Lifetimes and Lyapunov Modes of a Coupled Map Lattice Model for Turbulent Pipe Flow

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1. Motivation

Starting from the Navier-Stokes-Equation, the analysis of turbulence is a hard numerical and mathematical task.
Using a Dynamical Systems approach, e.g. a Coupled Map Lattice (CML), it is possible to understand basic features of turbulence. Our special interest lies in the understanding of the scaling of lifetimes of turbulent puffs with the system parameters.

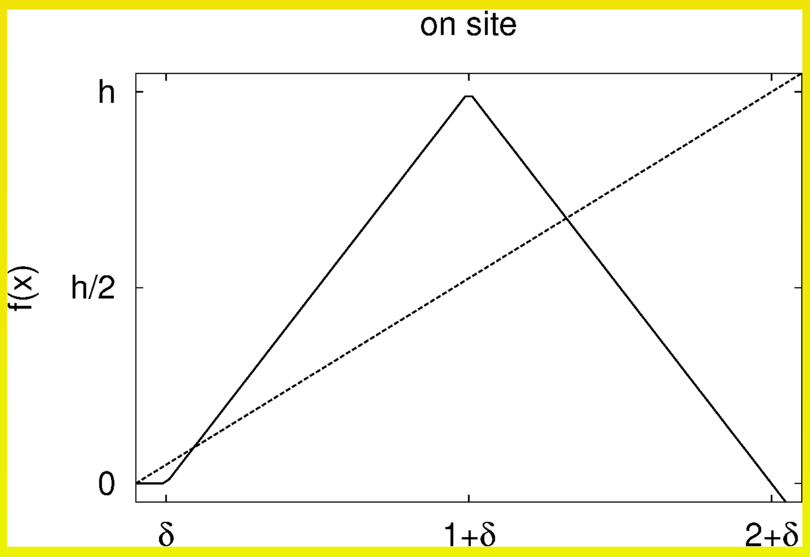
2. The Model

Our CML is given by the evolution equation

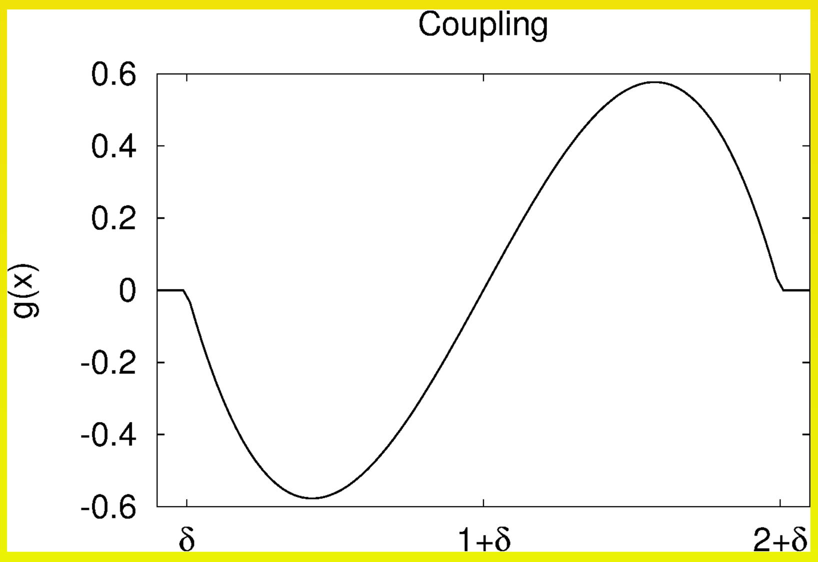
$$x(t+1,n) = \alpha g(x(t,n-1)) + f(x(t,n))$$

with periodic boundary conditions.

$$f(x) = \begin{cases} h(x-\delta) & \delta \leq x < 1+\delta \\ -h(x-2-\delta) & 1+\delta \leq x \\ 0 & x < \delta \end{cases} \qquad g(x) = \begin{cases} -1.5(x-\delta)(x-1-\delta)(x-2-\delta) & \text{if } \delta \leq x < 2+\delta \\ 0 & \text{else} \end{cases}$$



On site map f(x)

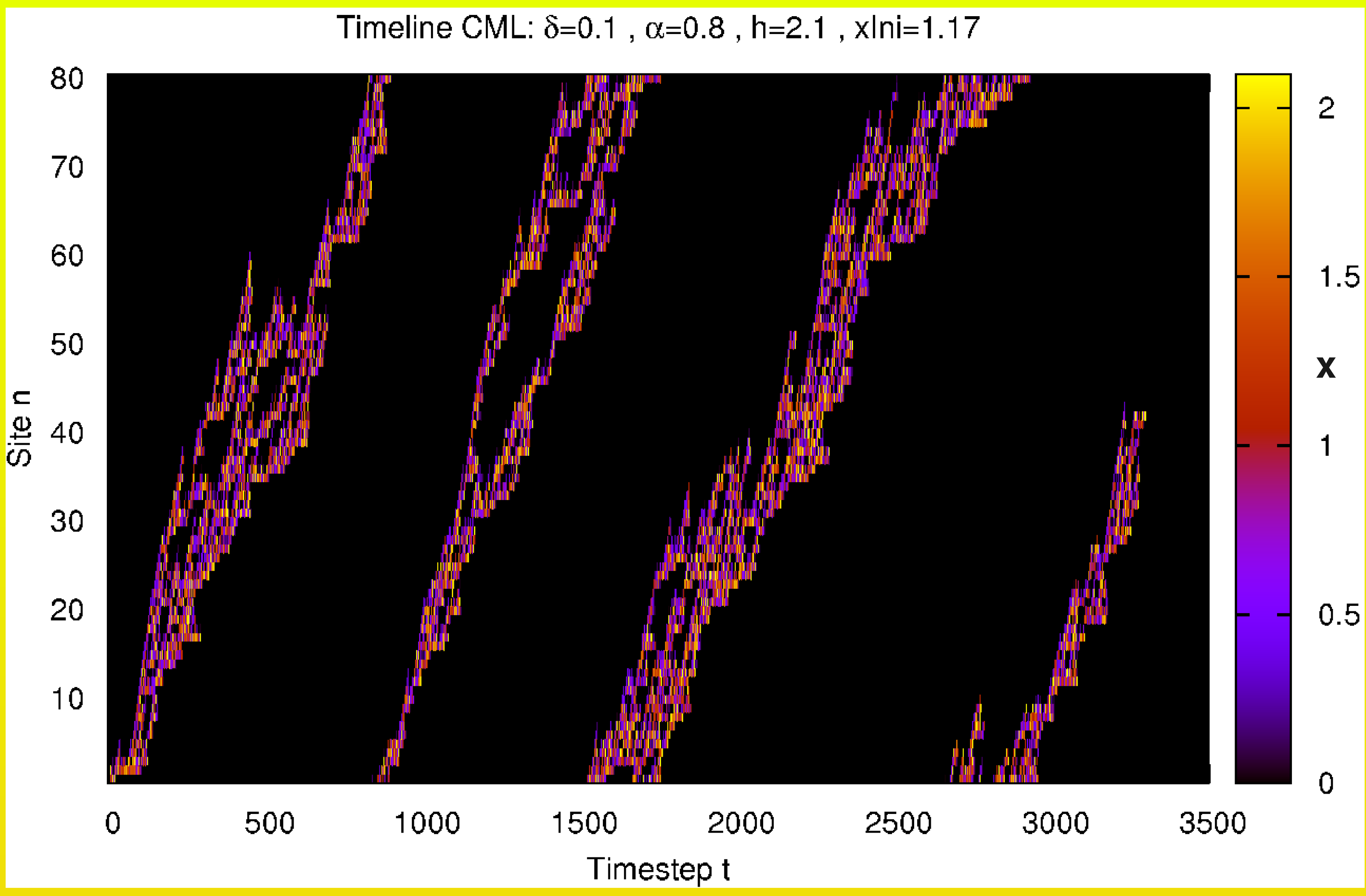


Coupling g(x)

The control parameters are the height h , the coupling strength α and the shift δ (will be fixed to $\delta=0.1$ throughout).

3. Trajectories

The initial condition is given by a finite perturbation at the first site. By iterating the system, this perturbations travels through the lattice and eventually decays after a certain lifetime, which can be interpreted as a turbulent puff.



One trajectory for given initial perturbation xlni=1.17 with a lifetime of ~3300. The colour indicates the value of each site. This trajectory plays the role of a turbulent puff for the dynamical system.

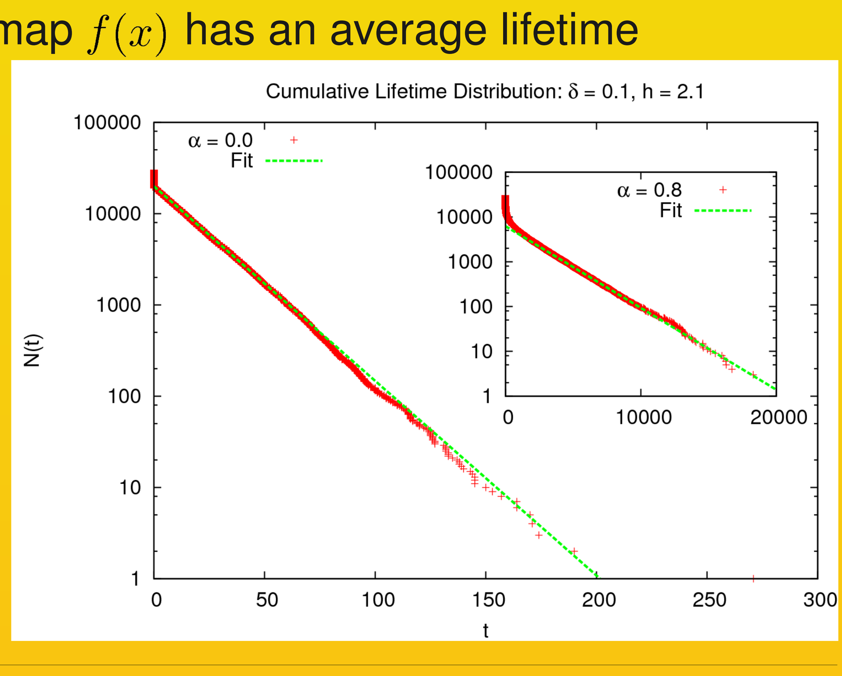
4. Lifetimes

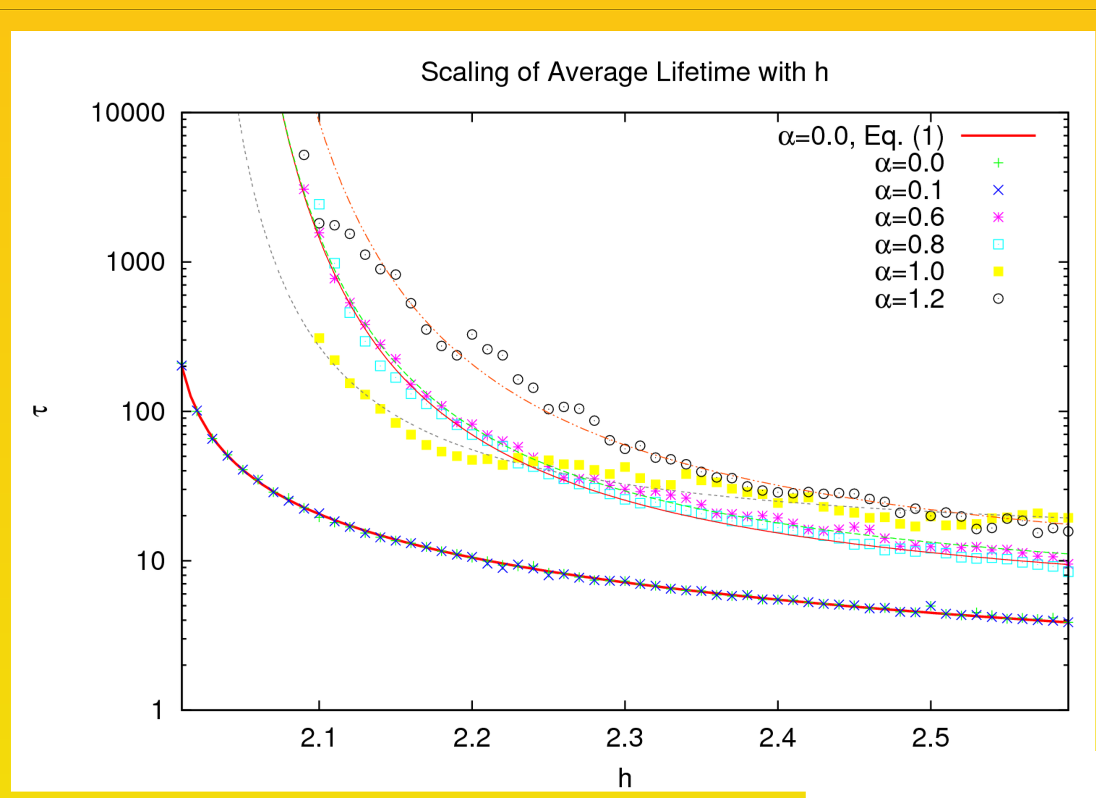
The lifetime for a fixed set of parameters can be computed by following trajectories of many initial conditions and calculating the exponential decay rate from the chaotic region. For the uncoupled lattice, the tent map $f(x)$ has an average lifetime

(1) $$\tau_s = \left(\ln\frac{h}{2}\right)^{-1}, \quad h > 2$$

In contrast, for super-long transients the average lifetime (cf. [5]) of the system should scale like

(2) $$\tau \propto \exp\left(C(h-2)^{\chi}\right), \quad \chi < 0, h > 2$$

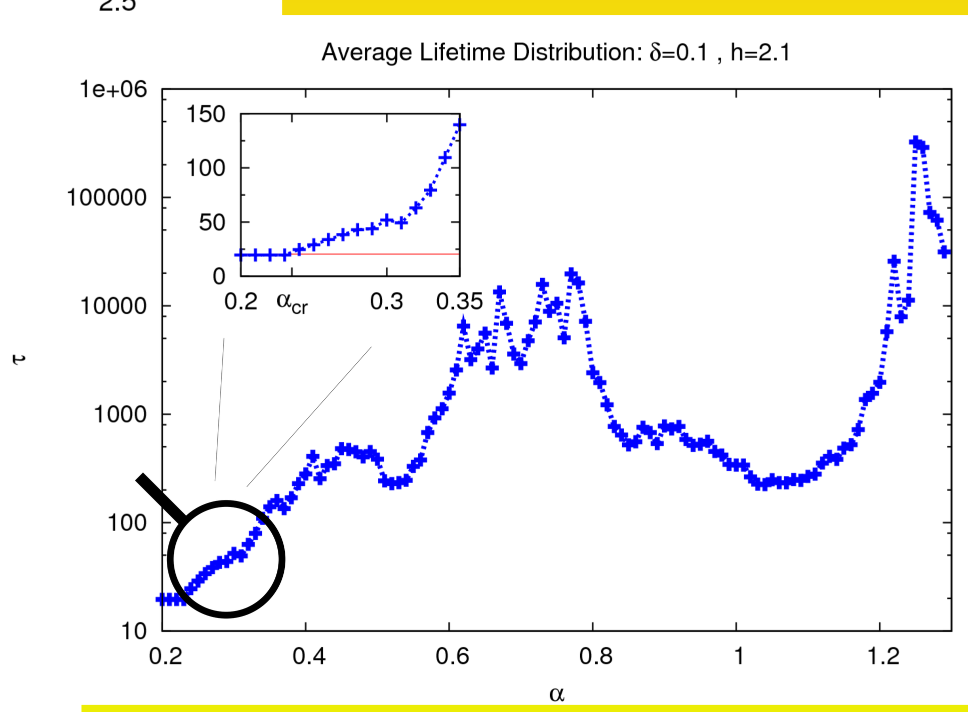




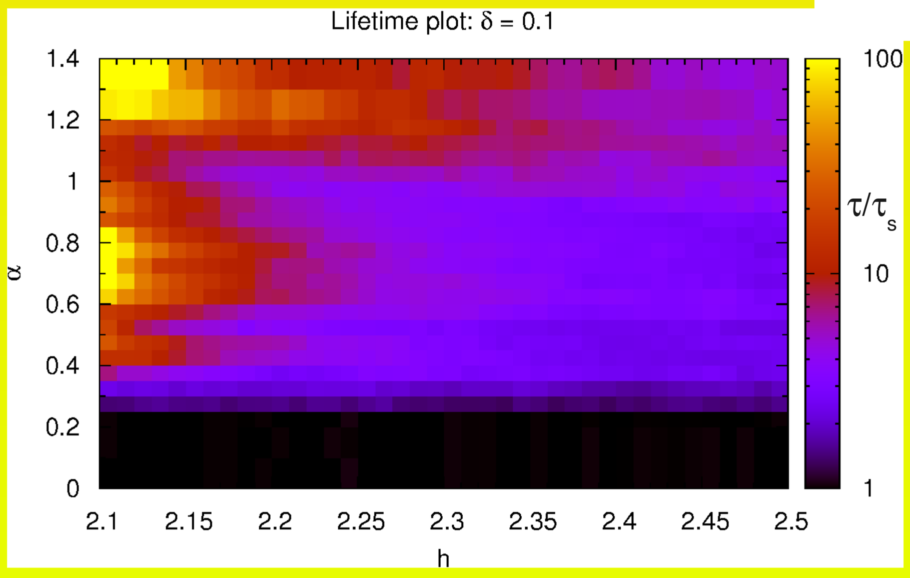
For zero coupling, the lifetime of the system follows the analytic solution Eq. (1).
In the coupled case, the average lifetime grows to significantly larger times, following the scaling law Eq. (2).

The scaling of the average lifetime for different coupling strength α .(N=30.000 initial conditions per data point). The lines are fits, according to Eqs. (1) and (2), respectively, where χ=-1 is fixed.

The average lifetime τ depends sensitively on the coupling strength α. Close to α=0.7 we observe very long average lifetimes of up to τ=20000.



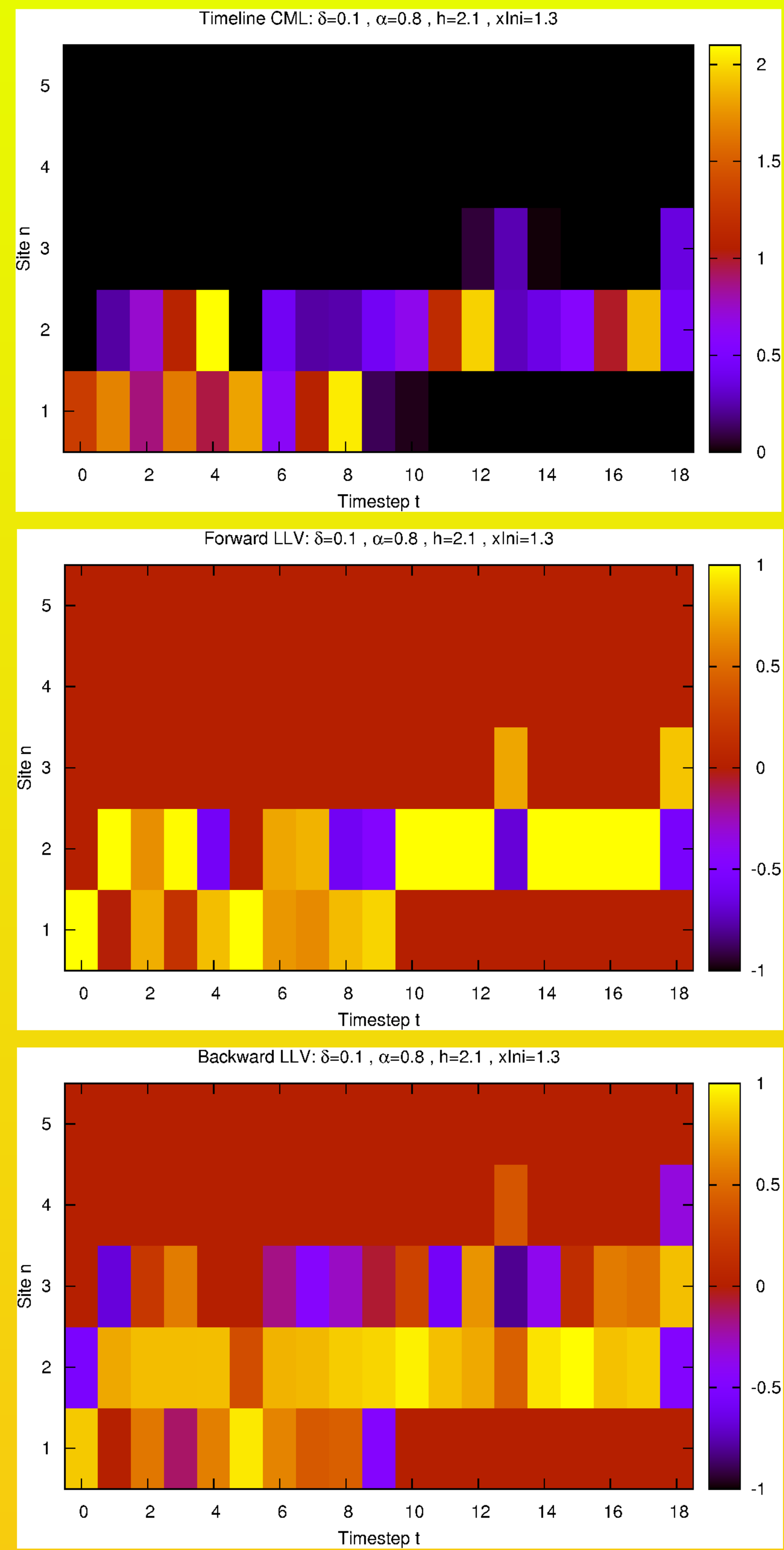
The scaling of the average lifetime with height for fixed coupling. (N=30000 initial conditions per data point). The inset shows a magnification of the transition region.



Phase space of the CML, where the colour represents the increase τ/τ_s of the average lifetime τ with respect to the single-site lifetime τ_s.

We summarize the h and α dependence of the lifetime in the false color plot to the left. The complex shape of the isolines mimics the complex parameter dependence of lifetimes, which is commonly observed in models of turbulent pipe flow [2].

5. Lyapunov vectors



Trajectory and the corresponding largest normalised Lyapunov vectors

The susceptibility to perturbations of a trajectory of the CML can be infered from the linearized dynamics ***M***. The growth of perturbations is governed by its singular values, i.e. it is dominated by the largest eigenvalue of ***(M* M)***. We wonder in what respect the corresponding eigenvectors of ***(M* M)*** and ***(M M*)*** characterize the most unstable directions of the forward and backwards dynamics, respectively.

6. Conclusion and Outlook

As shown above, the lifetime of the CML strongly depends on h and α. The behaviour is well understood in the uncoupled case, and we presently work on a theory for $\alpha \neq 0$. We also wish to explore the correspondence of the parameter dependence of our model system to features of real flows, like the dependence on Re and perturbation amplitues, and the interplay of convective and absolute stability in flows.

References:
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