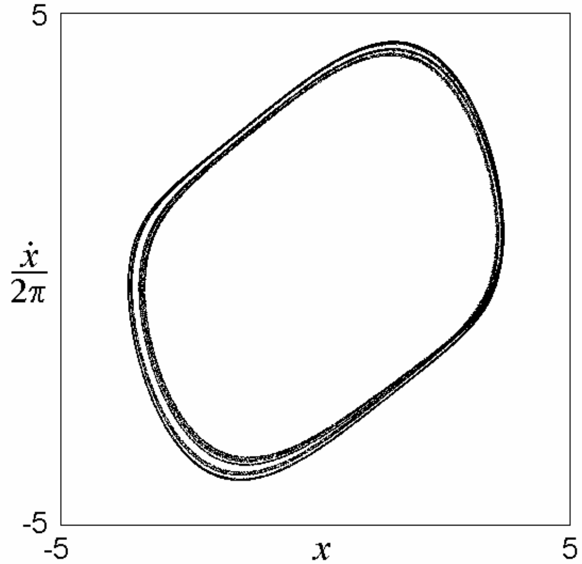
$$\ddot{x} - (A\cos(2\pi t/T) - x^2)\dot{x} + \omega_0^2 x = \varepsilon x(t-\tau)\dot{x}(t-\tau)\cos\omega_0 t$$

Approximately: $\varphi_{n+1} = 2\varphi_n + \mathrm{const}\,(\mathrm{mod}\,2\pi)$

$$\omega_0 = 2\pi, T = 2\pi\Omega^{-1}, \tau = \tfrac{3}{4}T, \ \ A = 5.5, \ \ \varepsilon = 0.2$$





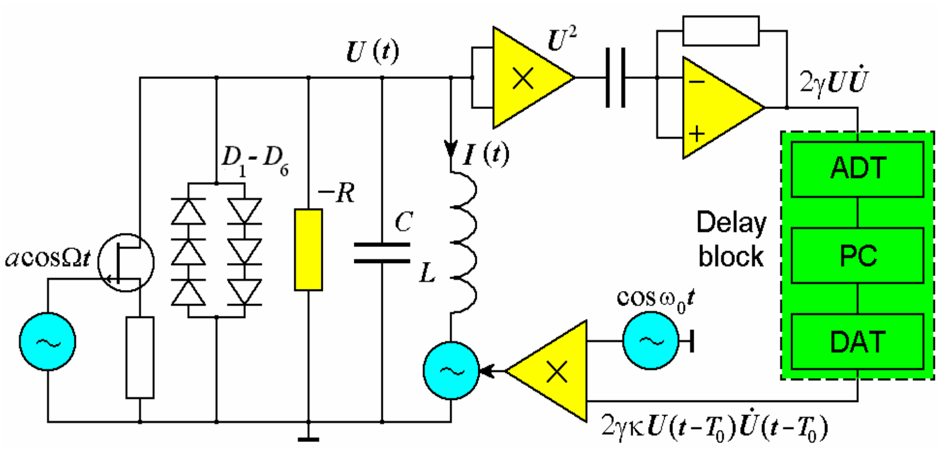


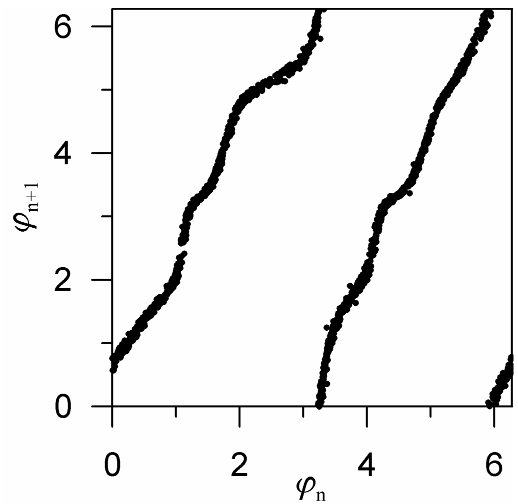
Lyapunov exponents: $\Lambda_1 = 0.6851$, $\Lambda_2 = -0.8844$, $\Lambda_3 = -4.9178$, $\Lambda_5 = -5.1171$. KY-dimension: $D = 1.77$

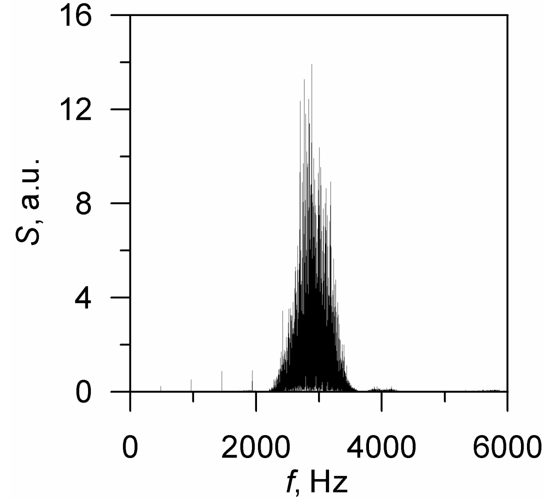
Experiment

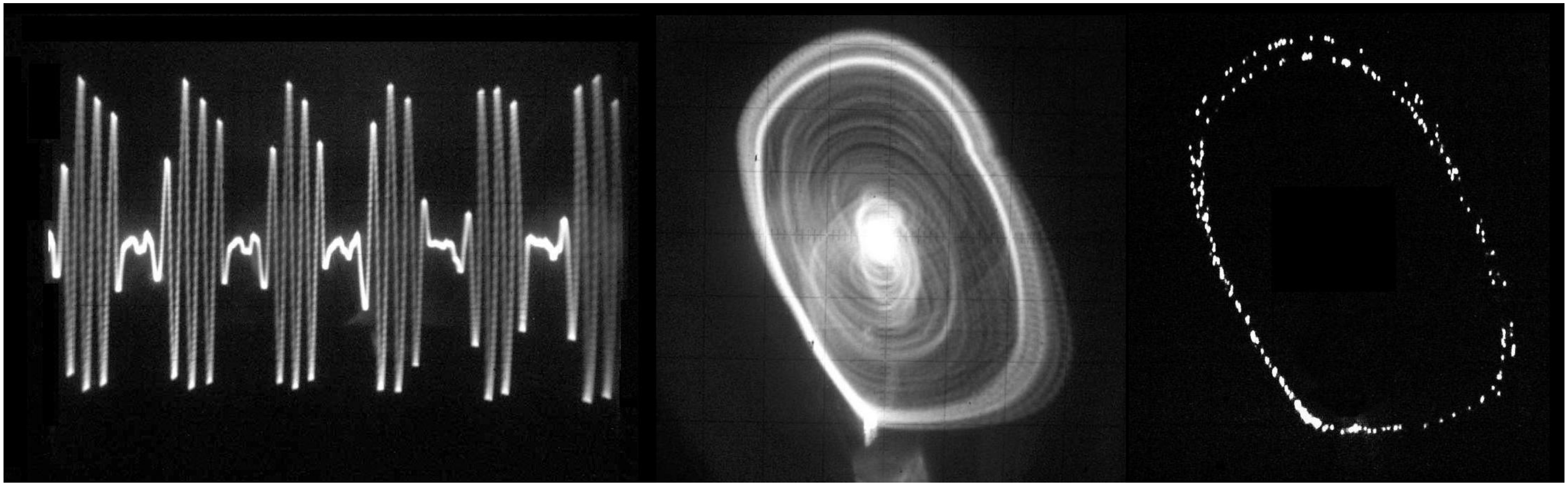
[S. P. Kuznetsov and V. I. Ponomarenko, Tech. Phys. Lett., 34, 2008, No 9, 771–773]

$$f_0 = \omega_0 / 2\pi = 3000\,\text{Hz},\;\; N = f_0 T = 6,\;\; T_0 = \tfrac{3}{4} T$$









Conclusion

- **Now, we get long-expected physical examples of systems with uniform hyperbolic attractors in their Poincaré maps. This is a "breakthrough to a land of hyperbolicity" because now one can construct many other examples exploiting the property of structural stability.**
- **There are several approaches to design systems with hyperbolic attractors, e.g. dynamics under short pulses, constructing dynamics as a sequence of distinct stages, manipulating phases of excitation trasferred between alternately excited oscillators, parametric generators, and in oscillators with delayed feedback.**
- **At last, some of the systems we propose allow implementation e.g. in electronics, mechanics, nonlinear optics.**
- **A possibility of physical realization of the hyperbolic chaos opens prospects for applications of the hyperbolic theory existed since now as a purely mathematical discipline.**

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