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Lyapunov Modes and Localization for Hard Particle Systems

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Contents

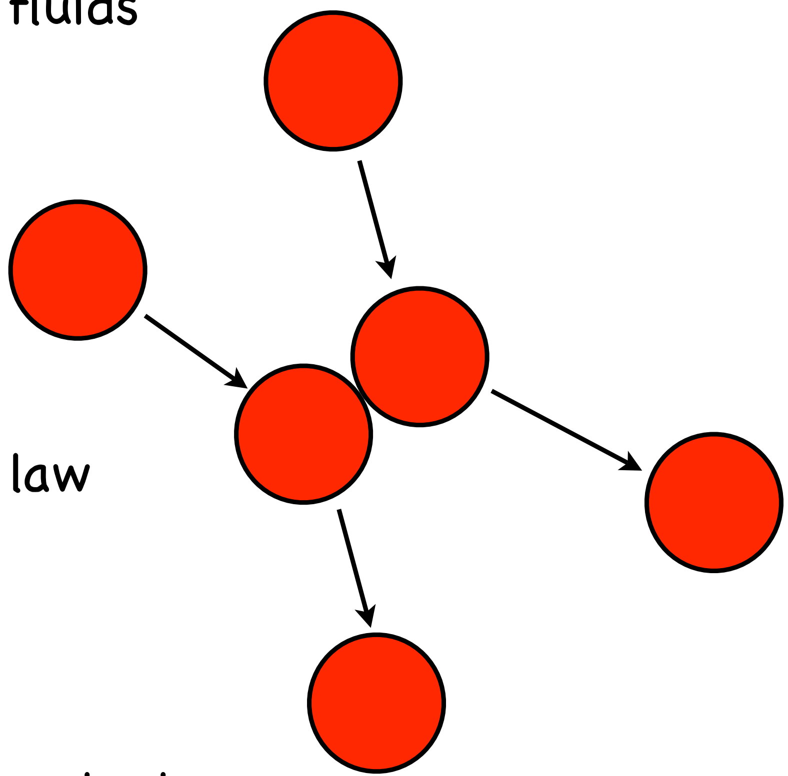
Equilibrium

- Quasi-one-dimensional (QOD) system
- Lyapunov spectrum has steps ⟺ modes
- Structure of the Lyapunov modes
- Localization of Lyapunov vectors
- Strong vs. weak localization
- Lyapunov vector Angle distributions

Hard Particle Systems

Hard Particle Systems

- Hard Spheres
- Models for atomic fluids
- Equation of state



Corrections to ideal gas law

$$P = \rho k_B T$$

Initial conditions are not important

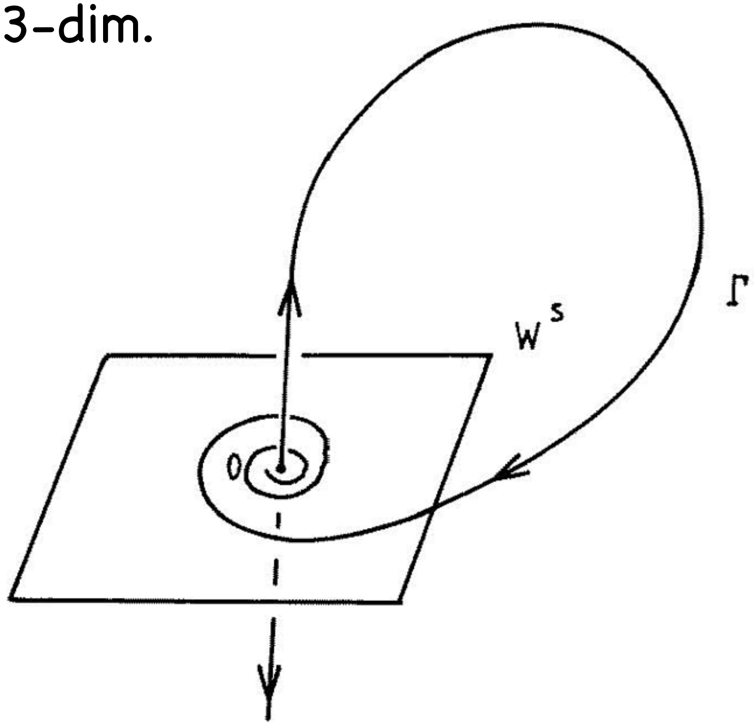
Properties

- Ergodicity - time averages = ensemble averages
- Single ergodic component or attractor
- Hyperbolicity --> "chaotic hypothesis"
- Structural stability

Tangency

- Homoclinic tangencies - example: saddle-focus stationary point in 3-dim.

Gonchenko, Turaev, Gaspard & Nicolis 97



$$\left(-\lambda \pm i\omega, \gamma\right)$$

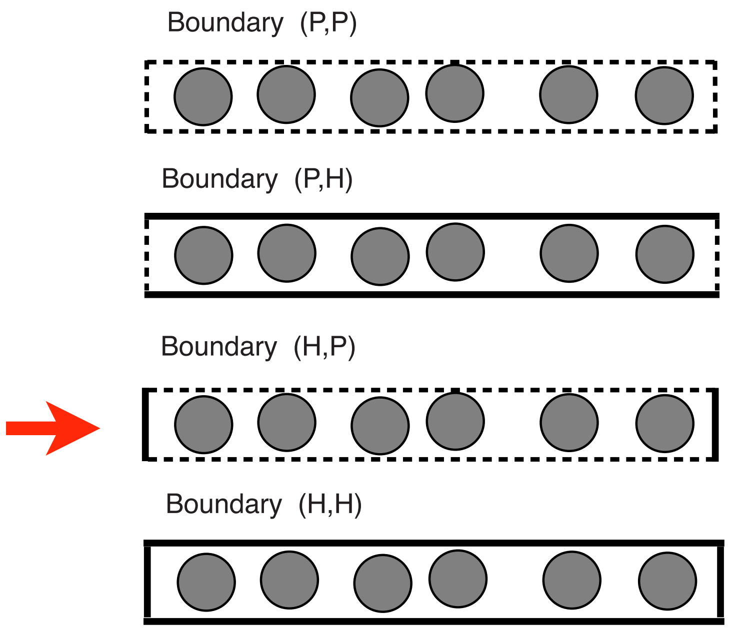
$$\lambda/\gamma < 1, \quad \lambda, \omega, \gamma > 0$$

- coexistence of many periodic orbits of different stability types (i.e. with different numbers of positive/negative Lyapunov exponents).

Gonchenko, Turaev & Shilnikov 07,08

Quasi-one-dimensional Systems

Schematic boundary conditions



Particle order is invariant, so particle index corresponds to x-coordinate.

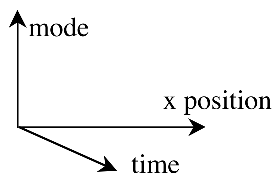
Four possible choices of boundary conditions. We mostly use (H,P) boundary conditions.

The number and pattern of steps changes with boundary conditions.

Modes observed in the long direction only - x-direction

Simplified presentation of the modes

NOTATION: $$\left(\underset{\{P,H\}}{X} \; , \; \underset{\{P,H\}}{Y} \right)$$



P = periodic H = hard wall

Calculations

- the Benettin et. al. algorithm to determine Lyapunov exponents and Gram-Schmidt Lyapunov vectors.
- the method of Ginelli et. al. to calculate covariant Lyapunov vectors.

Results

- step structure in the positive and negative Lyapunov exponents closest to zero
- Step structure => Lyapunov vectors with stable delocalized structure called Lyapunov modes.

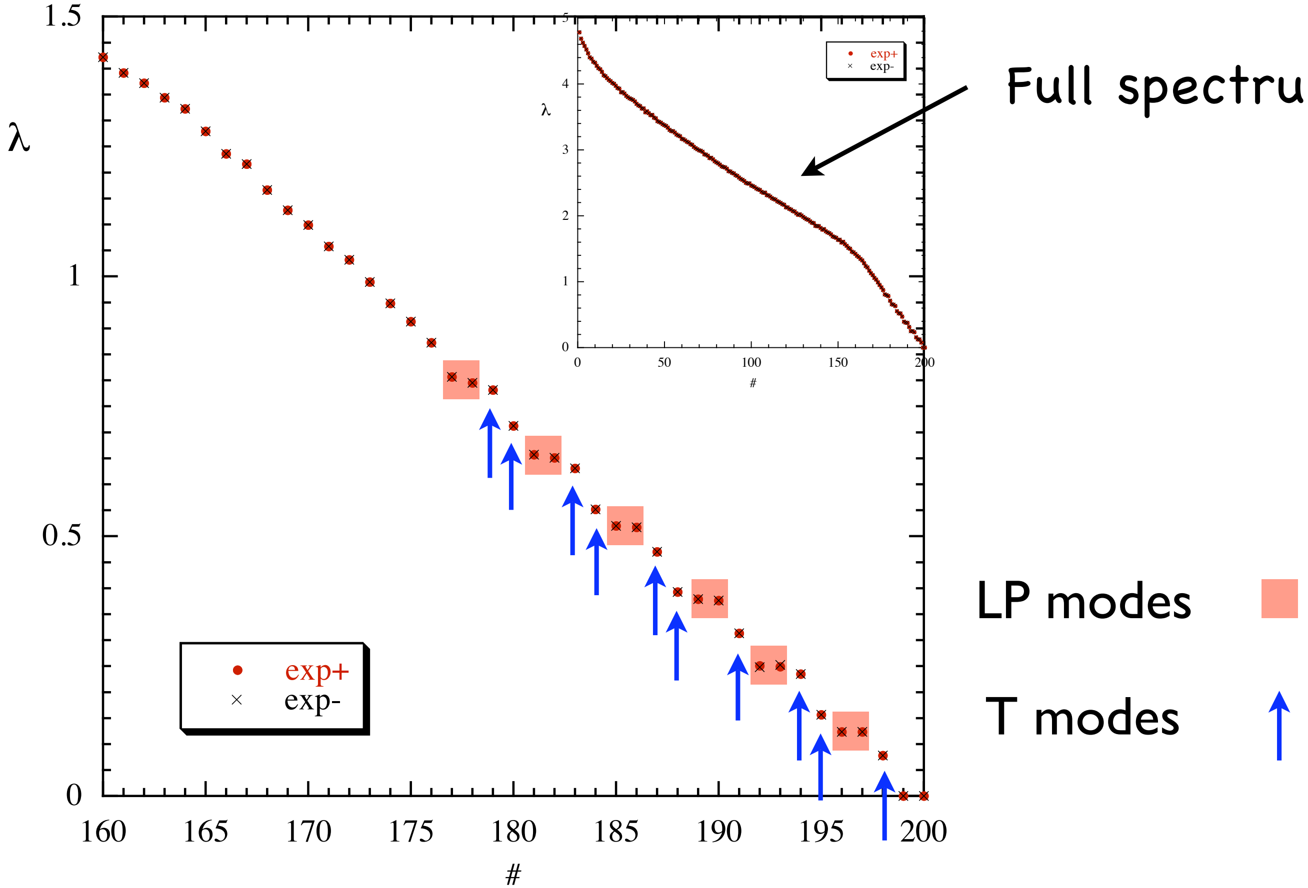
Lyapunov Exponents

Lyapunov Spectrum

Positive and negative branches

Pairing

$$\lambda_i + \lambda_{-i} = 0$$

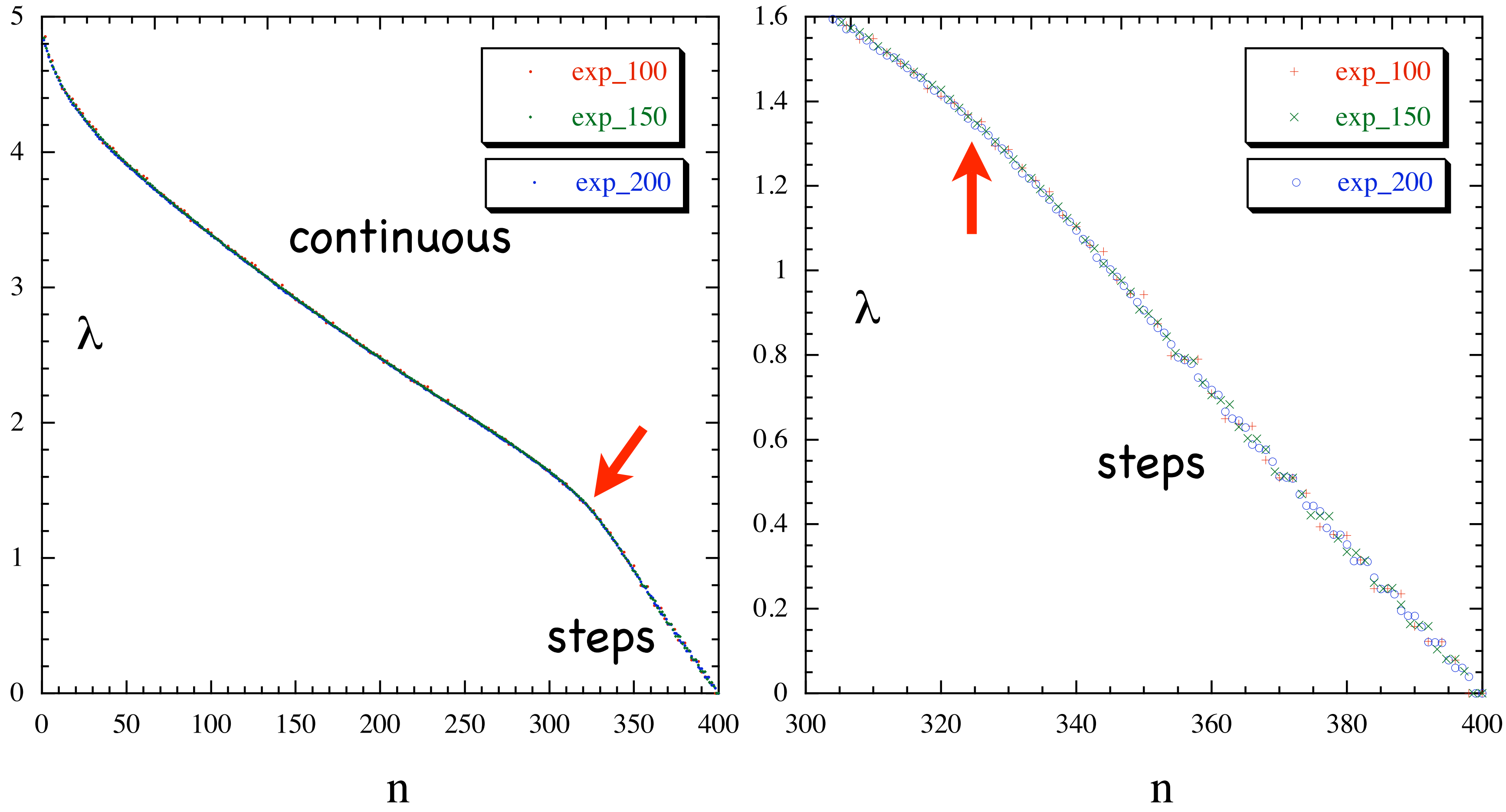


(H,P) boundary conditions

100 hard disks

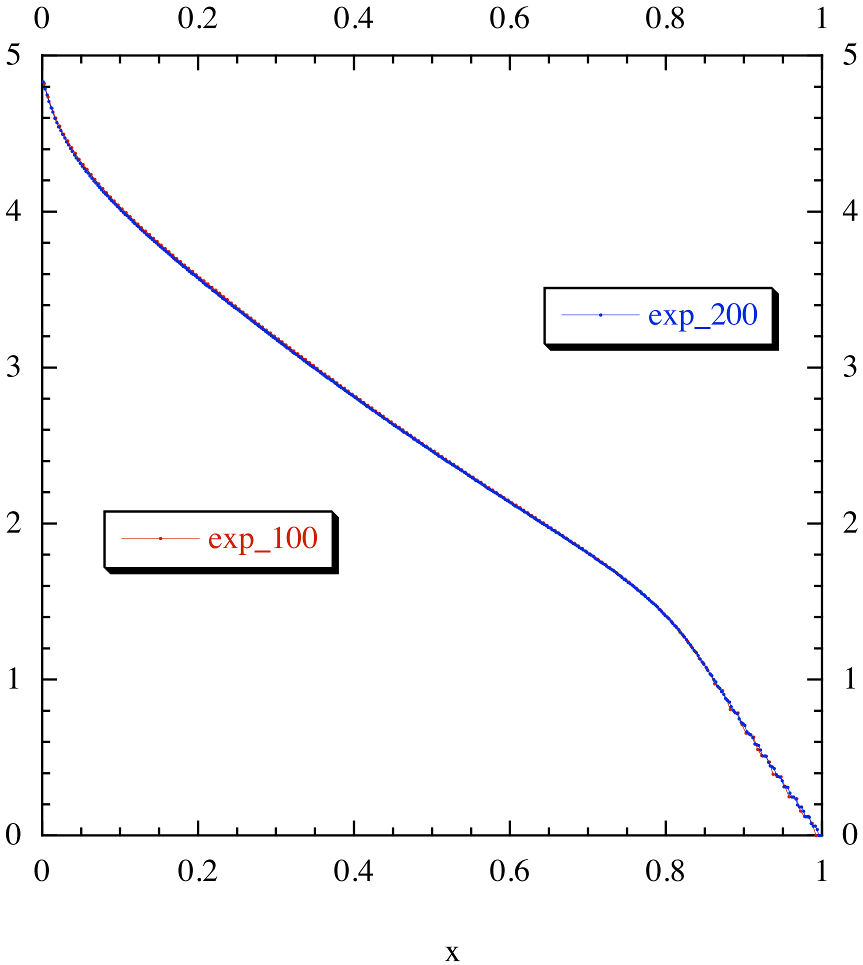
density=0.8

Convergence of Lyapunov Spectrum

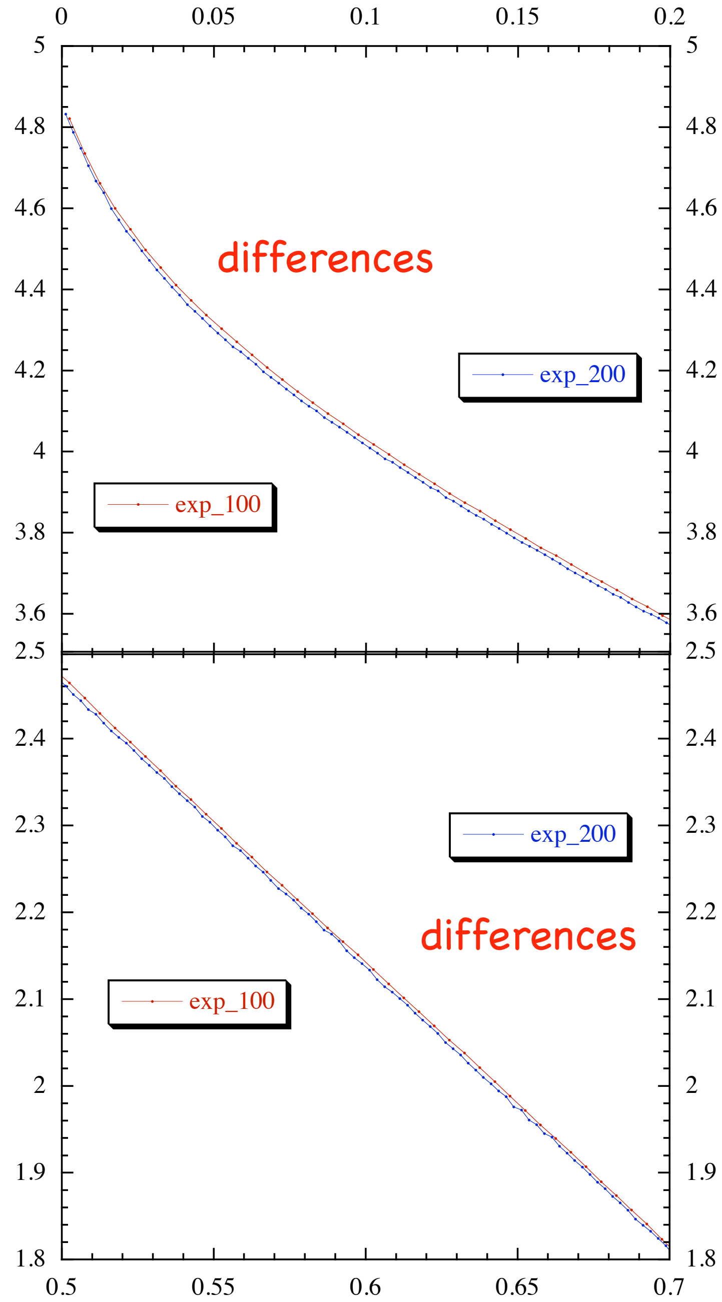


As a function of #/2N the spectrum is the same

Finite Size Scaling



$$x_j = \frac{1}{2N}\left(j-1\right)$$



How good is the data?

Continuous spectrum

	λ_i	λ_{-i}	$\lambda_i + \lambda_{-i}$
1	4.82138777	-4.82138777	0.00000000
2	4.73536921	-4.73536921	0.00000000
3	4.66161919	-4.66161919	0.00000000
4	4.59994888	-4.59994888	0.00000000
5	4.54831696	-4.54831696	0.00000000
6	4.49727392	-4.49727392	0.00000000
7	4.45375013	-4.45375013	0.00000000
8	4.41091824	-4.41091824	0.00000000
9	4.37316608	-4.37316608	0.00000000
10	4.33672523	-4.33672523	0.00000000
11	4.30343103	-4.30343103	0.00000000
12	4.27091694	-4.27091694	0.00000000
13	4.23881483	-4.23881483	0.00000000
14	4.20760679	-4.20760679	0.00000000
15	4.17766809	-4.17766809	0.00000000
16	4.14823580	-4.14823580	0.00000000
17	4.12064791	-4.12064791	0.00000000
18	4.09403896	-4.09403896	0.00000000
19	4.06858587	-4.06858587	0.00000000
20	4.04208279	-4.04208279	0.00000000

178	0.79464698	-0.79464698	0.00000000
179	0.78664899	-0.78664899	0.00000000
180	0.71133697	-0.71133697	0.00000000
181	0.65626800	-0.65626800	0.00000000
182	0.64845902	-0.64845902	0.00000000
183	0.63140100	-0.63140100	0.00000000
184	0.55199897	-0.55199897	0.00000000
185	0.51402801	-0.51402801	0.00000100
186	0.51095599	-0.51095599	-0.00000100
187	0.47261700	-0.47261700	0.00000000
188	0.39362499	-0.39362499	0.00000000
189	0.37898099	-0.37898001	0.00000000
190	0.37707400	-0.37707499	0.00000000
191	0.31479400	-0.31479400	0.00000000
192	0.24870700	-0.24870700	0.00000000
193	0.24779899	-0.24779899	0.00000000
194	0.23607300	-0.23607300	0.00000000
195	0.15742099	-0.15742099	0.00000000
196	0.12259600	-0.12259000	0.00000500
197	0.12232200	-0.12232700	-0.00000500
198	0.07876100	-0.07876100	0.00000000
199	0.00000000	0.00000000	0.00000000
200	0.00000000	0.00000000	0.00000000

How good is the data?

- $\lambda_i + \lambda_{-i} = 0$ accurate to 6-7 figures
- 2-point steps
- 0.122596 0.122322 for 196-7
- 0.786649 0.794647 for 178-9
- accurate to 2-4 significant figures

Symplectic structure

Symplectic eigenvalue theorem
- pairing of Lyapunov exponents

$$\{\lambda_j, -\lambda_j\}$$

Hamiltonian system

Conjugacy of Lyapunov vectors

$$\delta\Gamma_j = \begin{pmatrix} \delta q_j \\ \delta p_j \end{pmatrix} \qquad \delta\Gamma_{-j} = \begin{pmatrix} \delta q_{-j} \\ \delta p_{-j} \end{pmatrix} = \begin{pmatrix} -\delta p_j \\ \delta q_j \end{pmatrix}$$

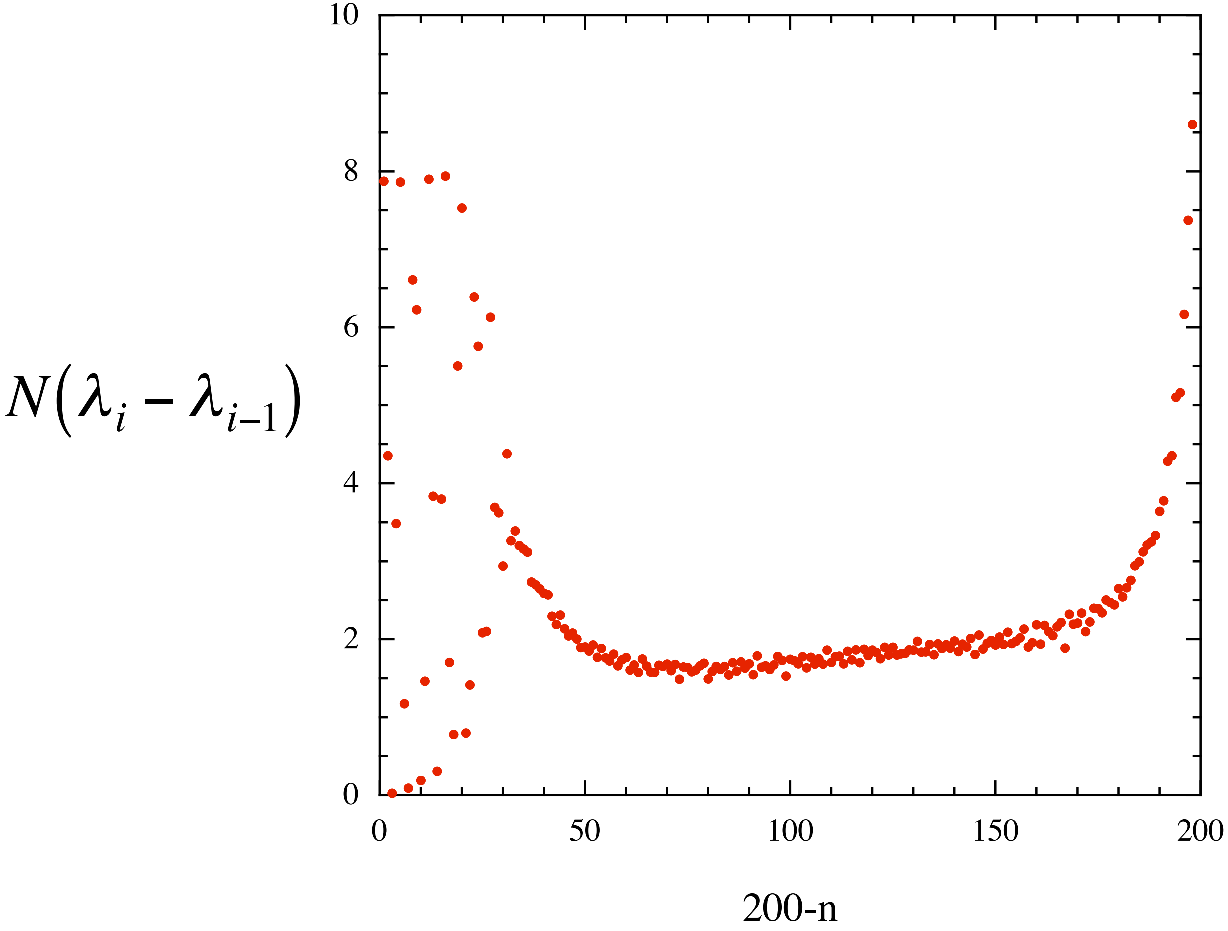
Benettin's algorithm preserves this structure

Notation for Lyapunov vectors →

$$\delta\Gamma = \begin{pmatrix} \delta q \\ \delta p \end{pmatrix} = \begin{pmatrix} \delta x \\ \delta y \\ \delta p_x \\ \delta p_y \end{pmatrix}$$

Slope of Spectrum

N=100



the error bar is much smaller than the difference between the exponents

Lyapunov Modes

Conserved Quantities

For each conserved quantity of the dynamics there is a zero Lyapunov exponent.
For (P,P):

Translational invariance

$$x_i \mapsto x_i + a \qquad y_i \mapsto y_i + b$$

Total momentum

$$\bar{p}_x = \frac{1}{N}\sum_{i=1}^{N} p_{xi} \qquad \bar{p}_y = \frac{1}{N}\sum_{i=1}^{N} p_{yi}$$

Energy

$$\tfrac{1}{2}\sum_{i=1}^{N} p_{xi}^2 + p_{yi}^2 = N\bar{e} = NT$$

(P,P) 6 zeros

(H,P) 4 zeros

There can be no exponential separation in the direction of the trajectory.

Time translational invariance $\bar{t}$

Conjugacy

$$\{\bar{x}, \bar{p}_x\} \qquad \{\bar{y}, \bar{p}_y\} \qquad \{\bar{e}, \bar{t}\}$$

Lyapunov Modes for Zero exponents

Lyapunov Vector notation

$$\delta\Gamma = \begin{pmatrix} \delta\mathbf{q} \\ \delta\mathbf{p} \end{pmatrix} = \begin{pmatrix} \delta x \\ \delta y \\ \delta p_x \\ \delta p_y \end{pmatrix}$$

Noether's theorem transformations generated by the conserved quantities

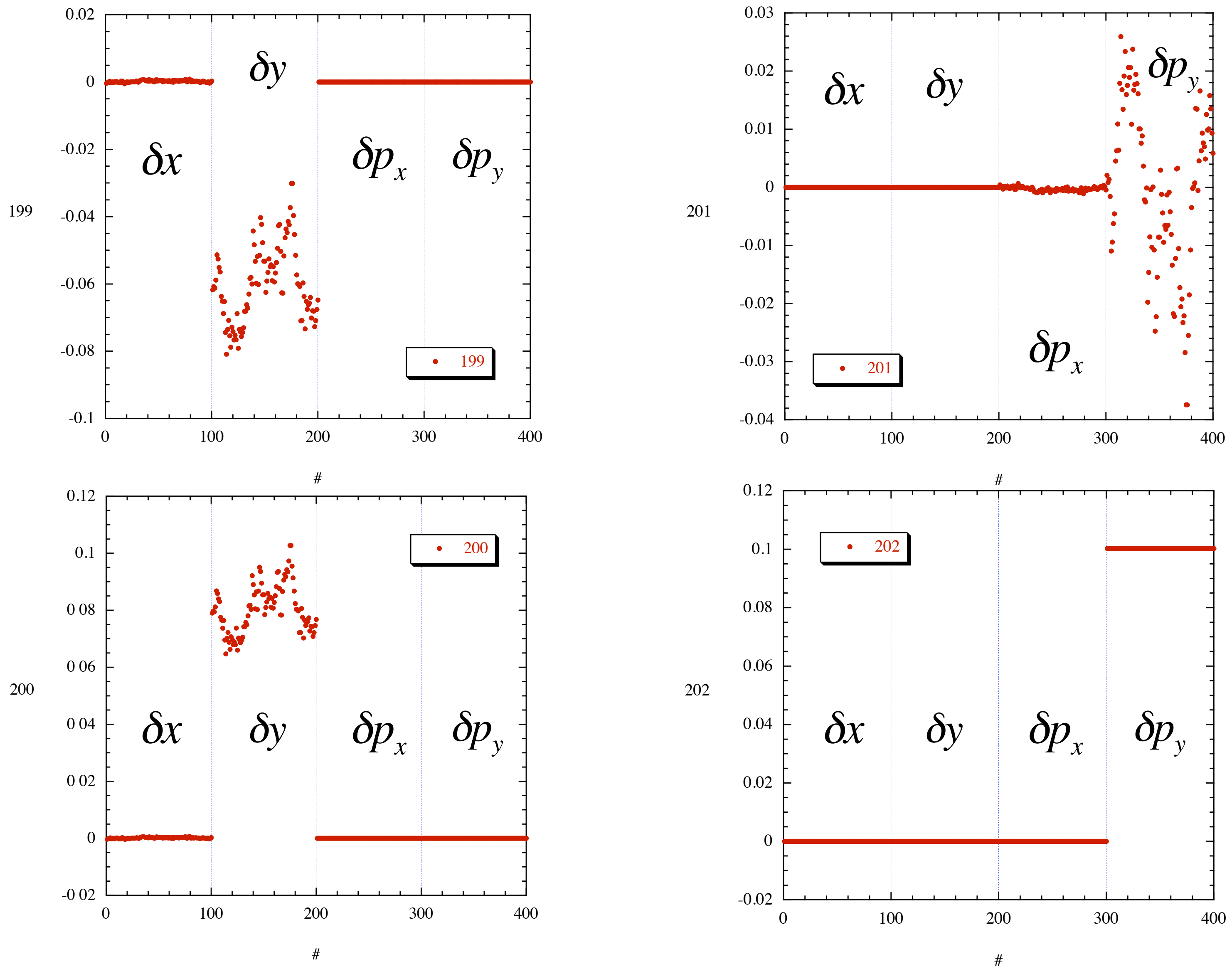
$$\delta\Gamma_x = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \delta\Gamma_y = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \delta\Gamma_t = \frac{1}{\sqrt{2NT}} \begin{pmatrix} p_x \\ p_y \\ 0 \\ 0 \end{pmatrix}$$

$$\delta\Gamma_{px} = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \delta\Gamma_{py} = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix} \qquad \delta\Gamma_e = \frac{1}{\sqrt{2NT}} \begin{pmatrix} 0 \\ 0 \\ p_x \\ p_y \end{pmatrix}$$

These form two independent sub-spaces $\{\delta\Gamma_x, \delta\Gamma_y, \delta\Gamma_t\}$ $\{\delta\Gamma_{px}, \delta\Gamma_{py}, \delta\Gamma_e\}$

Equilibrium

Instantaneous Zero Modes



Numerical Zero Modes are Linear combinations of basis vectors

So

$$\delta\Gamma_{199} = a\delta\Gamma_y + b\delta\Gamma_t = a\frac{1}{\sqrt{N}}\begin{pmatrix}0\\1\\0\\0\end{pmatrix} + b\frac{1}{\sqrt{2NT}}\begin{pmatrix}p_x\\p_y\\0\\0\end{pmatrix}$$

Therefore

$$\delta x_{199} = b\frac{1}{\sqrt{2N}}p_x$$

$$\delta y_{199} = a\frac{1}{\sqrt{N}} + b\frac{1}{\sqrt{2N}}p_y$$

$$T = 1$$

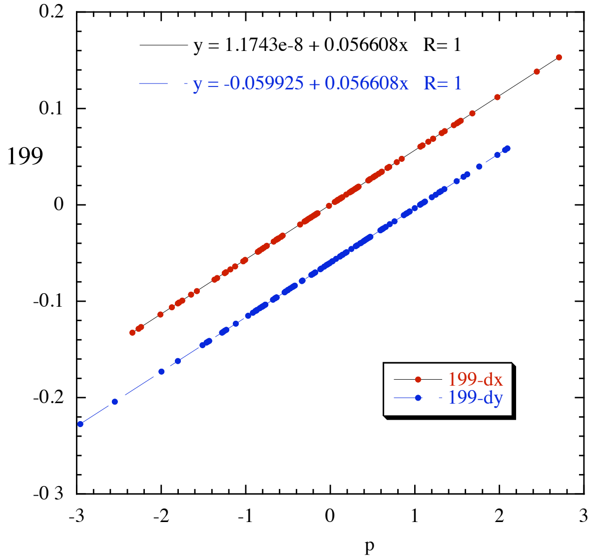
Plot:

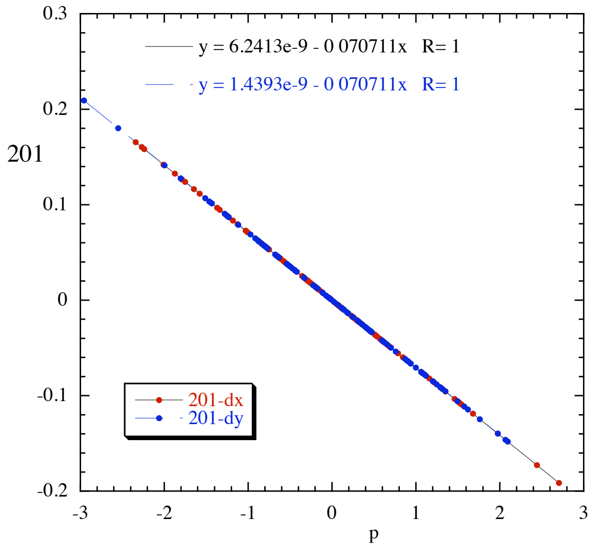
δx_{199}	versus	p_x	slope	$\frac{b}{\sqrt{2N}}$

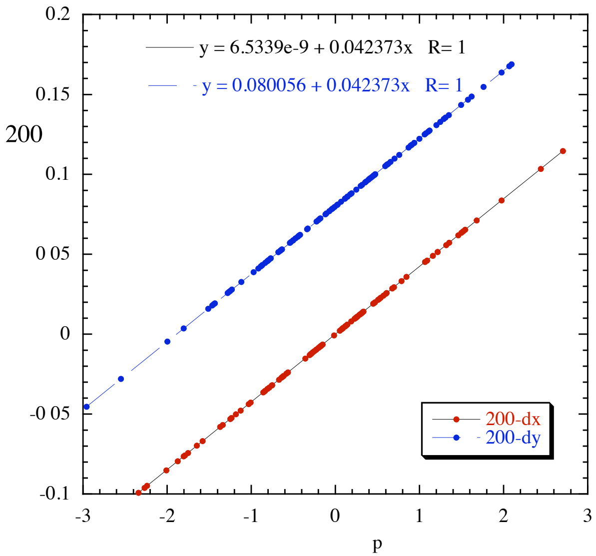
Plot:

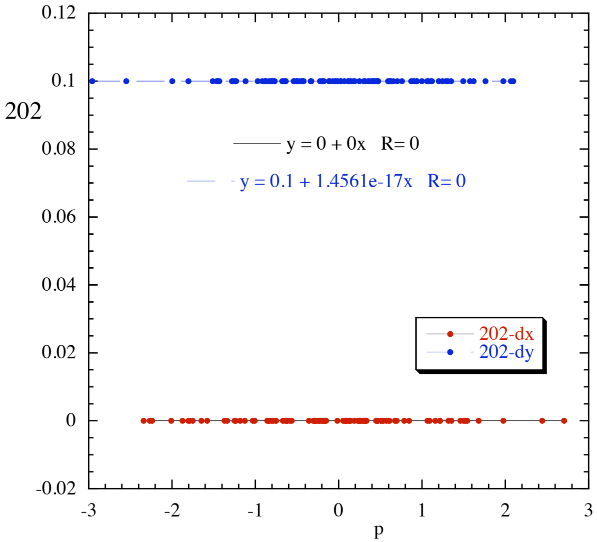
δy_{199}	versus	p_y	slope	$\frac{b}{\sqrt{2N}}$
			intercept	$\frac{a}{\sqrt{N}}$

Linear fits


199
y = 1.1743e-8 + 0.056608x R= 1
y = -0.059925 + 0.056608x R= 1
199-dx
199-dy
p
0.2
0.1
0
-0.1
-0.2
-0 3
-3
-2
-1
0
1
2
3


201
y = 6.2413e-9 - 0 070711x R= 1
y = 1.4393e-9 - 0 070711x R= 1
201-dx
201-dy
p
0.3
0.2
0.1
0
-0.1
-0.2
-3
-2
-1
0
1
2
3


200
y = 6.5339e-9 + 0.042373x R= 1
y = 0.080056 + 0.042373x R= 1
200-dx
200-dy
p
0.2
0.15
0.1
0 05
0
-0 05
-0.1
-3
-2
-1
0
1
2
3


202
y = 0 + 0x R= 0
y = 0.1 + 1.4561e-17x R= 0
202-dx
202-dy
p
0.12
0.1
0.08
0.06
0.04
0.02
0
-0.02
-3
-2
-1
0
1
2
3

vector	intercept	slope
199	-0.059925	0.056608
200	0.080056	0.042373
201	0	-0.070711
202	0.1	0

The zero modes
are
exactly linear
combinations
of the basis
vectors

	$\delta\Gamma_y$	$\delta\Gamma_t$	
vector	a	b	
199	-0.59925	0.80056	
200	0.80056	0.59925	
201	0	-1	$\delta\Gamma_e$
202	1	0	
	$\delta\Gamma_{py}$		

So far

1. Symmetries and conserved quantities
   determine the Zero modes

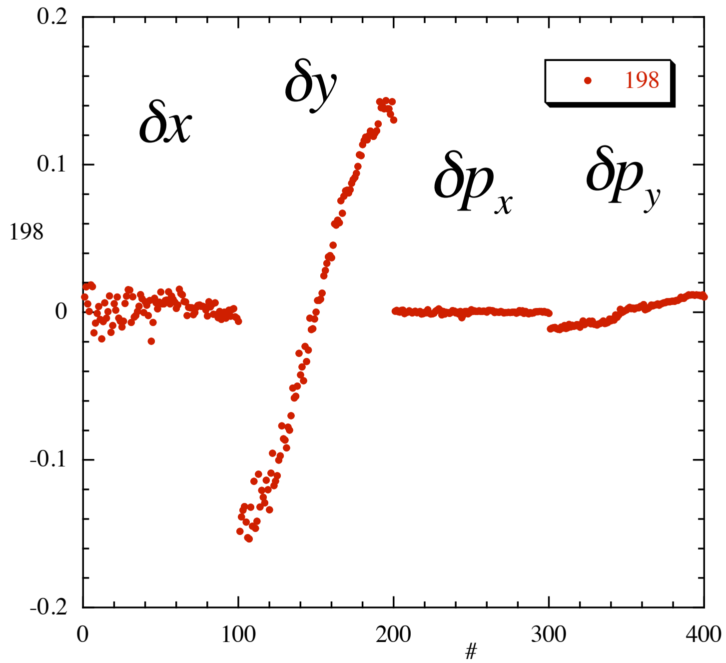
Next

2. Non-zero modes are Fourier analogues
   of the zero modes

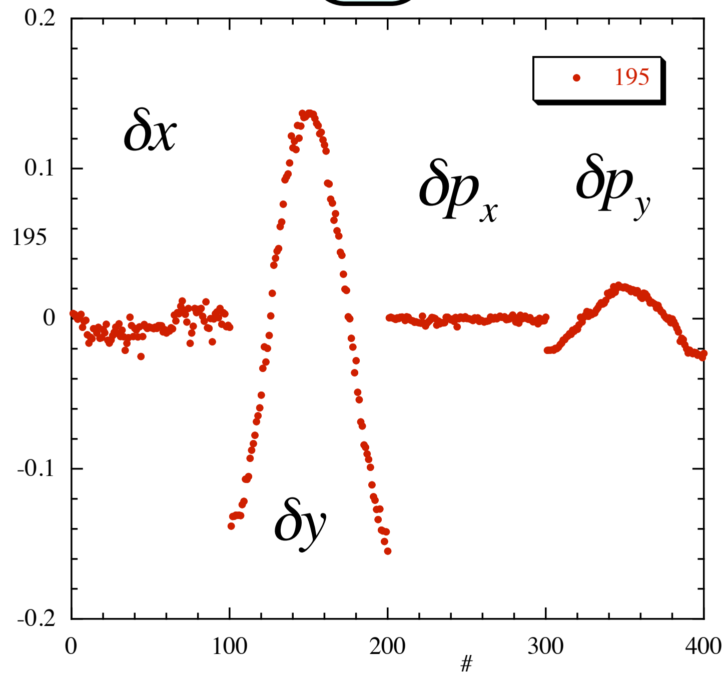
Transverse - Lyapunov Modes

Instantaneous Transverse modes

T1



T2



$$\delta p_x \sim \delta x$$

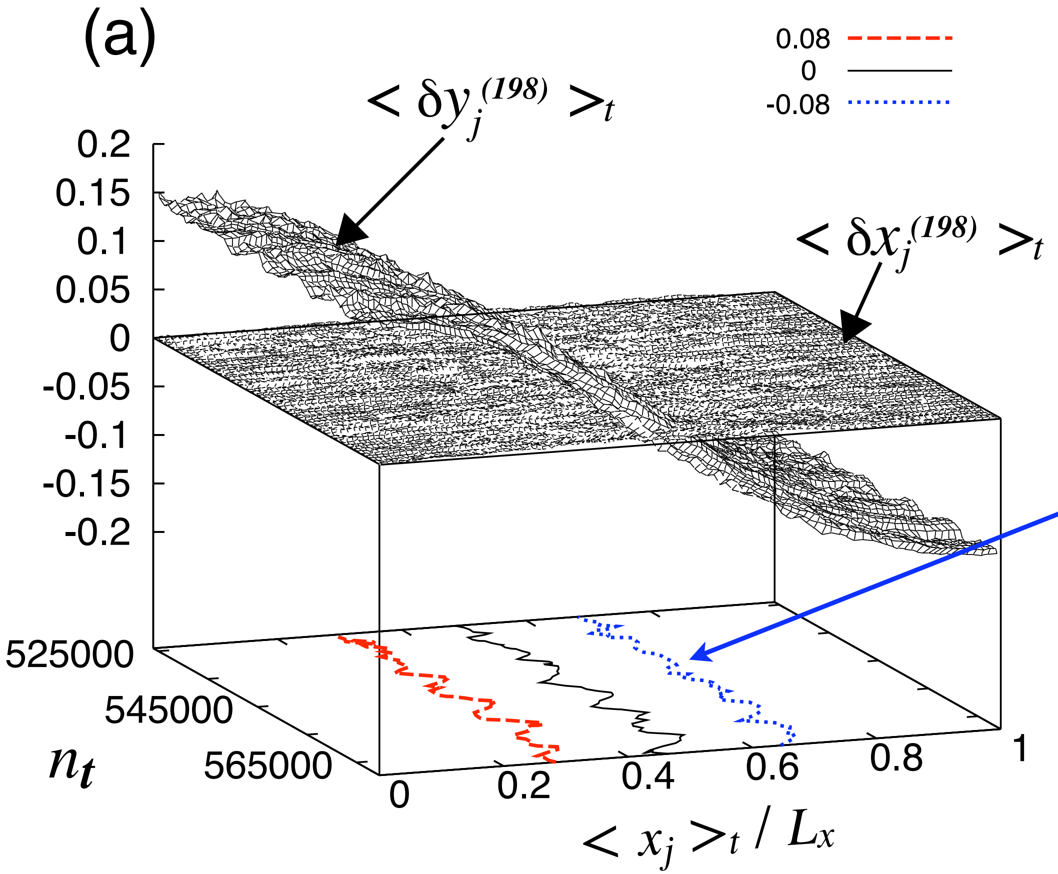
$$\delta p_y \sim \delta y$$

Consider δx and δy

Lyapunov modes

T1

Transverse



Small amplitude oscillations

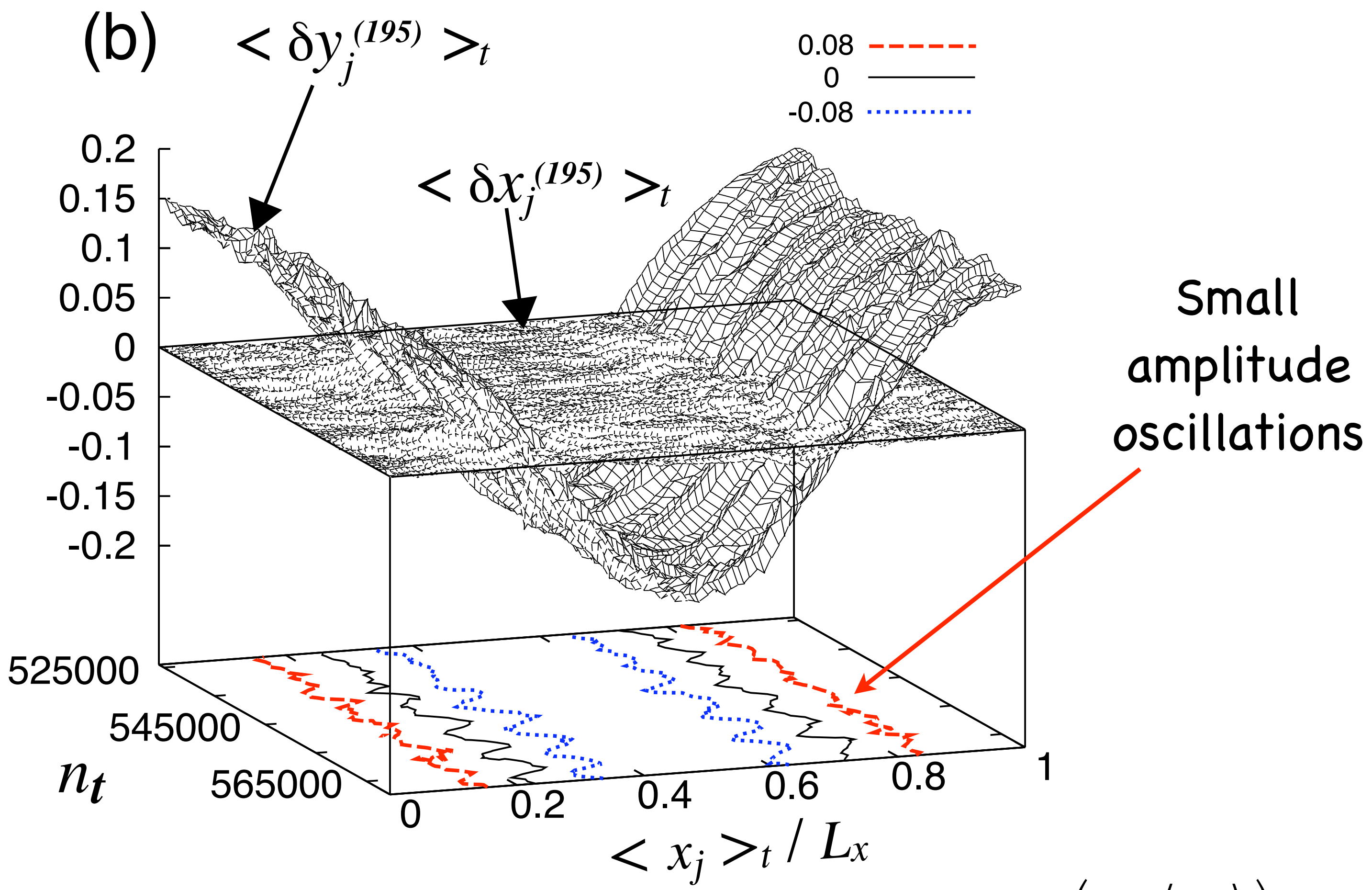
T1 mode

$$\delta x_j \sim 0 \quad \text{and} \quad \delta y_j \sim \cos\left(\frac{\pi \langle x_j \rangle}{L_x}\right)$$

Lyapunov modes

T2

Transverse



T2 mode

$$\delta x_j \sim 0 \quad \text{and} \quad \delta y_j \sim \cos\left(\frac{2\pi\langle x_j\rangle}{L_x}\right)$$

General Transverse modes - Tn

Functional form

$$\delta\Gamma_n^T = \begin{pmatrix} 0 \\ \gamma_n \cos k_n x_j \\ 0 \\ \gamma'_n \cos k_n x_j \end{pmatrix}$$

where

$$k_n = \frac{n\pi}{L_x}$$

coefficients γ_n and γ'_n

Projecting: instantaneous values of γ_n and γ'_n

Take numerical vector

$$\left(0, \delta y^{(T_n)}, 0, \delta p_y^{(T_n)}\right)$$

Orthogonality condition

$$\sum_{j=1}^{N} \cos k_m x_j \cos k_n x_j \approx \tfrac{N}{2}\delta_{m,n}$$

$$\gamma_n = \tfrac{2}{N}\sum_{j=1}^{N} \cos k_n x_j \; \delta y_j^{(T_n)}$$

$$\gamma'_n = \tfrac{2}{N}\sum_{j=1}^{N} \cos k_n x_j \; \delta p_{yj}^{(T_n)}$$

Average magnitude

$$\left\langle \left\| T_n \right\| \right\rangle = \sqrt{\tfrac{N}{2}\left\langle \gamma_n^2 + \gamma_n'^2 \right\rangle}$$

Normalization

If the average magnitude is one then the functional form is the whole vector

Transverse mode analysis

n mode	γ	γ'	magnitude	n mode	γ	γ'	magnitude
14 369	-0.08000120	-0.03966884	0.89302559	-14 432	-0.03966875	0.08000102	0.89302360
13 371	0.02481436	0.01145503	0.27343737	-13 430	0.01181519	-0.02559642	0.28204312
12 373	0.08371006	0.03602002	0.91135848	-12 428	0.03601995	-0.08370989	0.91135661
11 376	-0.08681836	-0.03451307	0.93431276	-11 425	0.03451301	-0.08681820	0.93431100
10 377	-0.08910981	-0.03241596	0.94826047	-10 424	0.03241590	-0.08910965	0.94825871
9 380	-0.09105360	-0.02996964	0.95861762	-9 421	0.02996959	-0.09105345	0.95861603
8 381	-0.08286711	-0.02447639	0.86408388	-8 420	0.02599364	-0.08800723	0.91767765
7 384	-0.09378257	-0.02445036	0.96919527	-7 417	0.02445033	-0.09378245	0.96919395
6 387	-0.09459071	-0.02121974	0.96943402	-6 414	-0.02121972	0.09459060	0.96943286
5 388	0.09641467	0.01813031	0.98105349	-5 413	-0.01813029	0.09641457	0.98105248
4 391	0.09733067	0.01469859	0.98434937	-4 410	0.01469858	-0.09733059	0.98434853
3 394	-0.09818094	-0.01119341	0.98817303	-3 407	0.01119373	-0.09818404	0.98820423
2 395	-0.09925888	-0.00754740	0.99545579	-2 406	-0.00754739	0.09925883	0.99545535
1 398	-0.09999366	-0.00388254	1.00069038	-1 403	0.00388254	-0.09999364	1.00069016

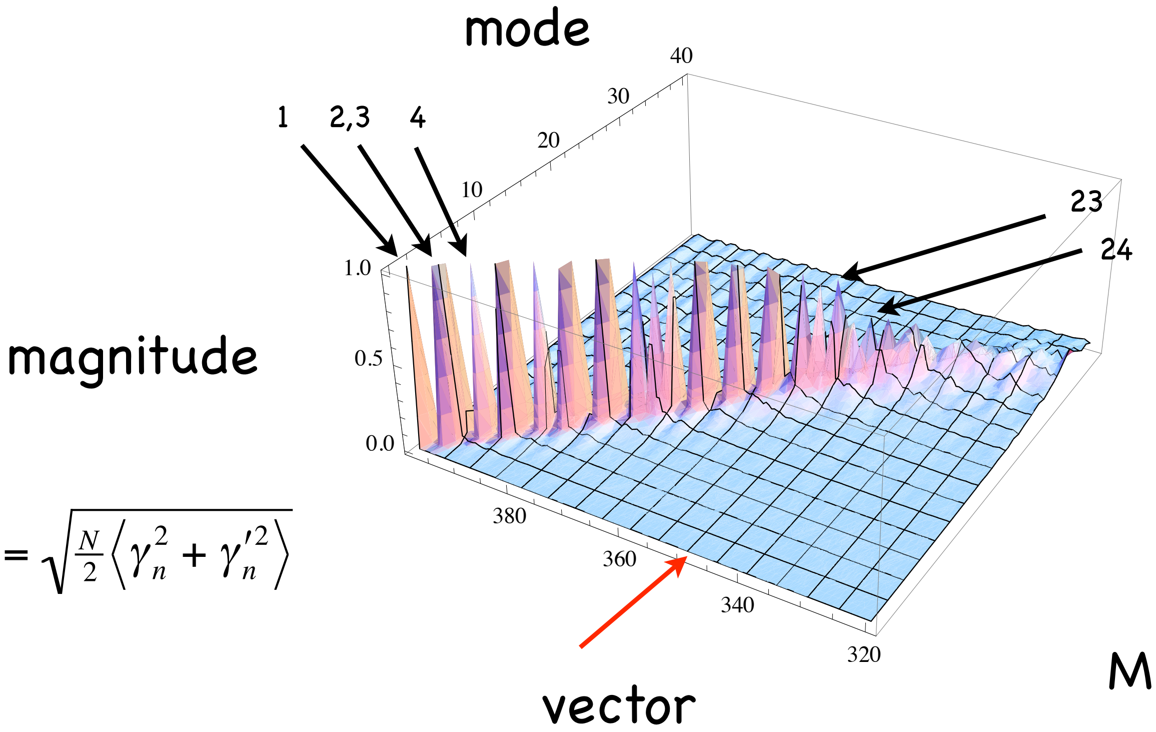
for $n > 0$ $\quad \gamma_n = \gamma'_{-n} \quad \gamma'_n = -\gamma_{-n}$

$$\delta\Gamma_{-j} = \begin{pmatrix} \delta q_{-j} \\ \delta p_{-j} \end{pmatrix} = \begin{pmatrix} -\delta p_j \\ \delta q_j \end{pmatrix}$$

T mode Analysis

	369	370	371	372	373	374	375	376	377	378	379	380	381	382	383	384	385	386	387	388	389	390	391	392	393	394	395	396	397	398		
0	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00
1	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.01	0.01	0.01	0.01	1.00	1.00
2	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.01	0.01	0.00	0.01	0.00	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	1.00	0.03	0.02	0.01	1.00
3	0.00	0.01	0.00	0.01	0.01	0.01	0.00	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.01	0.01	0.02	0.03	0.03	0.99	0.02	0.03	0.03	0.01	0.99
4	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.01	0.01	0.01	0.01	0.01	0.02	0.04	0.02	0.02	0.99	0.02	0.01	0.03	0.02	0.03	0.03	0.01	1.00
5	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.01	0.01	0.03	0.02	0.02	0.04	0.99	0.04	0.04	0.04	0.02	0.02	0.02	0.01	0.04	0.04	0.01	0.99
6	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.01	0.02	0.02	0.02	0.02	0.02	0.03	0.03	0.02	0.06	0.05	0.04	0.98	0.04	0.02	0.03	0.03	0.03	0.03	0.02	0.02	0.04	0.03	0.01	0.99
7	0.01	0.02	0.01	0.01	0.02	0.02	0.02	0.02	0.02	0.03	0.01	0.02	0.04	0.05	0.04	0.06	0.97	0.02	0.03	0.06	0.03	0.04	0.03	0.02	0.02	0.03	0.01	0.01	0.02	0.02	0.01	0.99
8	0.02	0.02	0.02	0.02	0.02	0.03	0.04	0.02	0.03	0.05	0.03	0.03	0.07	0.84	0.36	0.30	0.06	0.04	0.04	0.04	0.02	0.02	0.03	0.02	0.04	0.03	0.01	0.01	0.04	0.04	0.01	0.98
9	0.02	0.02	0.02	0.04	0.03	0.03	0.03	0.03	0.05	0.06	0.08	0.05	0.96	0.07	0.04	0.04	0.05	0.02	0.02	0.02	0.02	0.03	0.04	0.02	0.02	0.03	0.01	0.01	0.02	0.03	0.01	0.98
10	0.03	0.04	0.04	0.04	0.03	0.05	0.04	0.03	0.07	0.95	0.07	0.07	0.08	0.05	0.05	0.04	0.03	0.03	0.02	0.03	0.02	0.03	0.02	0.01	0.02	0.02	0.01	0.01	0.02	0.02	0.01	0.98
11	0.03	0.04	0.03	0.05	0.04	0.09	0.07	0.14	0.93	0.07	0.04	0.05	0.05	0.03	0.03	0.04	0.03	0.04	0.03	0.02	0.02	0.02	0.02	0.01	0.02	0.02	0.02	0.01	0.03	0.04	0.01	0.97
12	0.04	0.06	0.07	0.06	0.07	0.93	0.11	0.05	0.09	0.06	0.05	0.03	0.03	0.04	0.04	0.05	0.02	0.03	0.03	0.02	0.02	0.03	0.02	0.01	0.02	0.02	0.01	0.01	0.03	0.03	0.01	0.97
13	0.06	0.09	0.20	0.65	0.60	0.09	0.04	0.07	0.06	0.04	0.03	0.04	0.03	0.03	0.03	0.04	0.02	0.04	0.04	0.02	0.02	0.03	0.03	0.01	0.03	0.03	0.01	0.01	0.02	0.02	0.01	0.95
14	0.06	0.91	0.08	0.07	0.09	0.06	0.03	0.04	0.04	0.04	0.03	0.03	0.03	0.02	0.03	0.02	0.02	0.03	0.03	0.02	0.01	0.03	0.04	0.01	0.03	0.03	0.01	0.01	0.02	0.02	0.01	0.96
15	0.14	0.10	0.05	0.06	0.05	0.04	0.04	0.04	0.04	0.03	0.03	0.03	0.02	0.02	0.03	0.03	0.02	0.02	0.03	0.01	0.01	0.03	0.04	0.01	0.04	0.04	0.01	0.01	0.02	0.02	0.01	0.94
16	0.05	0.07	0.03	0.04	0.04	0.04	0.04	0.03	0.03	0.02	0.04	0.03	0.02	0.02	0.03	0.03	0.01	0.03	0.03	0.02	0.01	0.03	0.02	0.01	0.02	0.02	0.01	0.01	0.03	0.03	0.01	0.95
17	0.05	0.05	0.04	0.04	0.04	0.03	0.03	0.04	0.02	0.03	0.03	0.03	0.02	0.02	0.02	0.03	0.02	0.02	0.03	0.01	0.01	0.02	0.02	0.01	0.02	0.02	0.01	0.01	0.03	0.03	0.01	0.94
18	0.04	0.04	0.04	0.03	0.04	0.03	0.03	0.04	0.03	0.02	0.03	0.03	0.02	0.02	0.03	0.03	0.01	0.03	0.03	0.01	0.01	0.03	0.03	0.01	0.02	0.03	0.01	0.01	0.03	0.03	0.01	0.93
19	0.05	0.04	0.03	0.03	0.03	0.03	0.03	0.03	0.02	0.02	0.03	0.03	0.02	0.02	0.03	0.03	0.01	0.02	0.03	0.01	0.01	0.03	0.03	0.01	0.03	0.03	0.01	0.01	0.02	0.02	0.01	0.93
20	0.04	0.03	0.04	0.03	0.03	0.03	0.03	0.04	0.02	0.02	0.02	0.03	0.02	0.02	0.03	0.03	0.02	0.02	0.03	0.01	0.01	0.03	0.03	0.01	0.03	0.02	0.01	0.01	0.02	0.02	0.01	0.89
21	0.04	0.03	0.03	0.03	0.03	0.02	0.03	0.03	0.02	0.02	0.04	0.03	0.02	0.02	0.03	0.03	0.01	0.03	0.02	0.01	0.01	0.02	0.03	0.01	0.02	0.03	0.01	0.01	0.02	0.02	0.01	0.89
22	0.03	0.03	0.03	0.02	0.02	0.02	0.03	0.03	0.02	0.02	0.03	0.03	0.02	0.03	0.04	0.03	0.01	0.03	0.03	0.01	0.01	0.03	0.02	0.01	0.02	0.02	0.01	0.01	0.02	0.02	0.01	0.85
23	0.03	0.02	0.03	0.03	0.03	0.02	0.04	0.03	0.02	0.02	0.03	0.04	0.02	0.02	0.03	0.03	0.01	0.04	0.03	0.01	0.01	0.02	0.02	0.01	0.02	0.03	0.01	0.01	0.02	0.03	0.01	0.86
24	0.03	0.03	0.03	0.03	0.03	0.02	0.03	0.03	0.02	0.02	0.03	0.03	0.02	0.02	0.03	0.03	0.01	0.03	0.03	0.01	0.01	0.03	0.03	0.01	0.02	0.02	0.01	0.01	0.02	0.03	0.01	0.85
25	0.03	0.03	0.04	0.02	0.03	0.02	0.04	0.03	0.02	0.02	0.03	0.02	0.01	0.02	0.02	0.02	0.01	0.03	0.03	0.01	0.01	0.02	0.03	0.01	0.03	0.03	0.01	0.01	0.02	0.02	0.01	0.83
26	0.03	0.02	0.03	0.03	0.03	0.02	0.03	0.03	0.02	0.02	0.03	0.03	0.02	0.02	0.02	0.02	0.01	0.03	0.03	0.01	0.01	0.03	0.03	0.01	0.02	0.03	0.01	0.01	0.03	0.03	0.01	0.82
27	0.03	0.02	0.02	0.02	0.03	0.02	0.03	0.03	0.02	0.02	0.03	0.03	0.02	0.02	0.02	0.03	0.01	0.02	0.02	0.01	0.01	0.02	0.02	0.01	0.03	0.03	0.01	0.01	0.02	0.02	0.01	0.76
28	0.03	0.02	0.03	0.02	0.02	0.02	0.03	0.02	0.02	0.02	0.03	0.03	0.01	0.02	0.02	0.03	0.01	0.02	0.03	0.01	0.01	0.02	0.03	0.01	0.02	0.03	0.01	0.01	0.03	0.04	0.01	0.76
29	0.03	0.02	0.03	0.03	0.03	0.02	0.03	0.03	0.02	0.01	0.03	0.03	0.02	0.02	0.03	0.03	0.01	0.03	0.03	0.01	0.01	0.02	0.03	0.01	0.03	0.03	0.01	0.01	0.02	0.02	0.01	0.74
30	0.03	0.02	0.03	0.02	0.03	0.02	0.03	0.03	0.02	0.02	0.03	0.03	0.01	0.02	0.03	0.03	0.01	0.02	0.02	0.01	0.01	0.02	0.02	0.01	0.02	0.03	0.01	0.01	0.02	0.02	0.01	0.71
31	0.03	0.02	0.03	0.02	0.03	0.02	0.03	0.03	0.02	0.01	0.03	0.03	0.02	0.02	0.02	0.02	0.01	0.03	0.03	0.01	0.01	0.03	0.02	0.01	0.02	0.03	0.01	0.01	0.03	0.03	0.01	0.69
32	0.03	0.02	0.03	0.02	0.02	0.02	0.03	0.03	0.02	0.01	0.03	0.02	0.01	0.02	0.03	0.03	0.01	0.02	0.03	0.01	0.01	0.02	0.03	0.01	0.02	0.02	0.01	0.01	0.02	0.02	0.01	0.65
33	0.03	0.02	0.03	0.03	0.03	0.02	0.03	0.03	0.02	0.01	0.03	0.03	0.01	0.02	0.02	0.02	0.01	0.02	0.03	0.01	0.01	0.02	0.03	0.01	0.02	0.02	0.01	0.01	0.02	0.02	0.01	0.61

Transverse modes



$$\langle \| T_{23} \| \rangle > \frac{1}{2} > \langle \| T_{24} \| \rangle$$

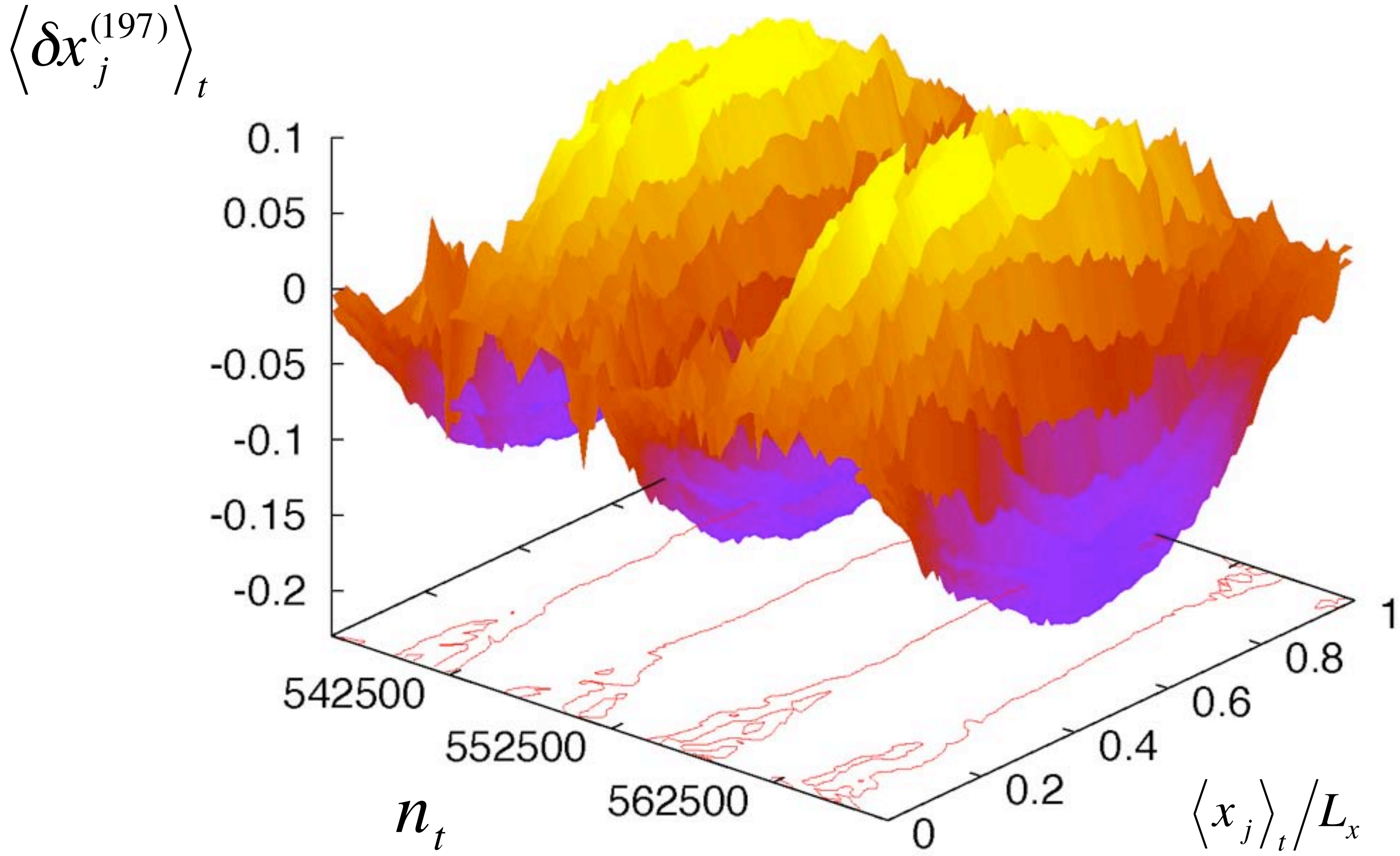
$$\langle \| T_n \| \rangle = \sqrt{\frac{N}{2} \langle \gamma_n^2 + \gamma_n'^2 \rangle}$$

Modes disappear ~350

T mode projection for N=200

LP - Lyapunov Modes

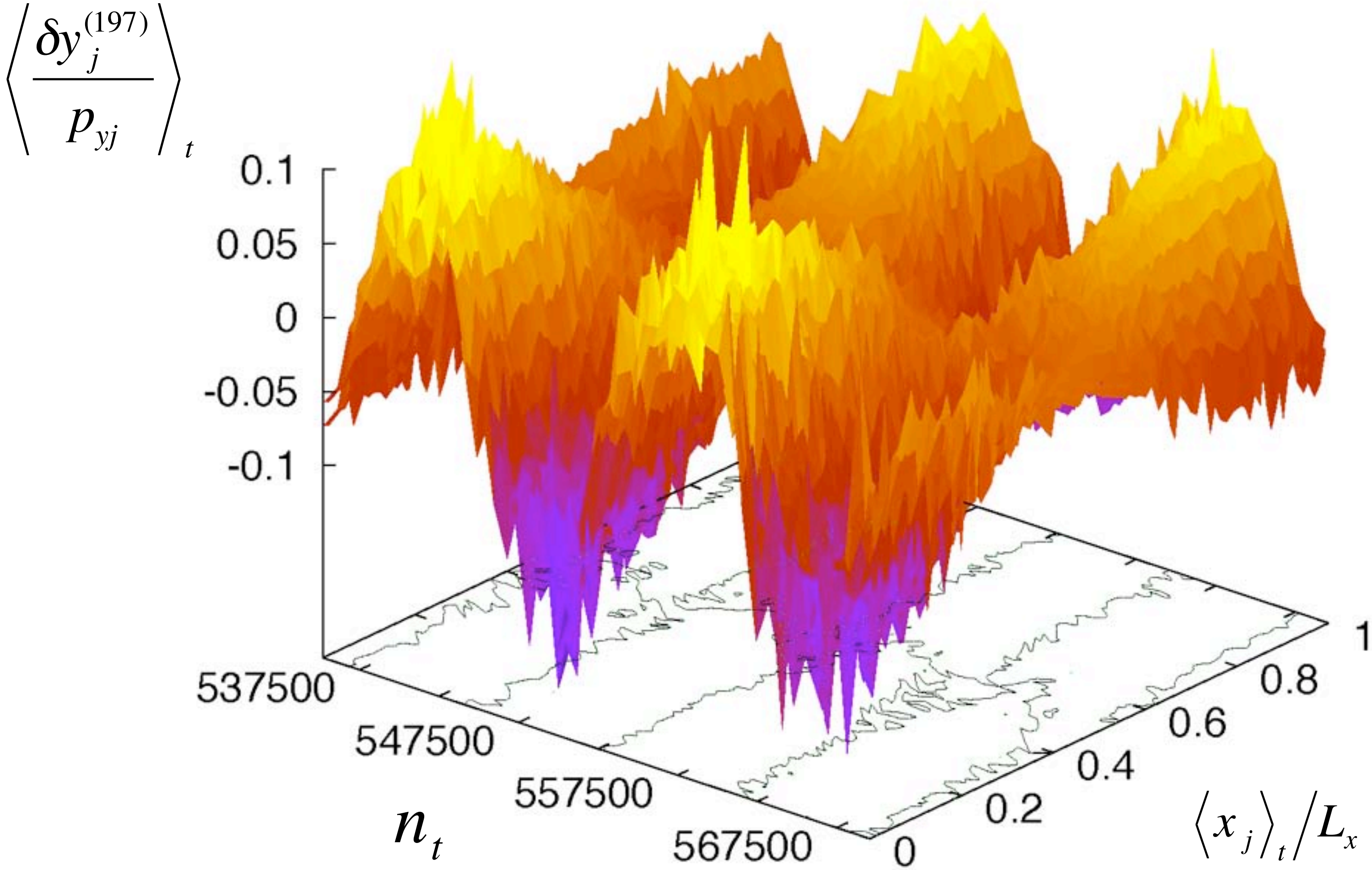
Lyapunov modes First 2-point step Longitudinal



L mode

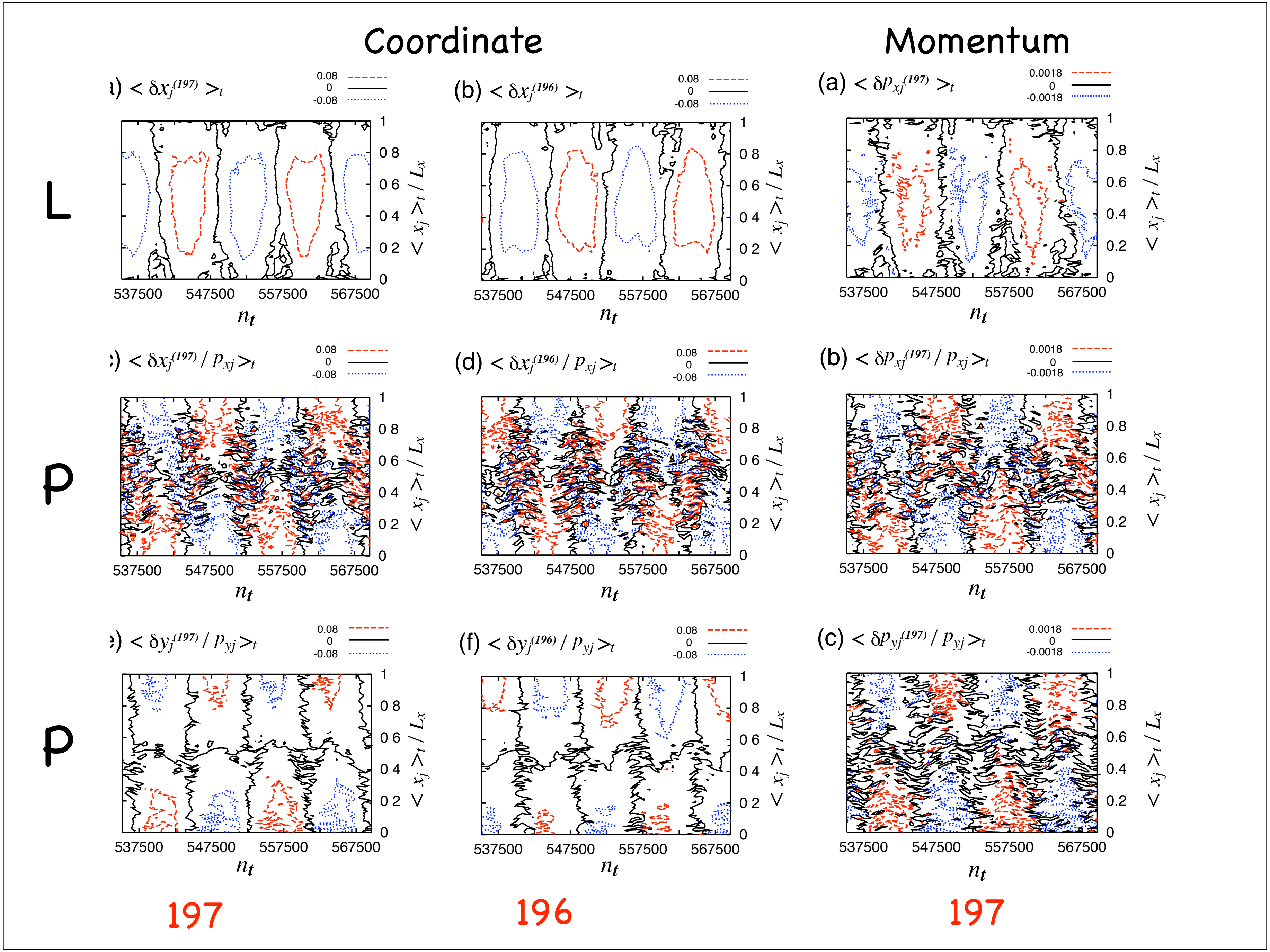
$$\delta x_j \sim \sin(k_1 x_j) \cos(\omega t)$$

Lyapunov modes First 2-point step Momentum



P mode

$$\delta y_j \sim p_{yj} \cos(k_1 x_j) \sin(\omega t)$$



Coordinate
Momentum
L
P
P
) < δxj(197) >t
(b) < δxj(196) >t
(a) < δpxj(197) >t
) < δxj(197) / pxj >t
(d) < δxj(196) / pxj >t
(b) < δpxj(197) / pxj >t
) < δyj(197) / pyj >t
(f) < δyj(196) / pyj >t
(c) < δpyj(197) / pyj >t
0.08
0
-0.08
0.0018
0
-0.0018
537500
547500
557500
567500
nt
< xj >t / Lx
197
196
197

LP modes - 2-point steps

$$\delta\Gamma^{(LP1_n)} = \sin\omega_n t \begin{pmatrix} \alpha_n \sin k_n x_j \\ 0 \\ \alpha'_n \sin k_n x_j \\ 0 \end{pmatrix} + \cos\omega_n t \begin{pmatrix} \beta_{xn} p_{xj} \cos k_n x_j \\ \beta_{yn} p_{yj} \cos k_n x_j \\ \beta'_{xn} p_{xj} \cos k_n x_j \\ \beta'_{yn} p_{yj} \cos k_n x_j \end{pmatrix}$$

$$\delta\Gamma^{(LP2_n)} = \cos\omega_n t \begin{pmatrix} \alpha_n \sin k_n x_j \\ 0 \\ \alpha'_n \sin k_n x_j \\ 0 \end{pmatrix} + \sin\omega_n t \begin{pmatrix} \beta_{xn} p_{xj} \cos k_n x_j \\ \beta_{yn} p_{yj} \cos k_n x_j \\ \beta'_{xn} p_{xj} \cos k_n x_j \\ \beta'_{yn} p_{yj} \cos k_n x_j \end{pmatrix}$$

↑ longitudinal

↑ momentum

LP Normalization

$$\left\|\delta\Gamma^{(LP_n)}\right\|^2 = \frac{N}{2}\left(\alpha_n^2 + \alpha_n'^2\right)\cos^2\omega_n t + \frac{N}{2}\left(\left(\beta_{xn}^2 + \beta_{xn}'^2\right)T_x + \left(\beta_{yn}^2 + \beta_{yn}'^2\right)T_y\right)\sin^2\omega_n t$$

Define an L and P normalization separately

$$\left\langle\left\|\delta\Gamma^{(L_n)}\right\|^2\right\rangle = \frac{N}{2}\left(\left\langle\alpha_n^2\right\rangle + \left\langle\alpha_n'^2\right\rangle\right)$$

$$\left\langle\left\|\delta\Gamma^{(P_n)}\right\|^2\right\rangle = \frac{N}{2}\left\{\left\langle\left(\beta_{xn}^2 + \beta_{xn}'^2\right)T_x\right\rangle + \left\langle\left(\beta_{yn}^2 + \beta_{yn}'^2\right)T_y\right\rangle\right\}$$

$$\left\|\delta\Gamma^{(LP_n)}\right\|^2 = \left\|\delta\Gamma^{(L_n)}\right\|^2\cos^2\omega_n t + \left\|\delta\Gamma^{(P_n)}\right\|^2\sin^2\omega_n t$$

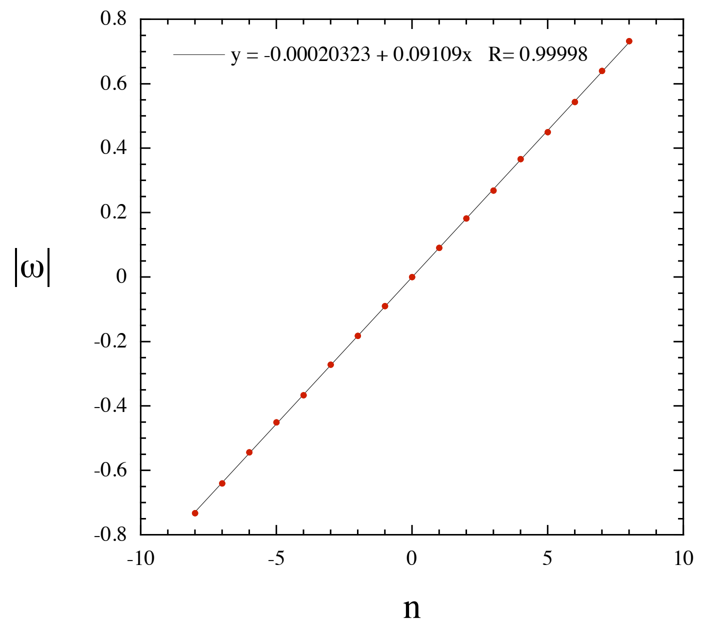
LP mode analysis

mode vectors	α_n	α'_n	β_{xn}	β_{yn}	β'_{xn}	β'_{yn}	$\|L_n\|$	$\|P_n\|$
-8 429 431	0.0192	0.0536	0.0132	0.0118	0.0362	0.0326	0.5696	0.5187
-7 426 427	0.0236	0.0740	0.0169	0.0149	0.0512	0.0458	0.7768	0.7234
-6 422 423	0.0225	0.0810	0.0165	0.0142	0.0562	0.0496	0.8408	0.7805
-5 418 419	0.0201	0.0842	0.0151	0.0135	0.0591	0.0539	0.8653	0.8252
-4 415 416	0.0177	0.0899	0.0138	0.0125	0.0635	0.0589	0.9159	0.8858
-3 411 412	0.0143	0.0940	0.0113	0.0106	0.0667	0.0636	0.9509	0.9344
-2 408 409	0.0100	0.0965	0.0084	0.0078	0.0697	0.0669	0.9698	0.9725
-1 404 405	0.0051	0.0984	0.0043	0.0041	0.0696	0.0682	0.9849	0.9759
0 0 0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
1 397 396	0.0984	0.0051	0.0696	0.0682	0.0043	0.0041	0.9849	0.9759
2 393 392	0.0965	0.0100	0.0696	0.0669	0.0084	0.0078	0.9698	0.9725
3 390 389	0.0940	0.0143	0.0667	0.0636	0.0113	0.0106	0.9509	0.9344
4 386 385	0.0899	0.0177	0.0635	0.0589	0.0138	0.0125	0.9159	0.8858
5 383 382	0.0818	0.0195	0.0576	0.0527	0.0147	0.0132	0.8408	0.8057
6 379 378	0.0810	0.0225	0.0562	0.0496	0.0165	0.0142	0.8408	0.7805
7 375 374	0.0740	0.0236	0.0512	0.0458	0.0169	0.0149	0.7768	0.7234
8 372 370	0.0535	0.0192	0.0361	0.0325	0.0132	0.0118	0.5683	0.5172

LP mode frequencies

n	periods	$\Delta\theta$	ω
-8	21	65.790964	0.245910
-7	41	127.587867	0.476892
-6	-35	-110.415803	-0.412707
-5	-28	-89.485346	-0.334474
-4	23	72.362080	0.270471
-3	17	53.864262	0.201331
-2	11	34.488821	0.128911
-1	-5	-17.215220	-0.064346
0	0	0.0	0.0
1	-6	-17.983489	-0.067218
2	-11	-34.330859	-0.128320
3	-17	-53.639963	-0.200493
4	-23	-71.537765	-0.267390
5	-28	-89.318952	-0.333852
6	-34	-107.492284	-0.401779
7	-41	-127.624799	-0.477030
8	-15	-46.952352	-0.175496

ratio of L and P as a function of t gives frequencies



LP - coefficients

$$\alpha_n \approx \frac{1}{\sqrt{(N/2)\left(1+(an)^2\right)}}$$

drop x,y dependence

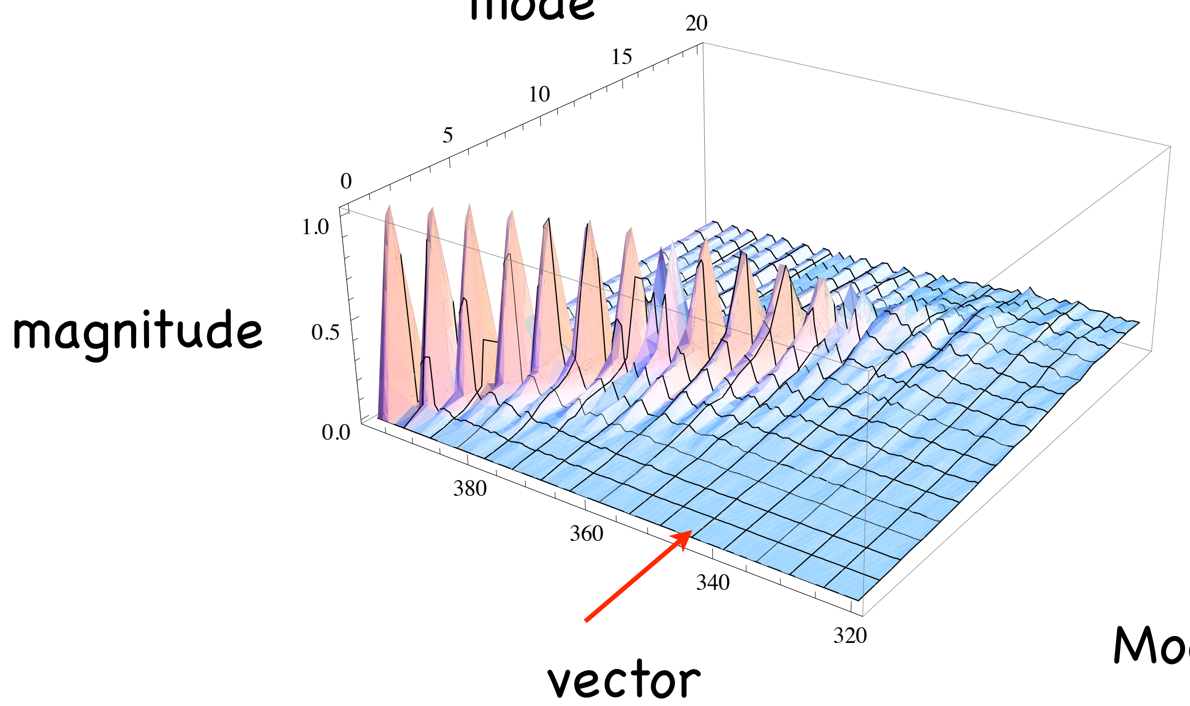
$$\beta_n \approx \frac{1}{\sqrt{NT\left(1+(bn)^2\right)}}$$

$$k_n = \frac{2\pi n}{L} \qquad \omega_n = n\omega_0$$

4 parameters $\quad a,b,g,\omega_0 = f\left(\rho,T,N\right)$

L-modes

$$\left\langle \left\| \delta\Gamma^{(L_n)} \right\|^2 \right\rangle = \frac{N}{2}\left(\left\langle \alpha_n^2 \right\rangle + \left\langle \alpha_n'^2 \right\rangle \right)$$

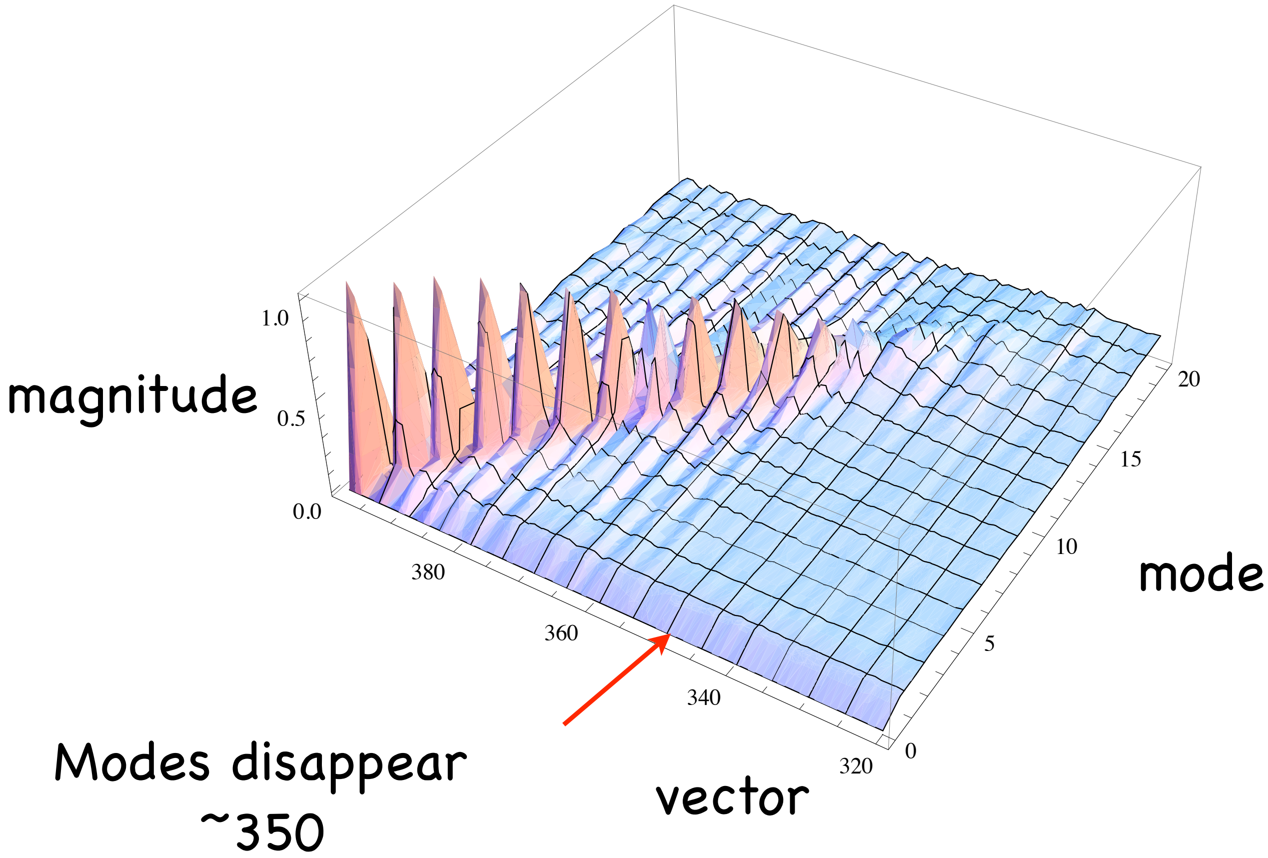


Modes disappear
~350

L mode projection for N=200

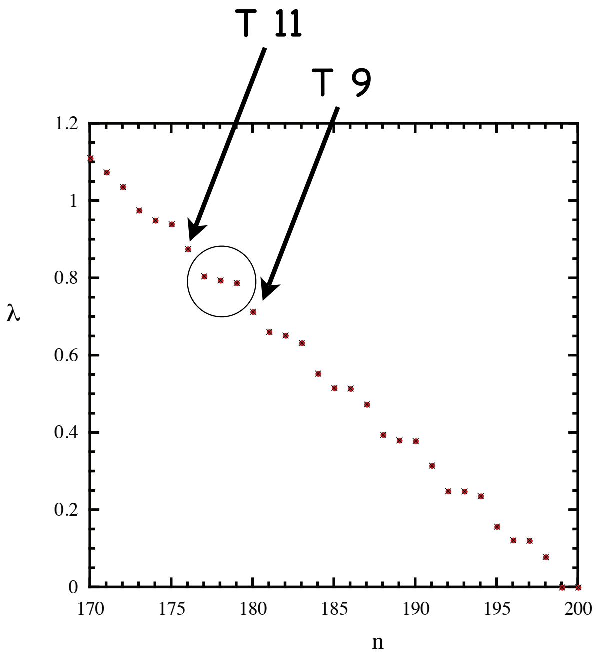
P-modes

$$\left\langle \left\| \delta\Gamma^{(P_n)} \right\|^2 \right\rangle = \frac{N}{2} \left\{ \left\langle \left(\beta_{xn}^2 + \beta_{xn}'^2 \right) T_x \right\rangle + \left\langle \left(\beta_{yn}^2 + \beta_{yn}'^2 \right) T_y \right\rangle \right\}$$



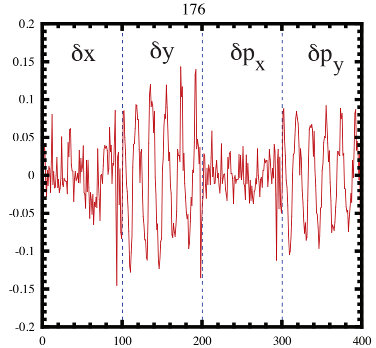
P mode projection for N=200

Mode Mixing



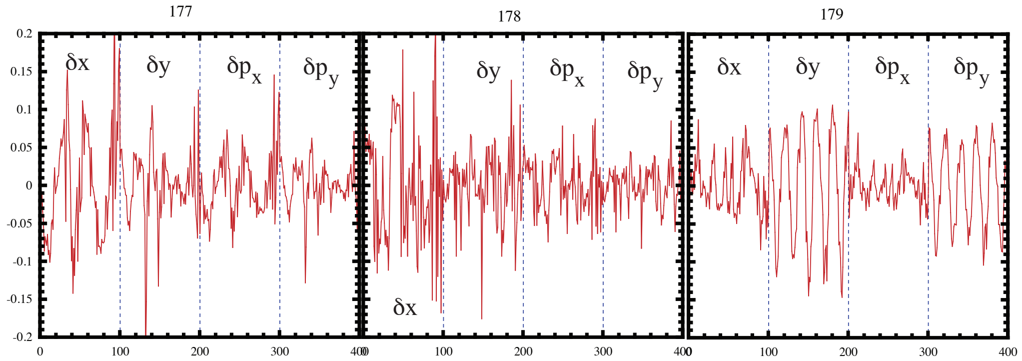
N=100

Transverse mode
n=11

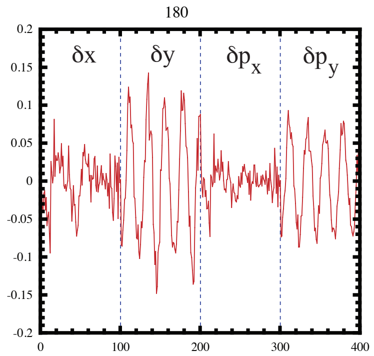


T 11

T 10 ?

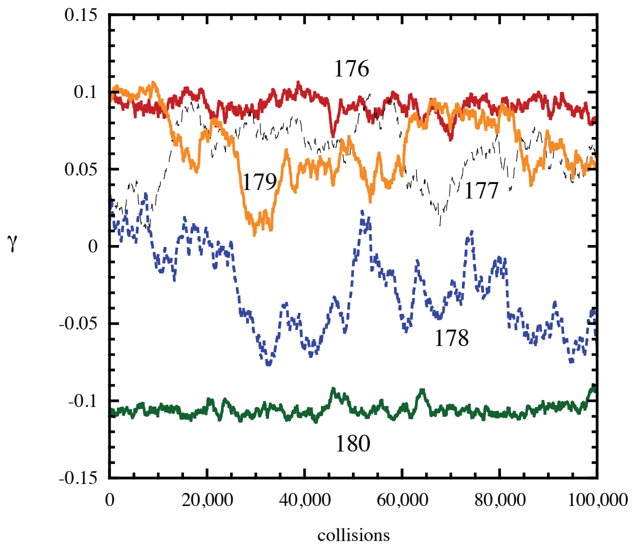


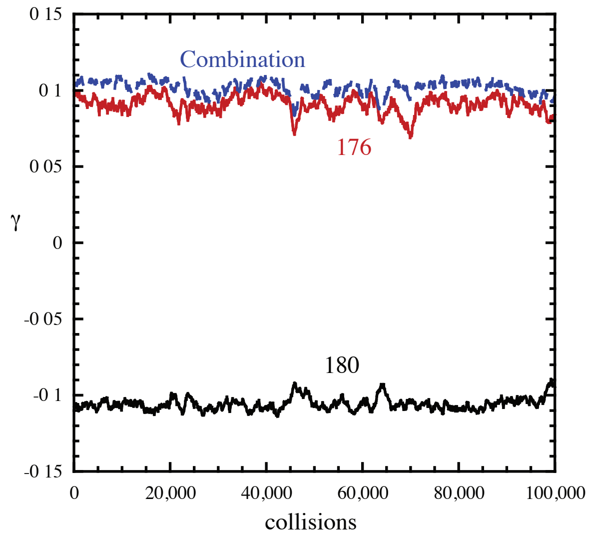
Transverse mode
n=9



T 9

Projecting coefficients





for T_9 and T_11

$$\gamma \approx \pm 0.1$$

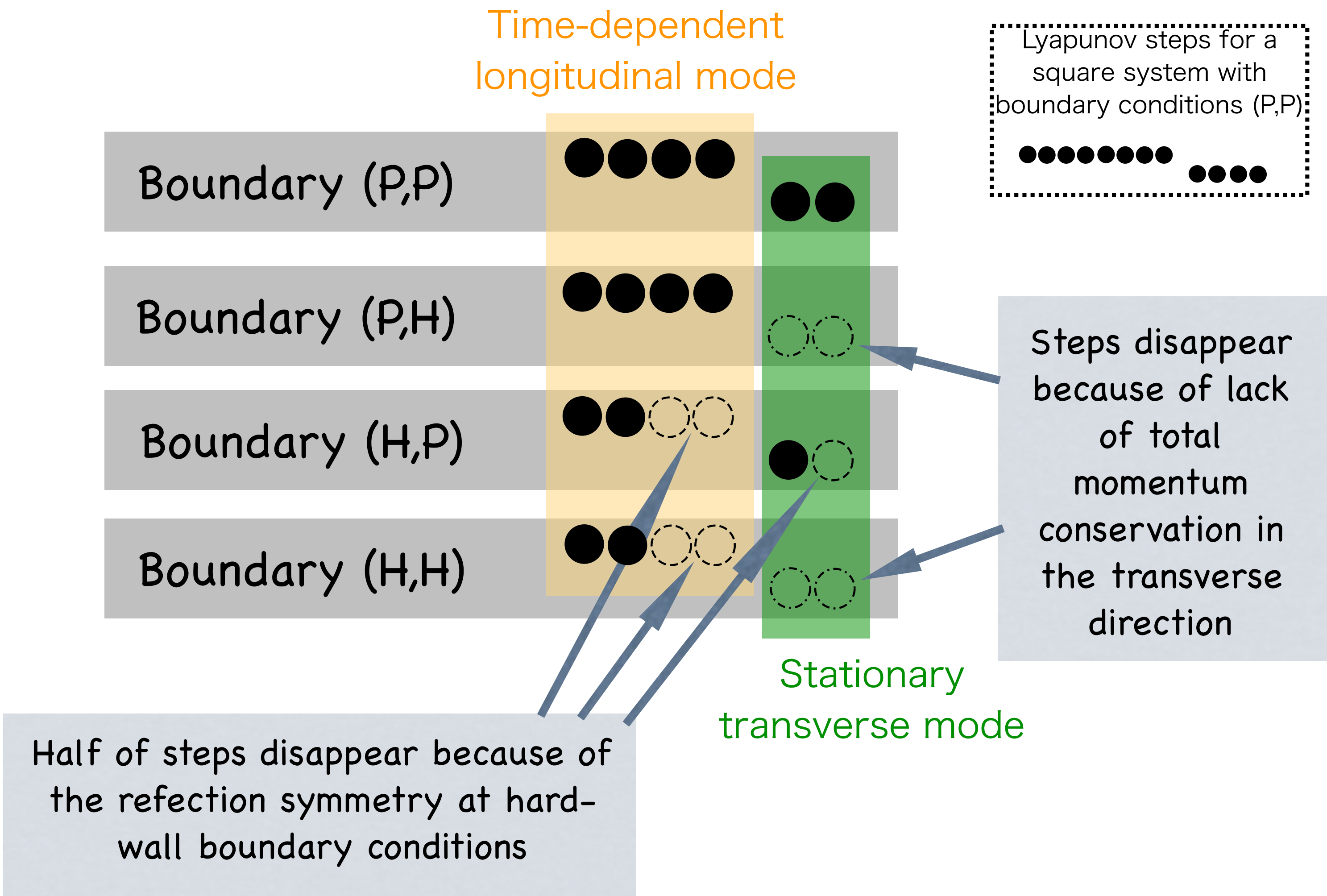
for T_10 components in three vectors 177, 178 & 179

$$\sqrt{\left(\gamma_{177}^2 + \gamma_{178}^2 + \gamma_{179}^2\right)} \approx 0.1$$

Convergence ?

Is mixing usual when exponents are close ?

Comparison of Boundary Conditions



Lyapunov Localization

Localization

contribution of particle i to Lyapunov vector j

$$\chi_i^{(j)} = \left(\delta x_i^{(j)}\right)^2 + \left(\delta y_i^{(j)}\right)^2 + \left(\delta p_{xi}^{(j)}\right)^2 + \left(\delta p_{yi}^{(j)}\right)^2$$

must be 0 < $\chi_i^{(j)}$ < 1 as Lyapunov vector is normalized

$$S(t) = -\sum_{i=1}^{N} \chi_i^{(j)} \ln \chi_i^{(j)}$$

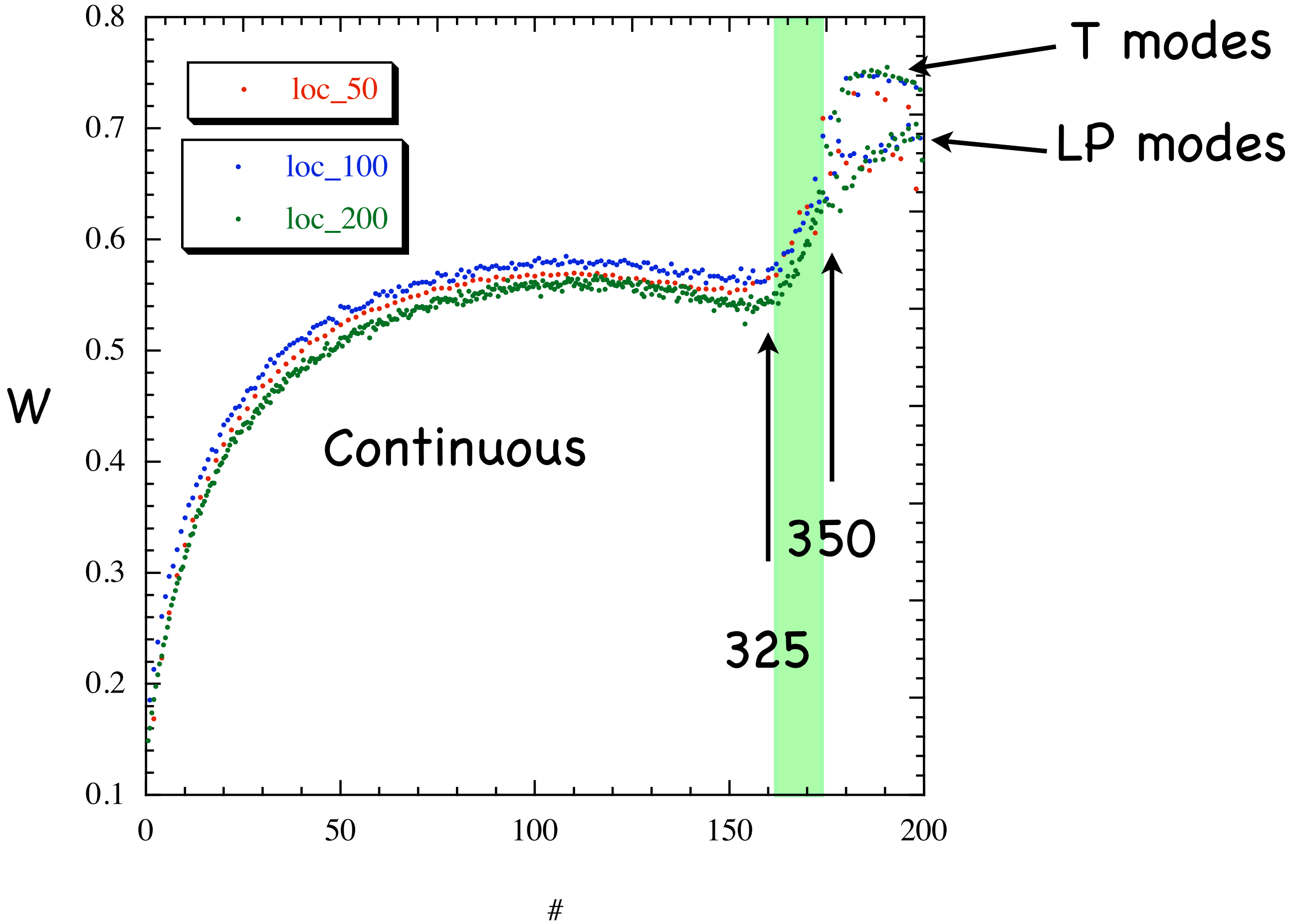
localization 'width'

$$W^{(j)}(t) = \frac{1}{N} \exp\left(-\sum_{i=1}^{N} \chi_i^{(j)} \ln \chi_i^{(j)}\right)$$

$$0 < W^{(j)}(t) < 1$$

Distribution of localizations or mean localizaton

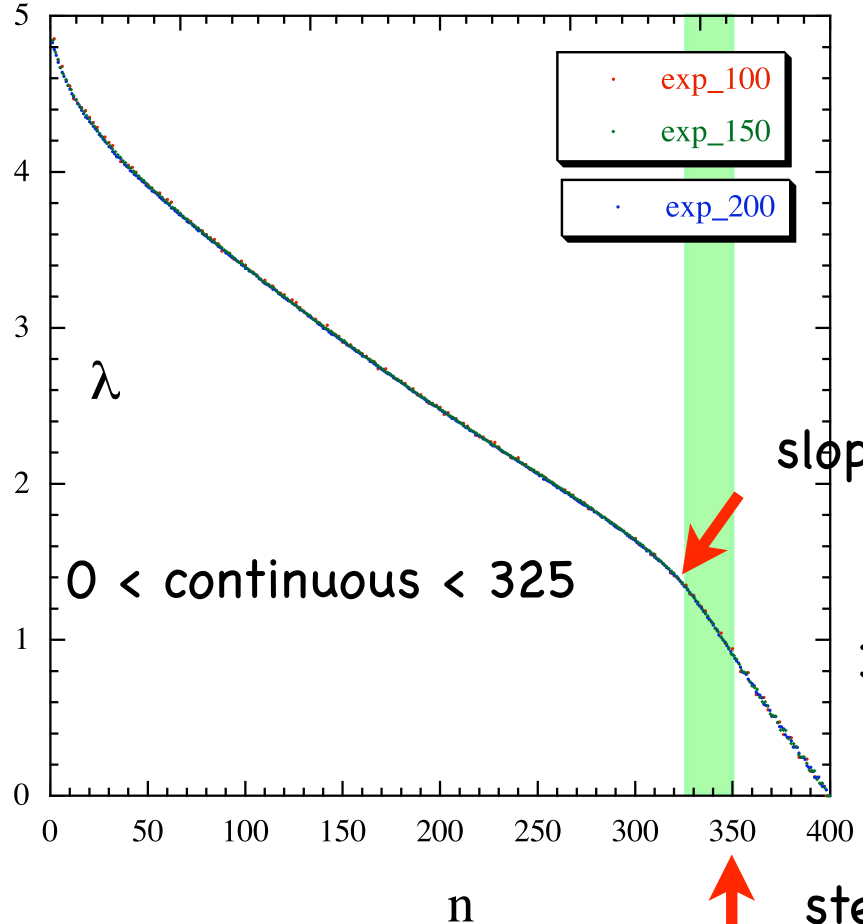
Mean Localization



GS vectors
Density=0.8

Extensivity ?

Probably yes!



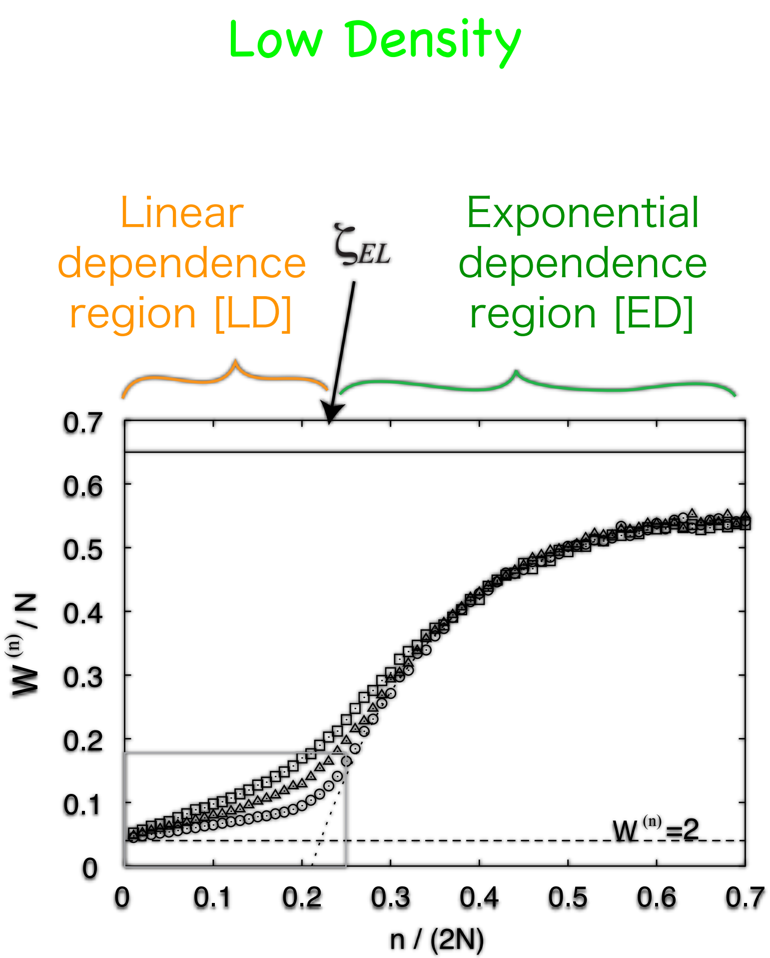
slope change 325

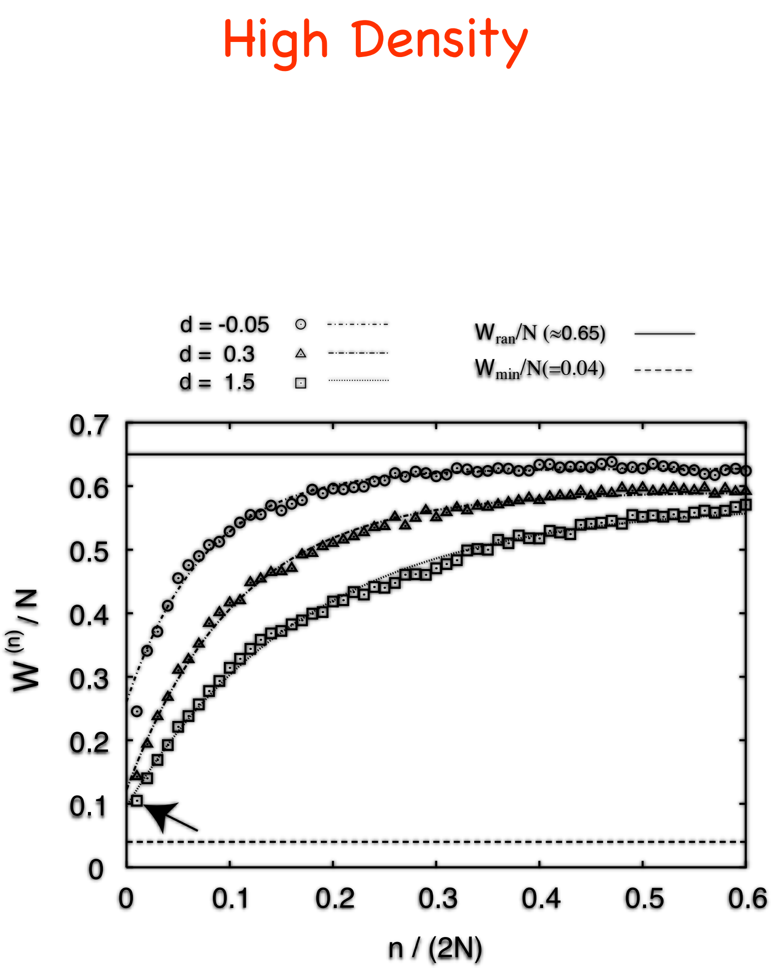
0 < continuous < 325

350 < modes < 400

steps finish at 350

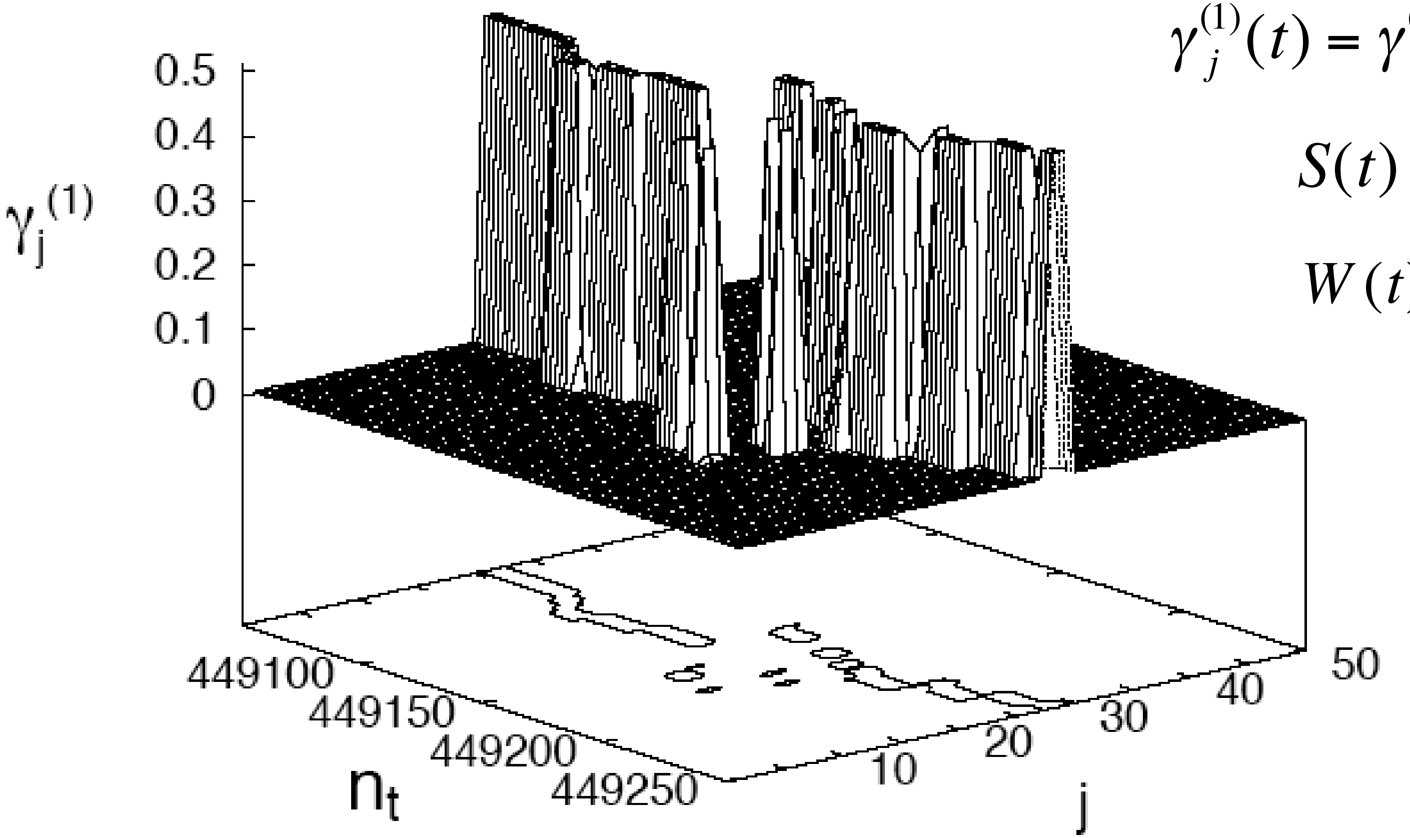
Localization spectra





Localization for the first Lyapunov vector

Quasi-one-dimensional System

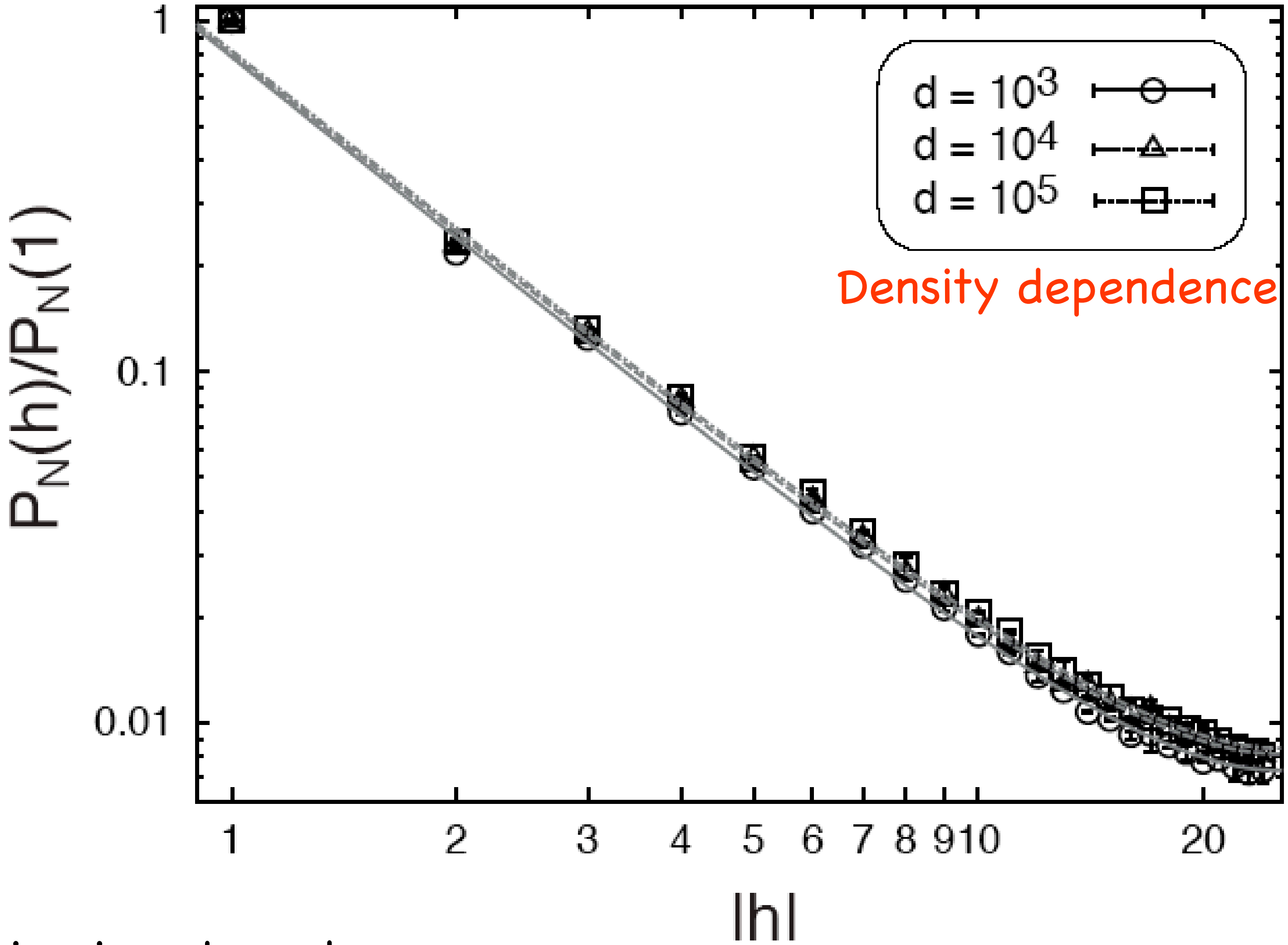


$$\gamma_j^{(1)}(t) = \gamma_{j+1}^{(1)}(t) = 0.5$$

$$S(t) = \ln 2$$

$$W(t) = \frac{2}{N}$$

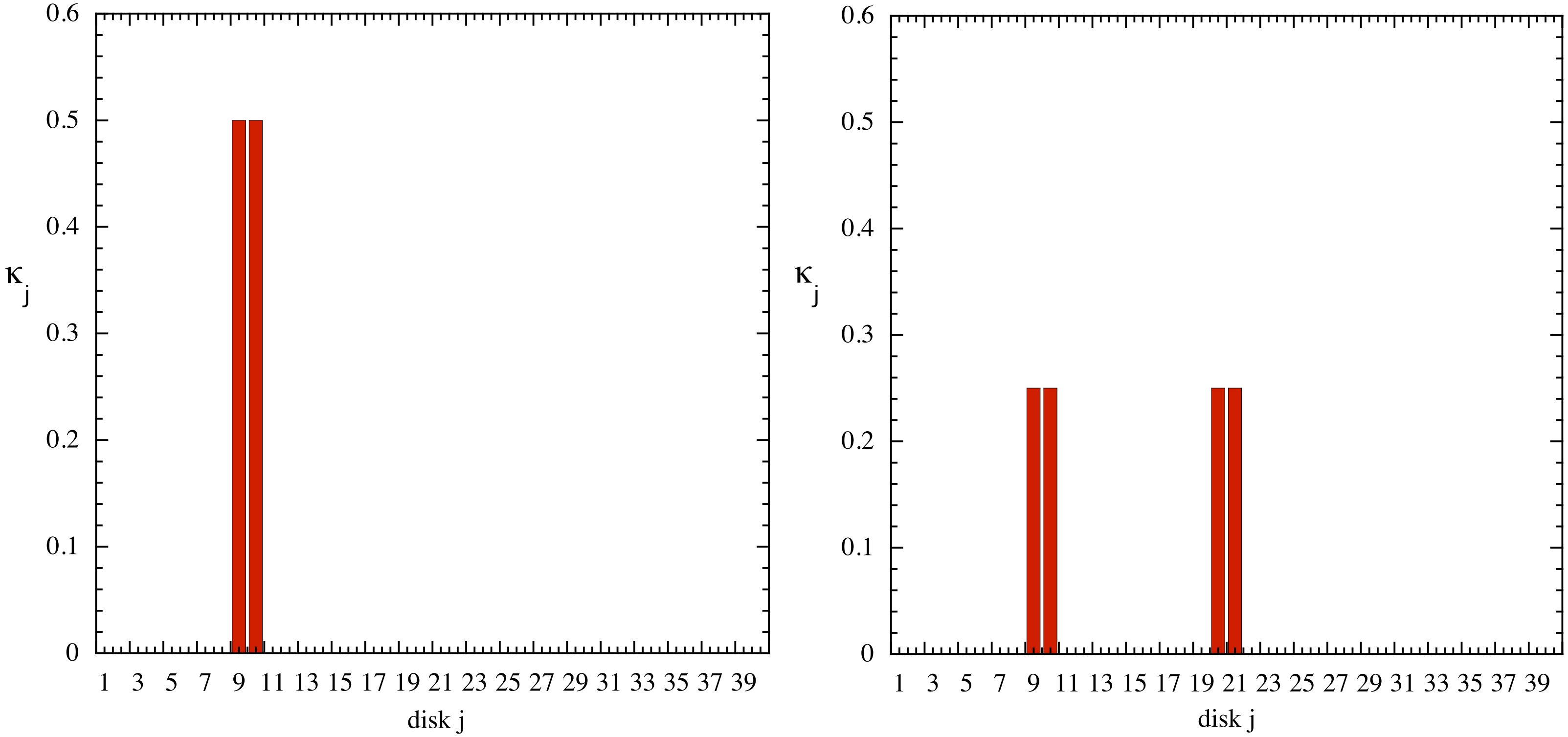
Normalized Hopping Rate



Hopping is not random

Strong Localization

Largest vector at low density



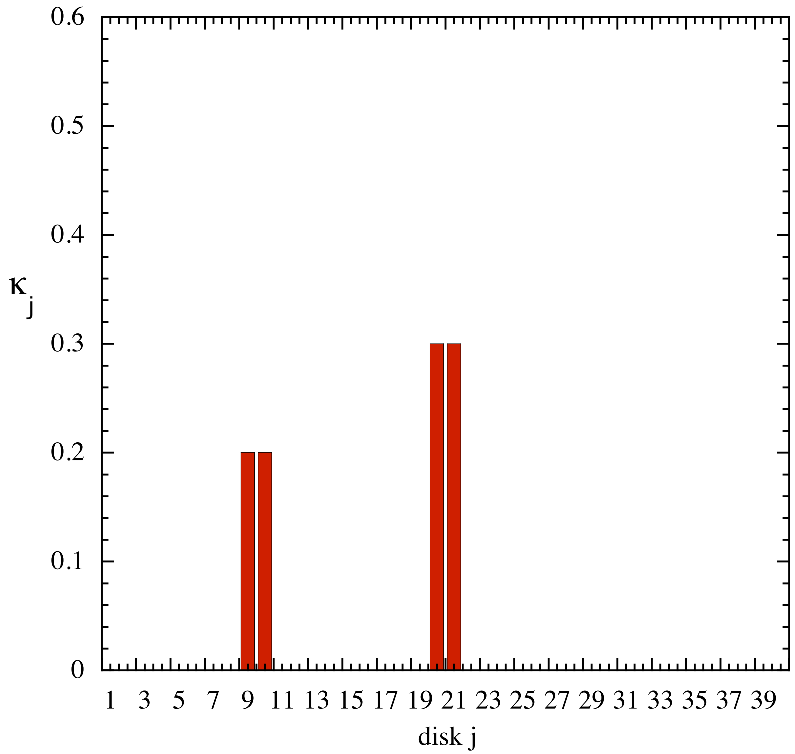
$$W = \frac{2}{N}$$

$$W = \frac{4}{N}$$

As density->0, 1st 10 vectors

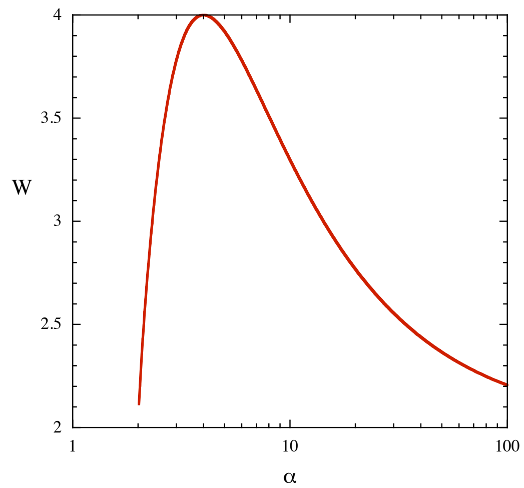
Covariant are linear combinations of GS

Variation on Strong localization



$$\chi_i = \frac{1}{\alpha} \qquad \chi_j = \frac{1}{\beta}$$

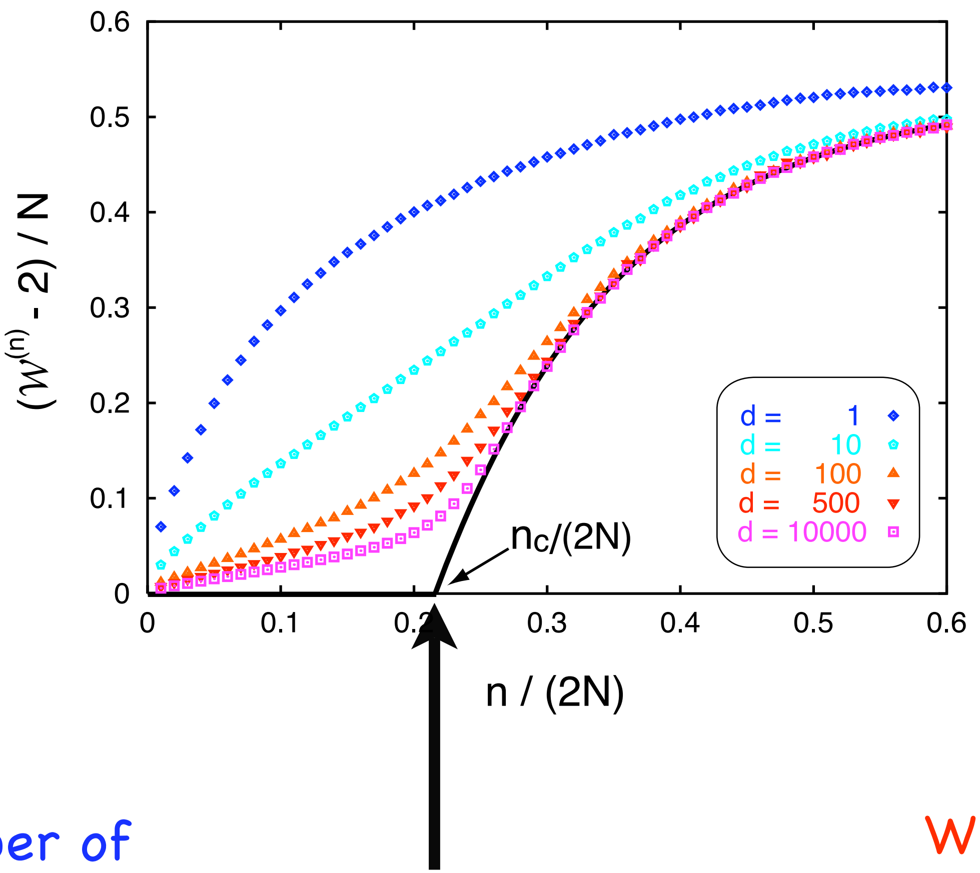
$$W = \frac{1}{N}\alpha(\alpha - 1)^{(1/\alpha - 1)}$$



Needs some distribution for α

$$2 > W \geq 4$$

Localization in Low density Limit



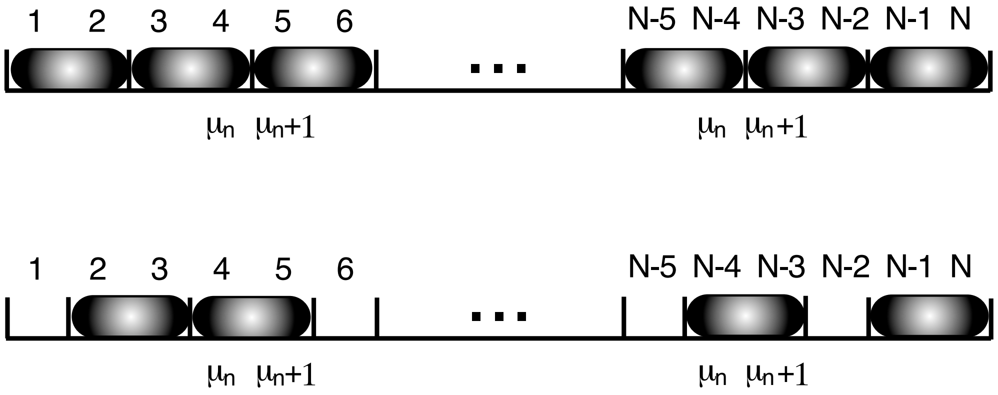
A fixed number of strongly localized vectors

0.219±0.005

Why does this happen?

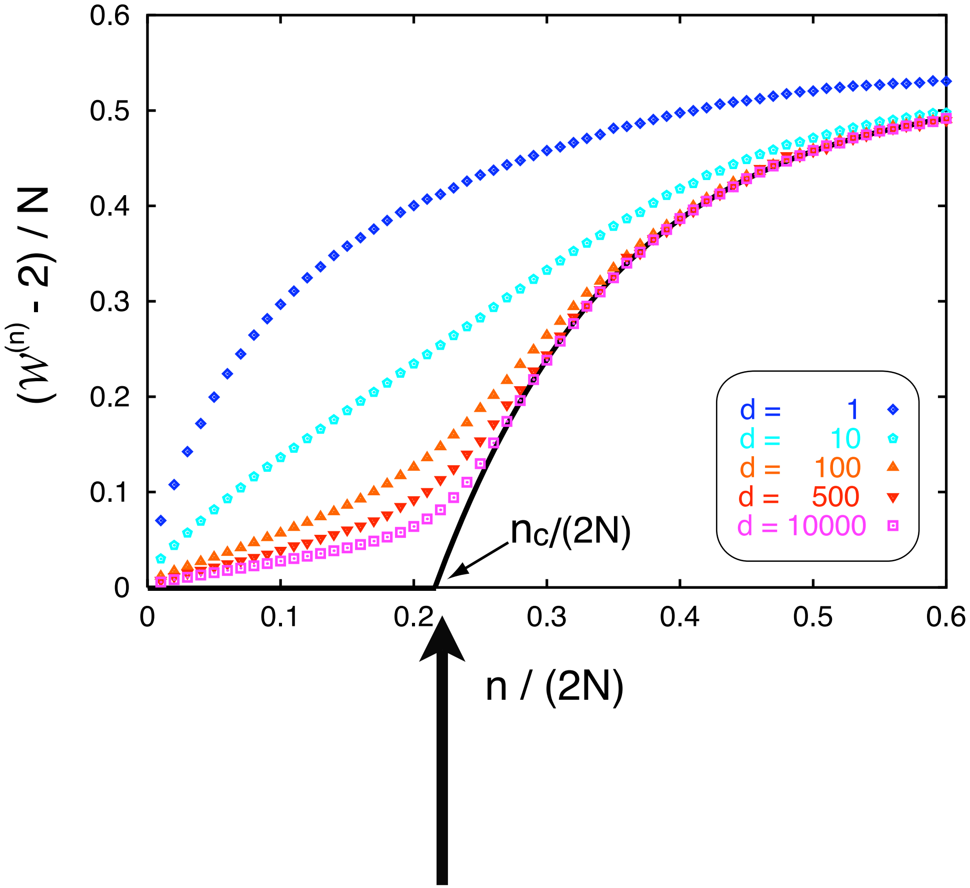
Randomly Distributed Brick Model

Conjecture: The number of most localized Lyapunov vectors at low density is equal to the number of exponents in the linear region



The number of exponents in the linear region is the average number of randomly dropped bricks in one layer

Localization in Low density Limit



Randomly distributed brick model gives 0.216

Simulation gives (0.219±0.005)

Localization Distributions

20 QOD hard disks

Zero modes 39, 40

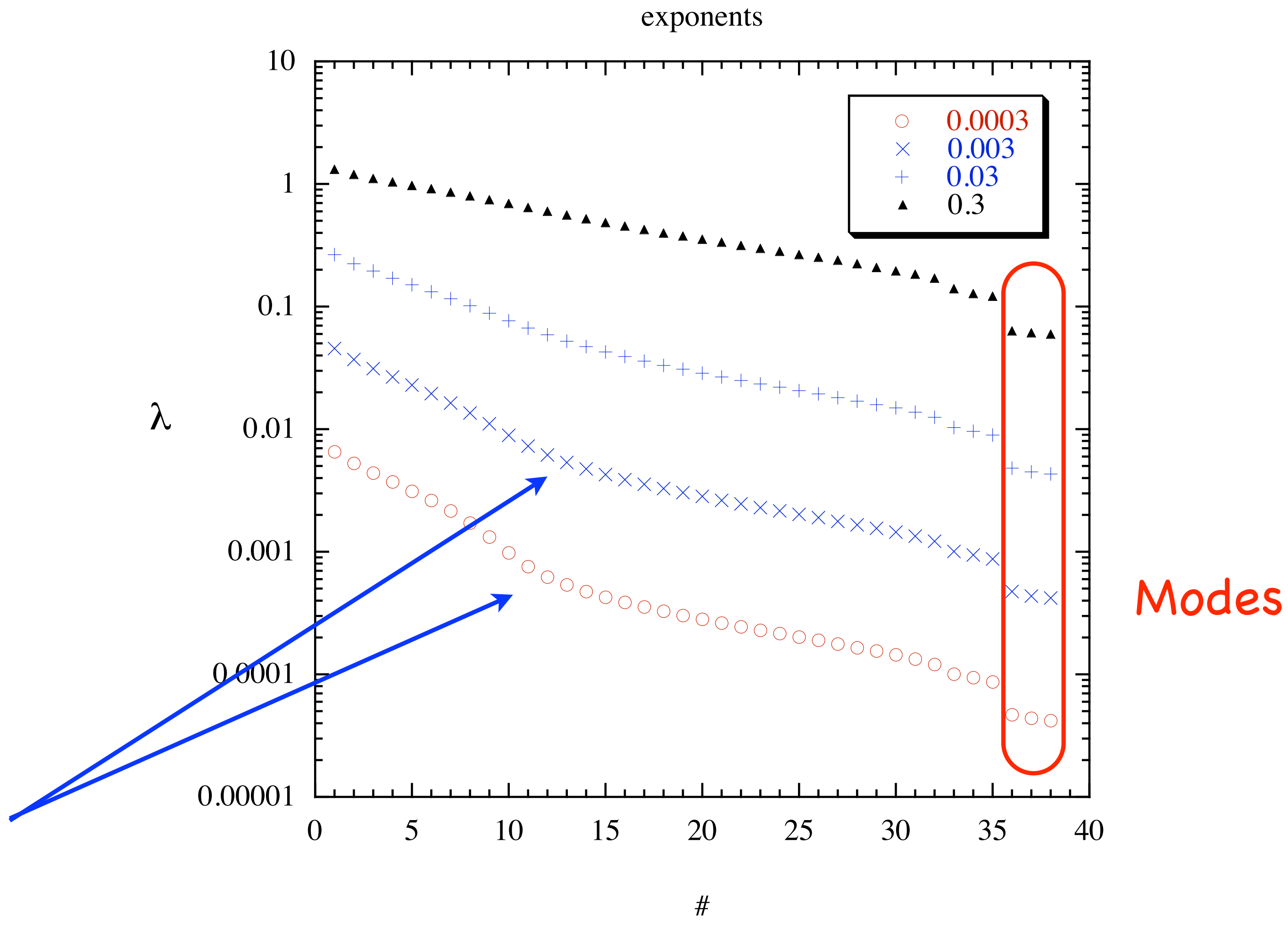
LP modes 37, 38

T modes 36

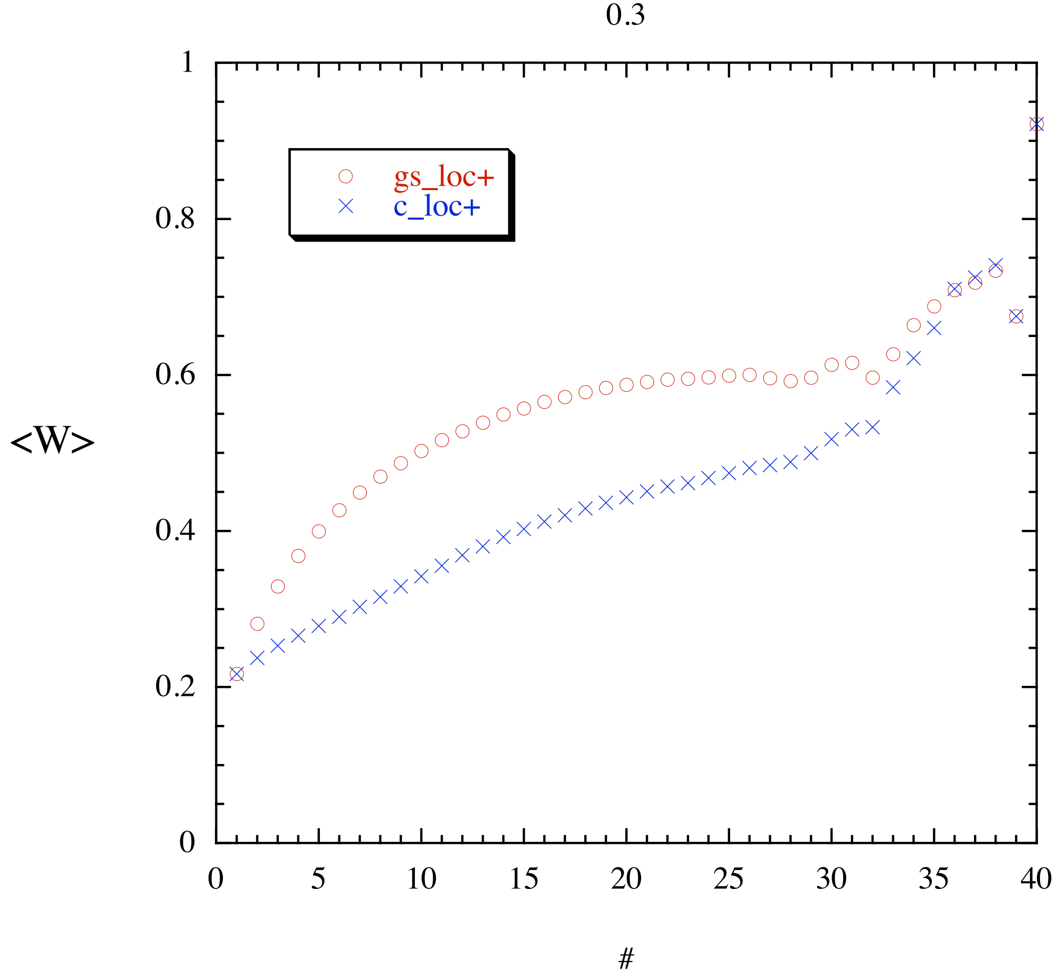
density = 0.0003 ↕ density = 0.3

Strong modes 80% ↕ Weak modes 40%

Lyapunov Exponents N=20



Average localization N=20



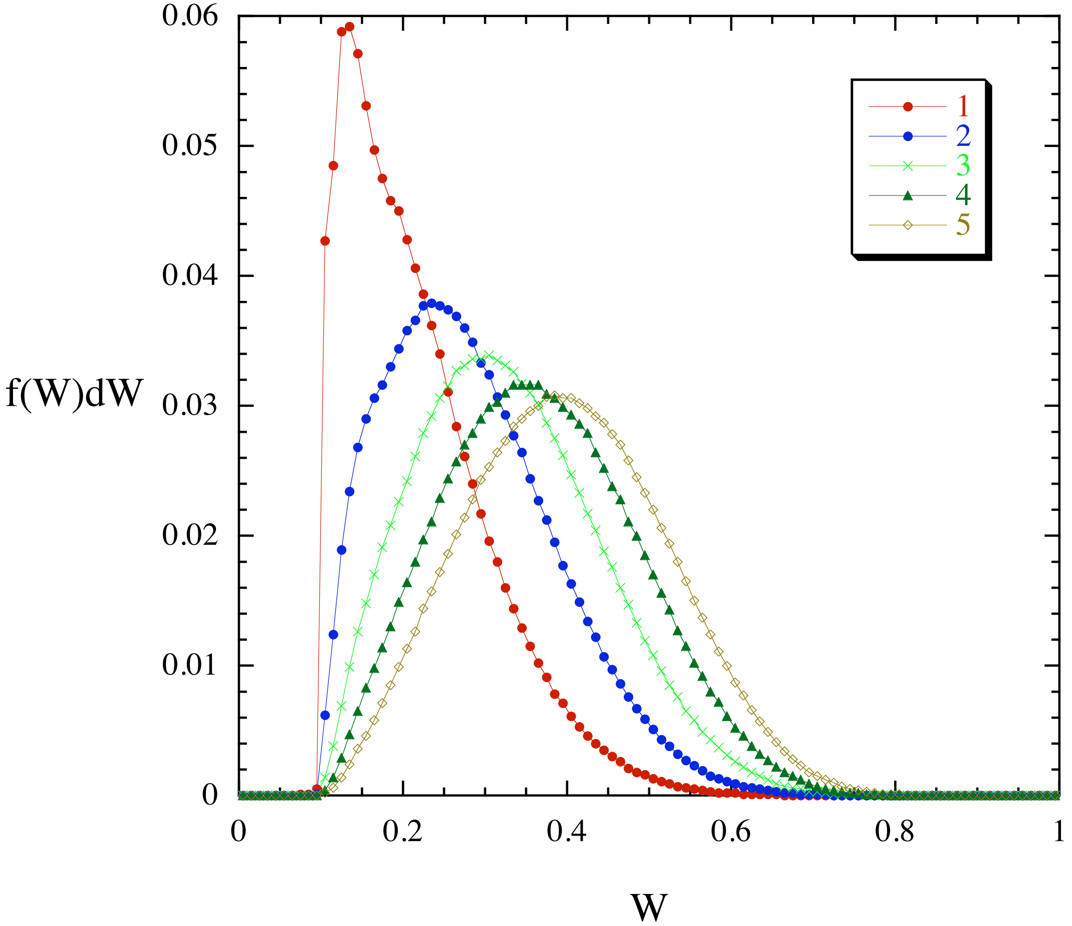
density = 0.3

Localization Distributions

Gram-Schmidt vectors

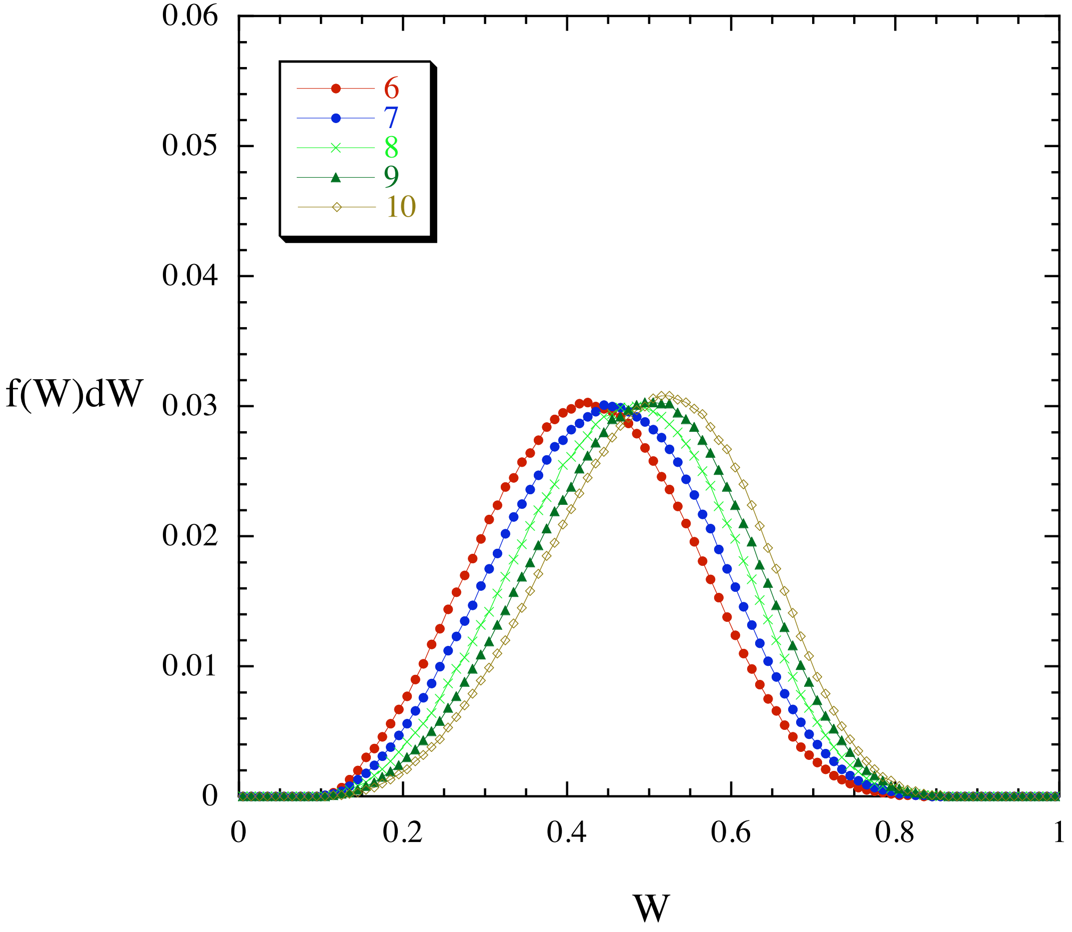
Density = 0.3

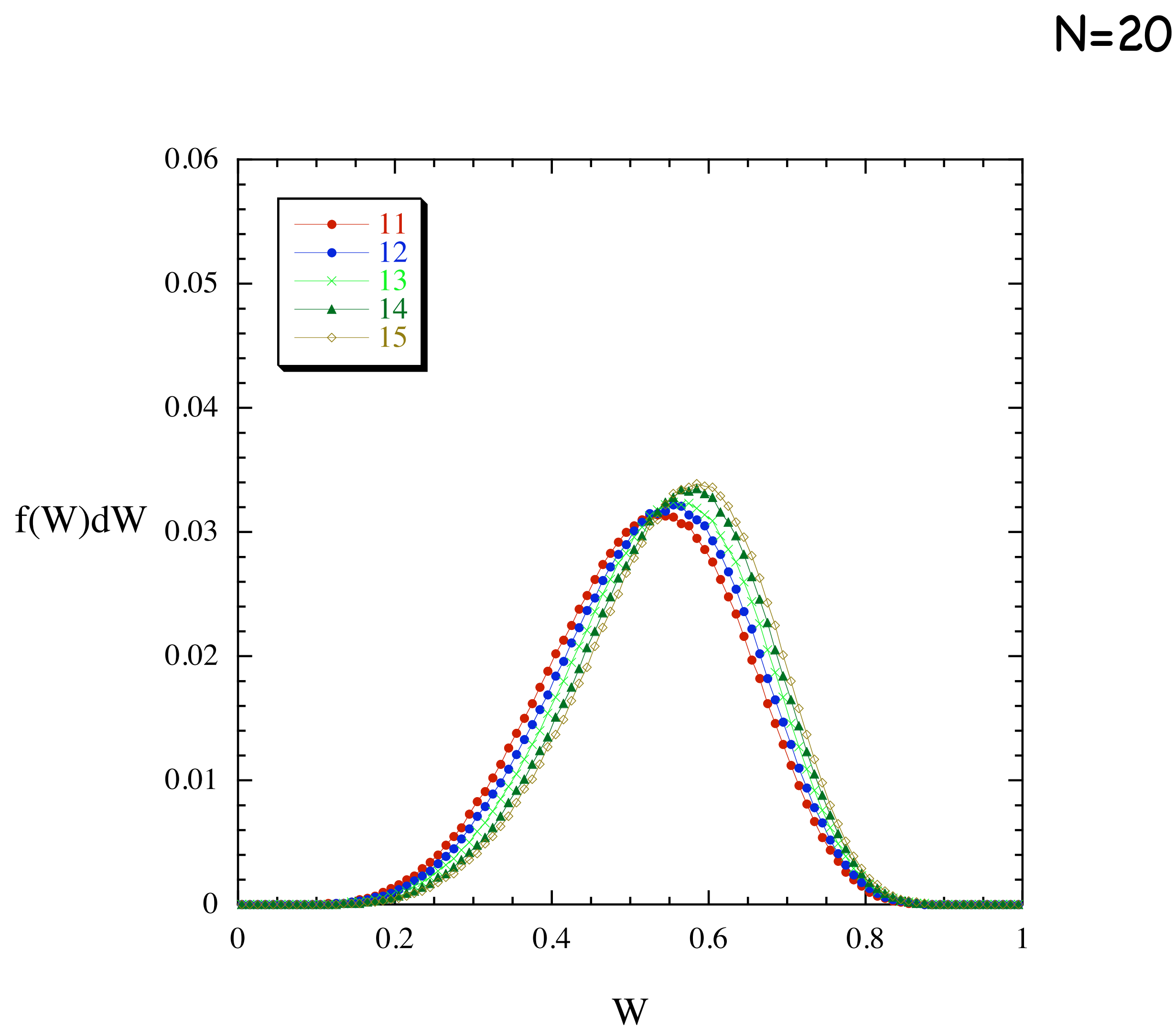
Localization Distributions N=20



Gram-Schmidt vectors Density = 0.3

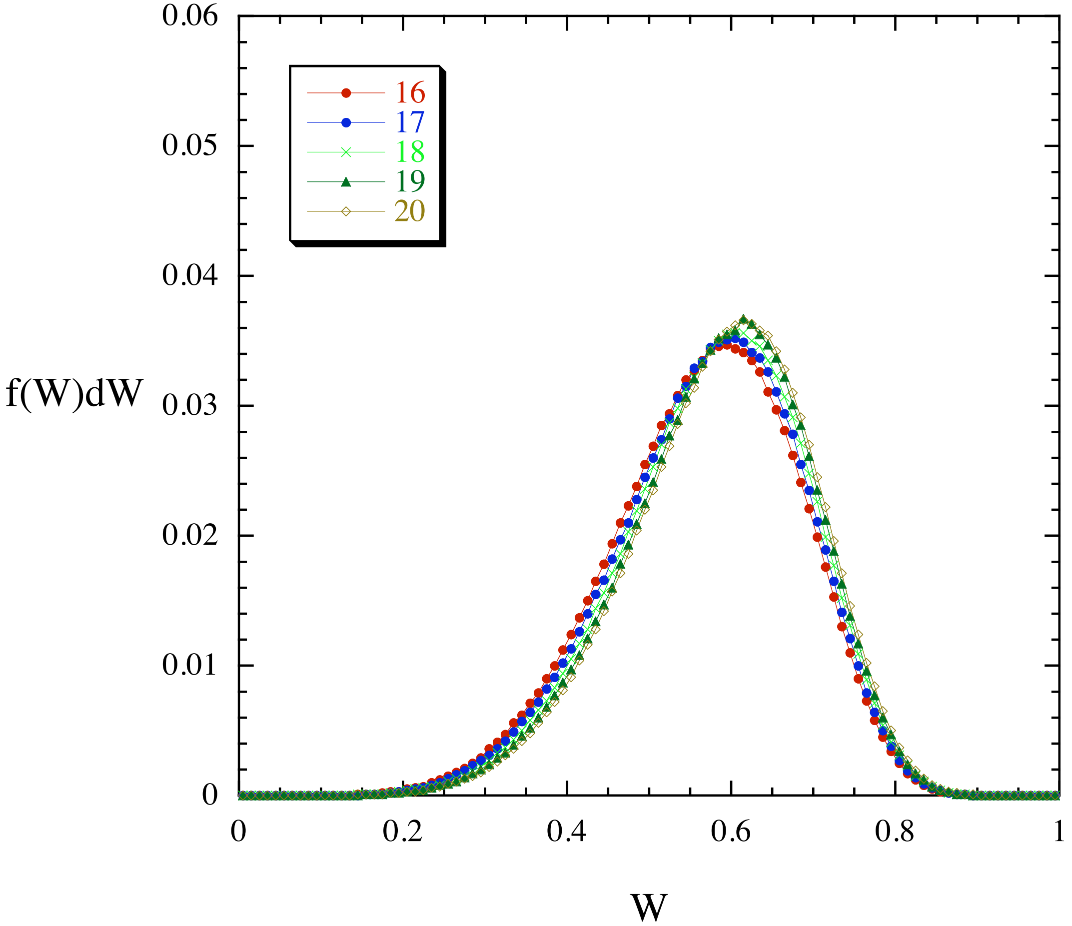
N=20



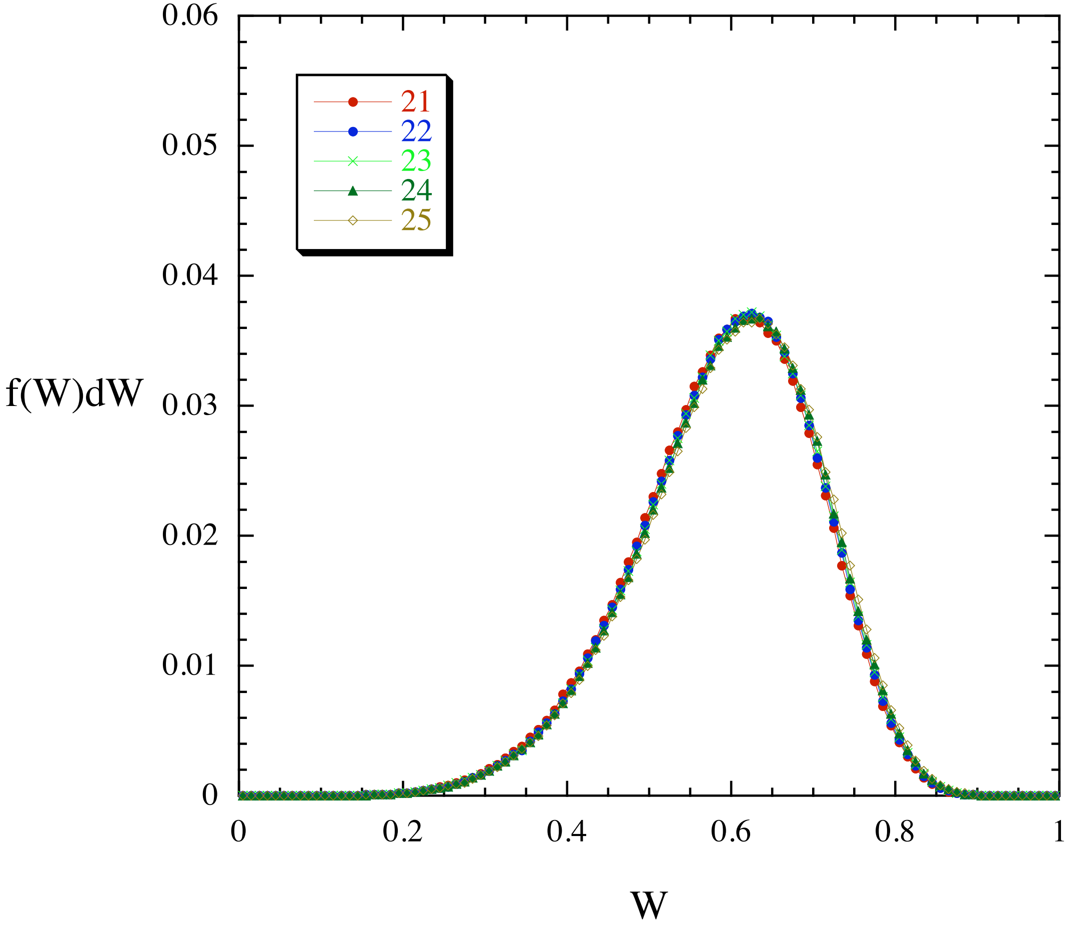


N=20
11
12
13
14
15
f(W)dW
0.06
0.05
0.04
0.03
0.02
0.01
0
0.2
0.4
0.6
0.8
1
W

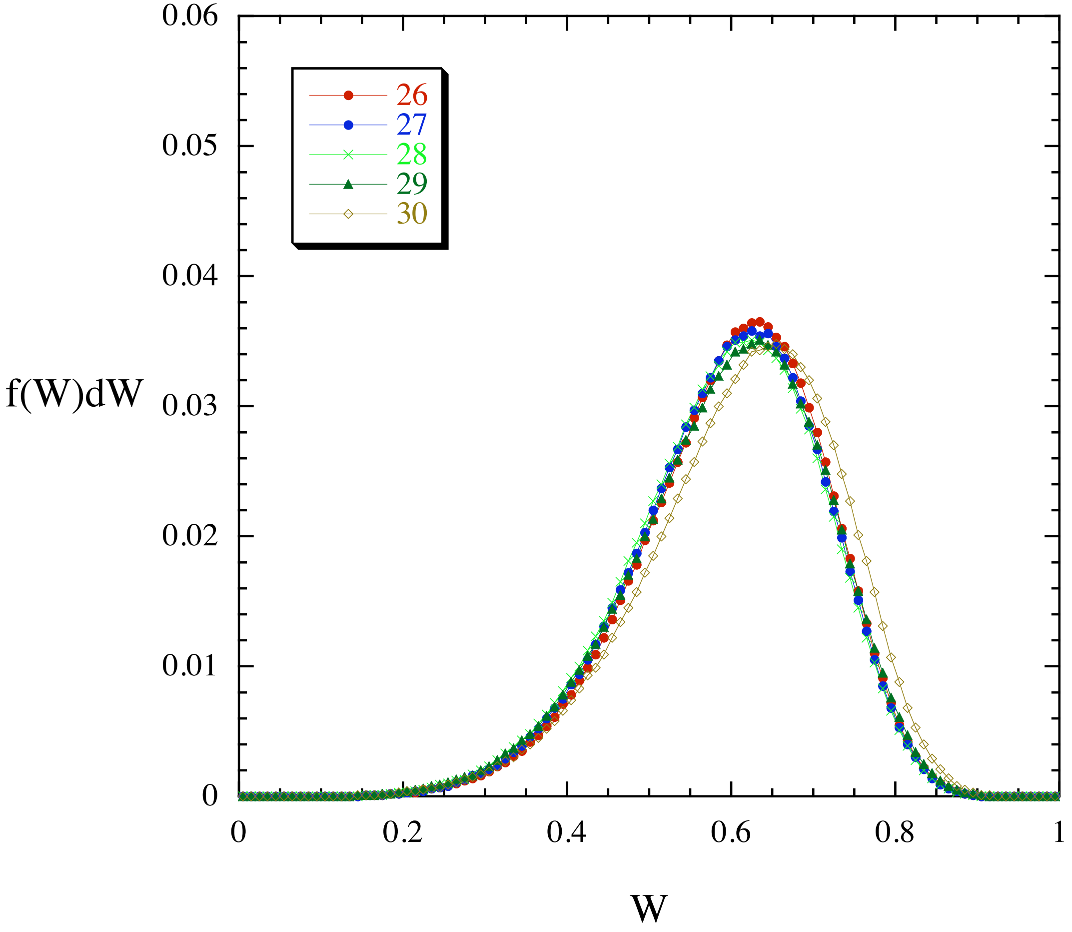
N=20

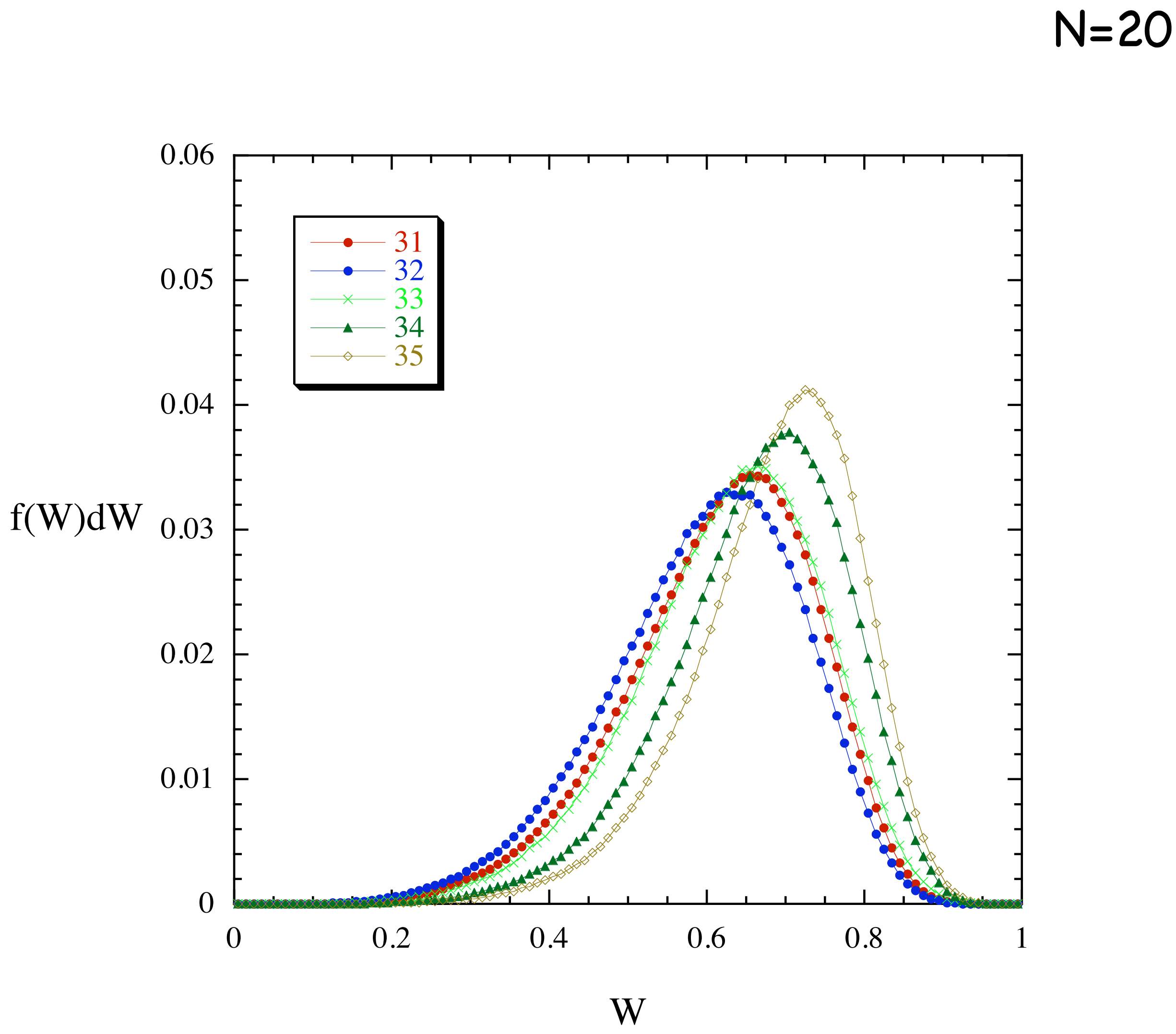


N=20

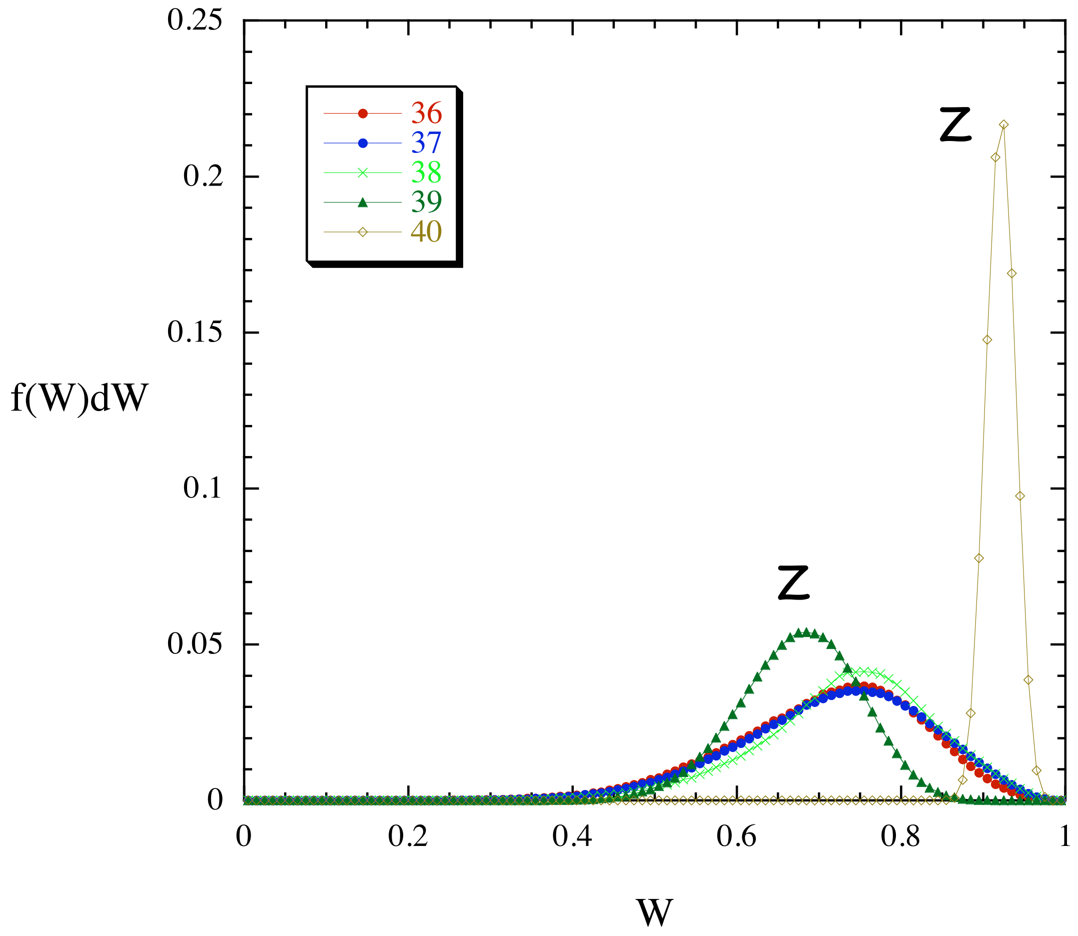


N=20




N=20
f(W)dW
W
0
0.01
0.02
0.03
0.04
0.05
0.06
0.2
0.4
0.6
0.8
1
31
32
33
34
35

N=20



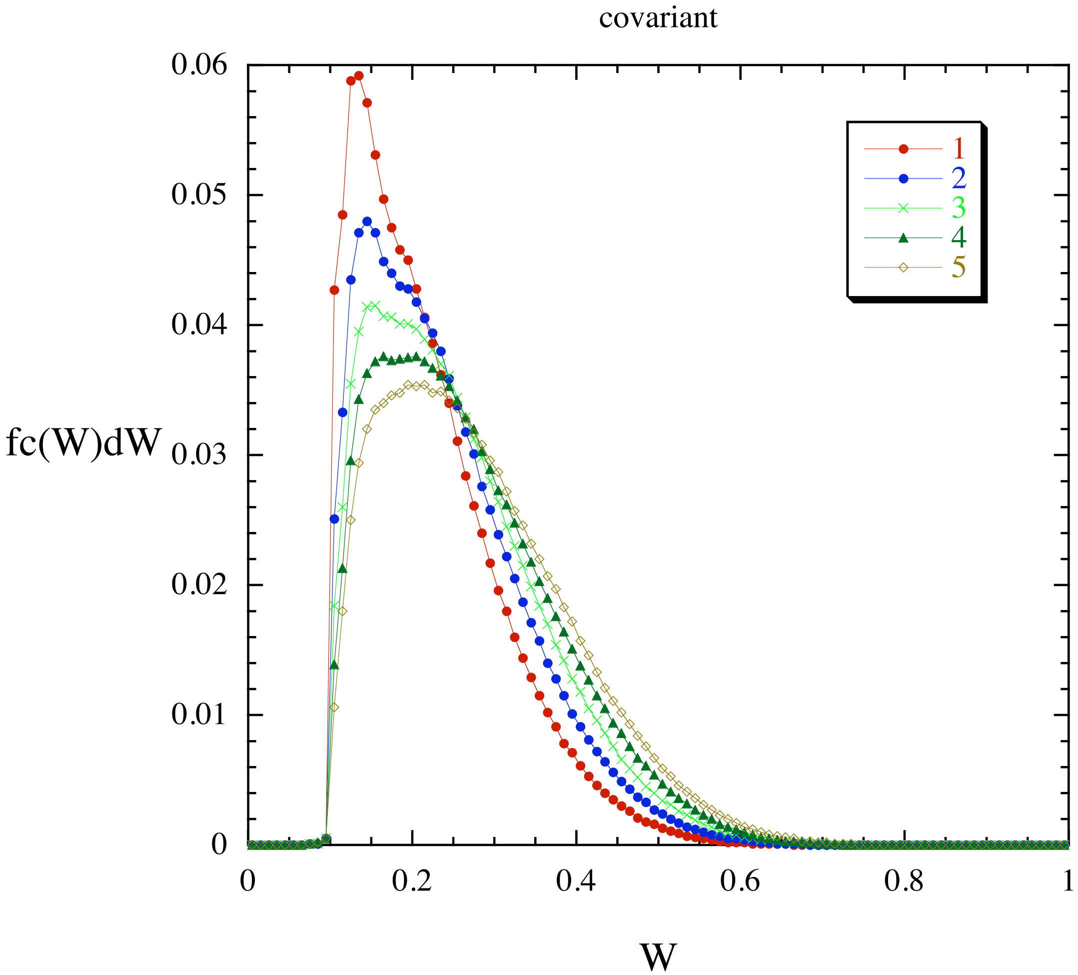
Zero modes 39, 40 LP modes 37, 38 T mode 36

Localization Distributions

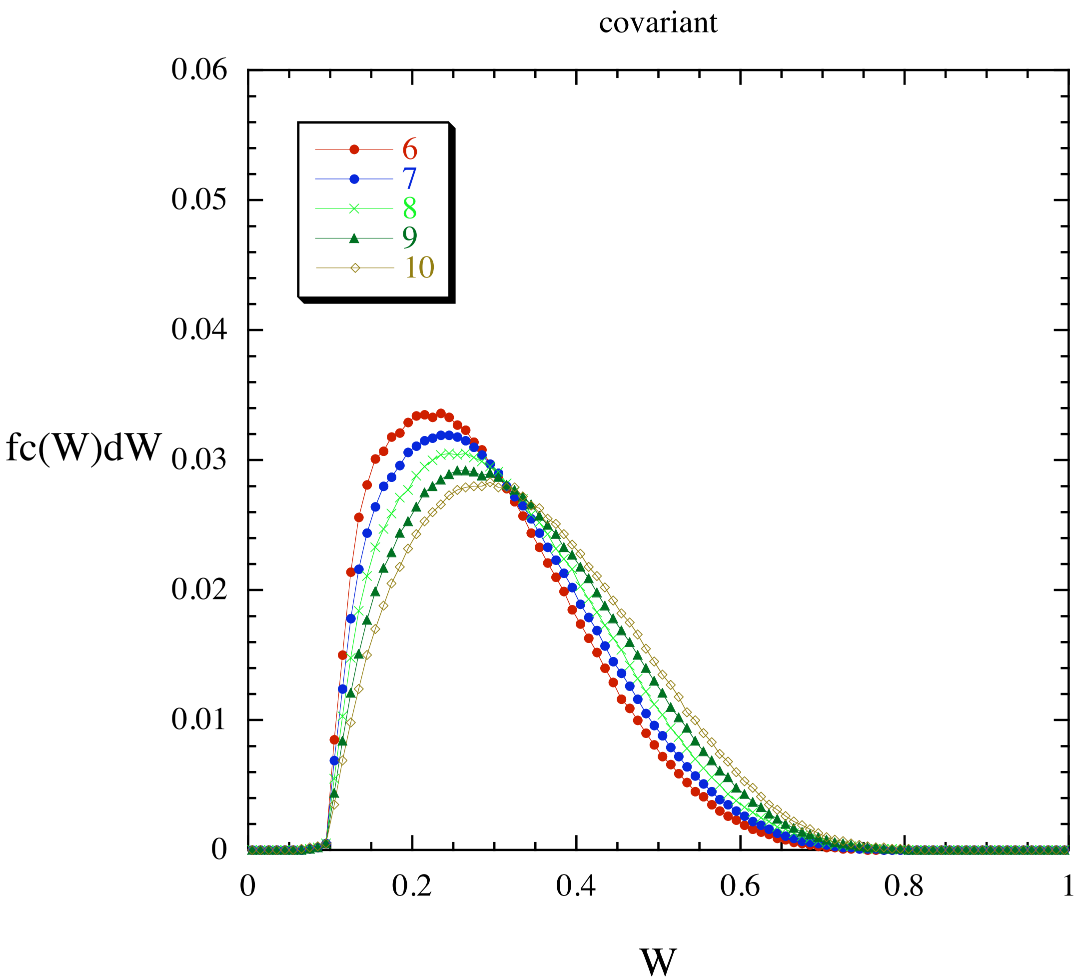
Covariant vectors

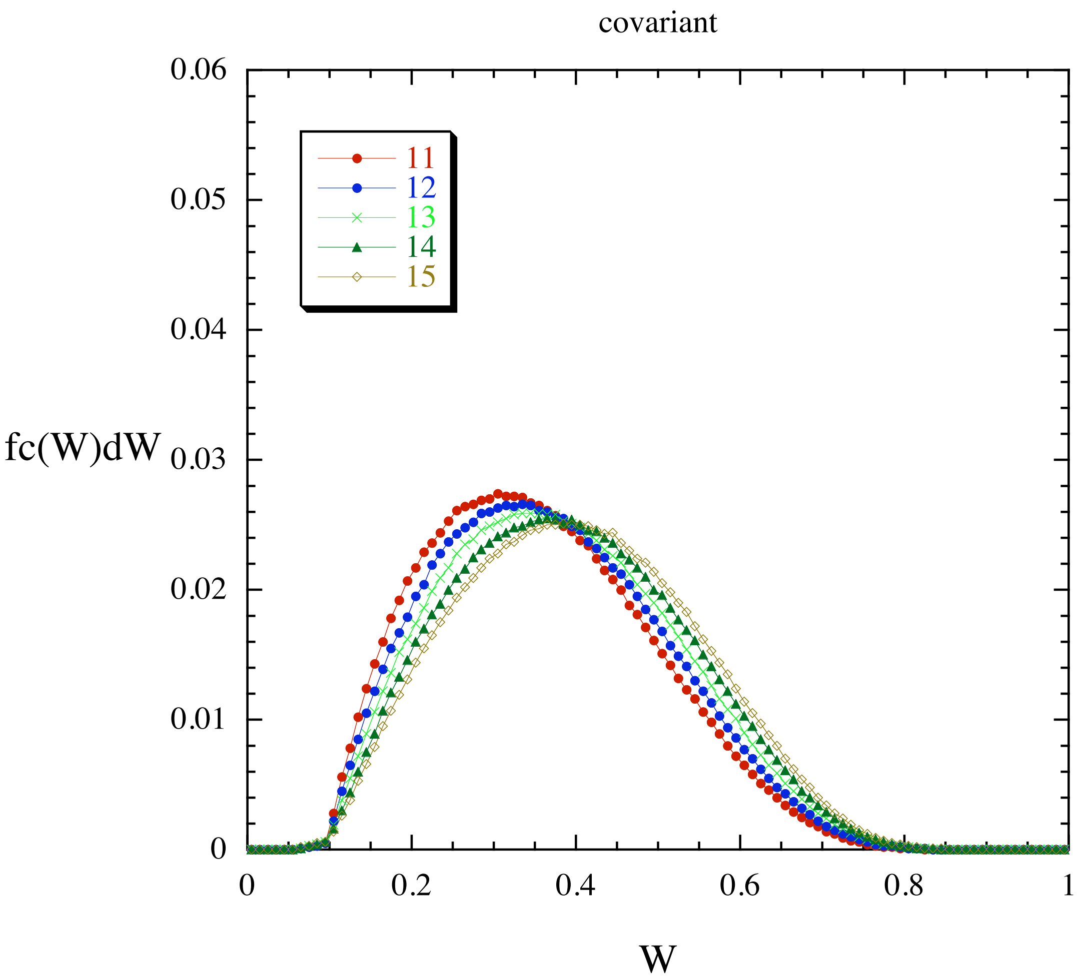
Density = 0.3

Localization Distributions N=20

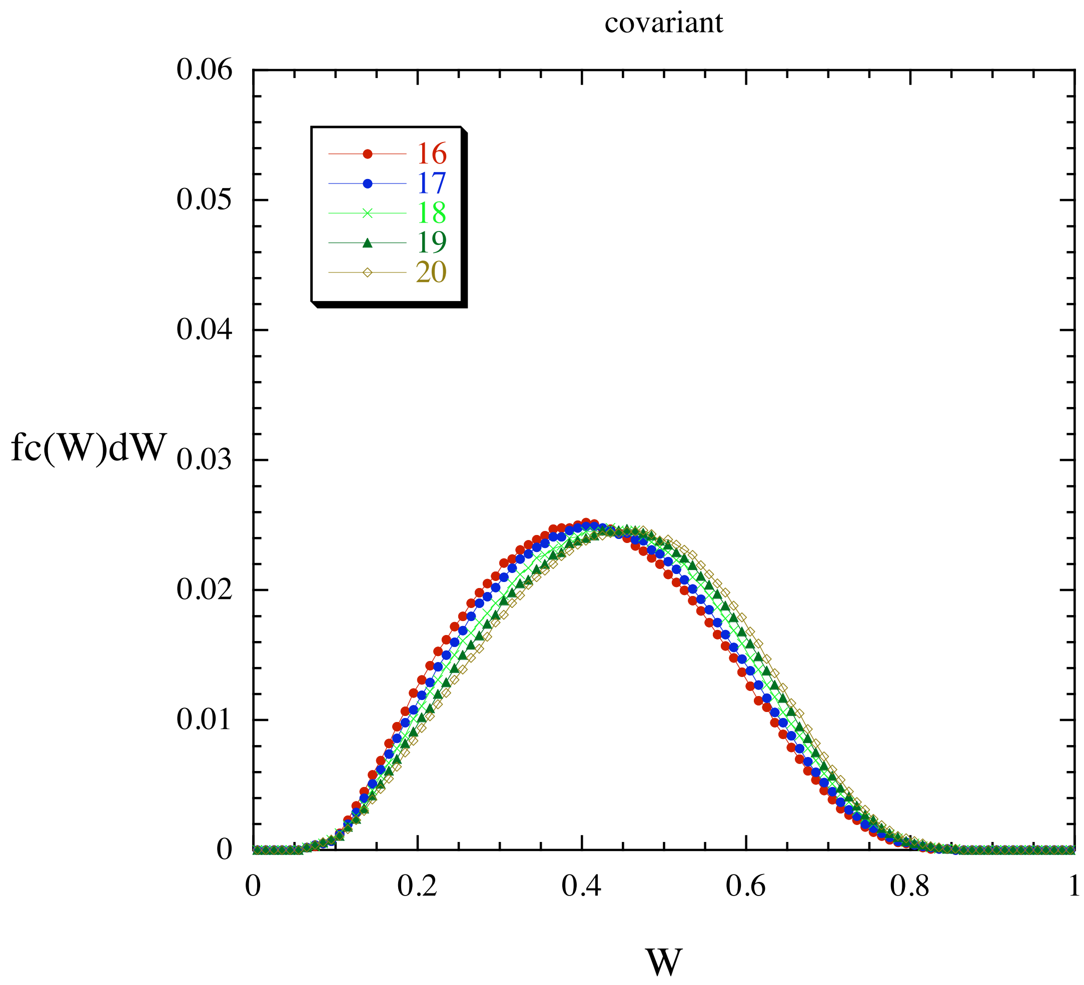


Covariant vectors Density = 0.3

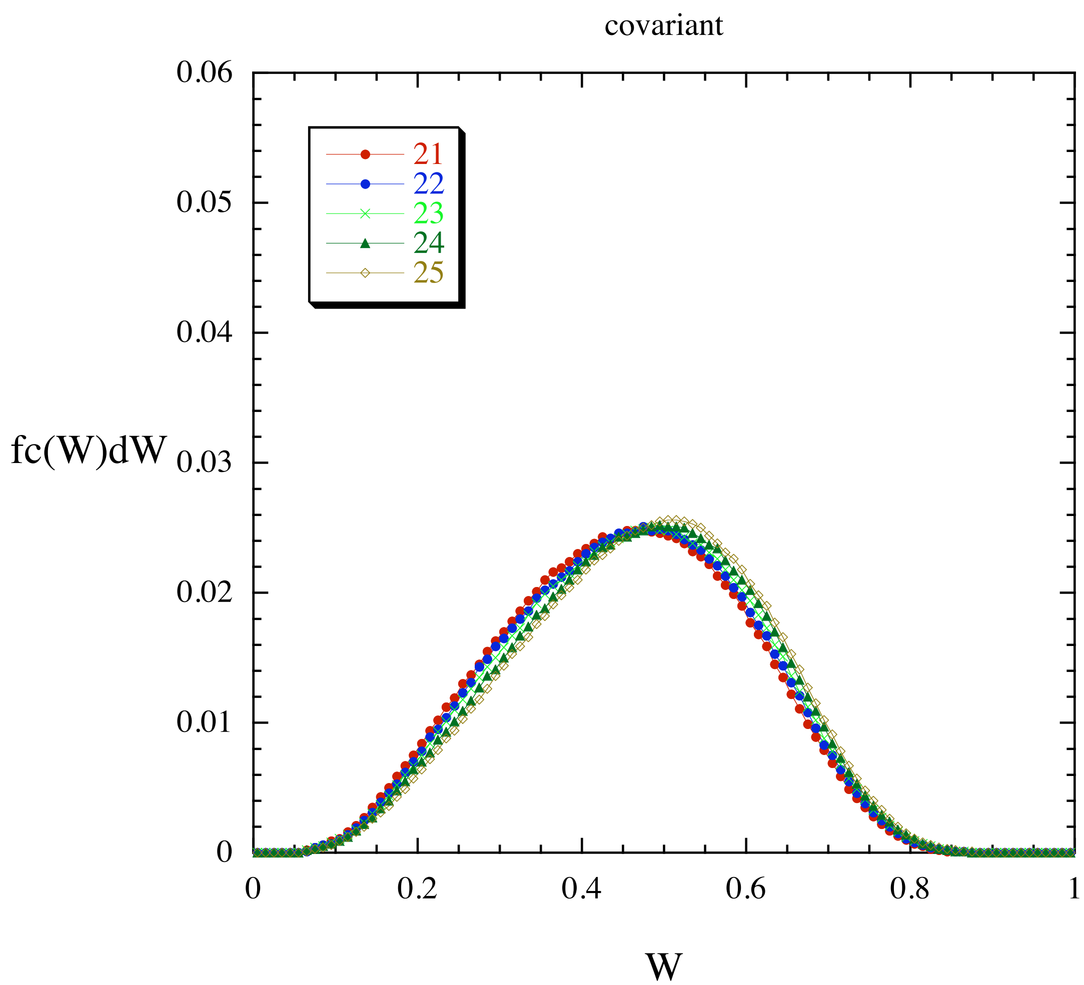

covariant
0.06
0.05
0.04
0.03
0.02
0.01
0
fc(W)dW
0
0.2
0.4
0.6
0.8
1
W
6
7
8
9
10

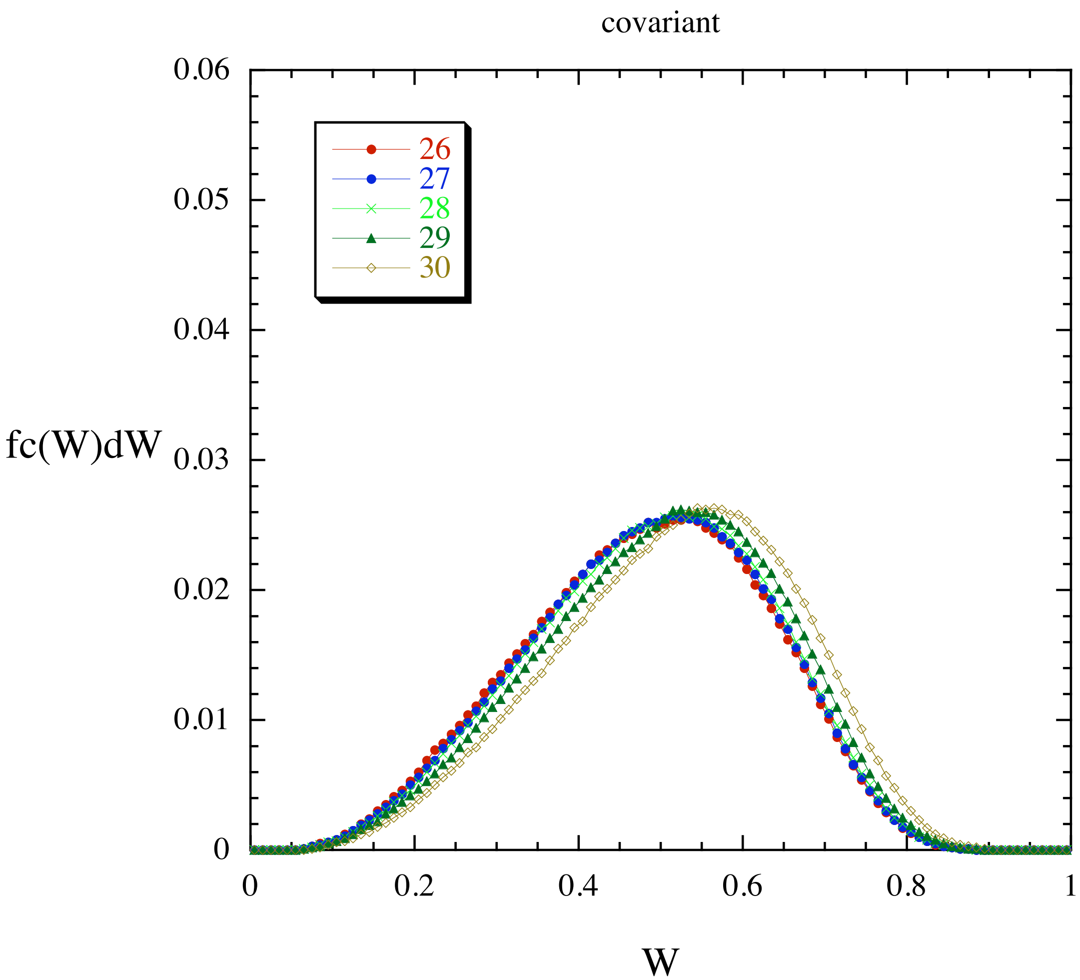


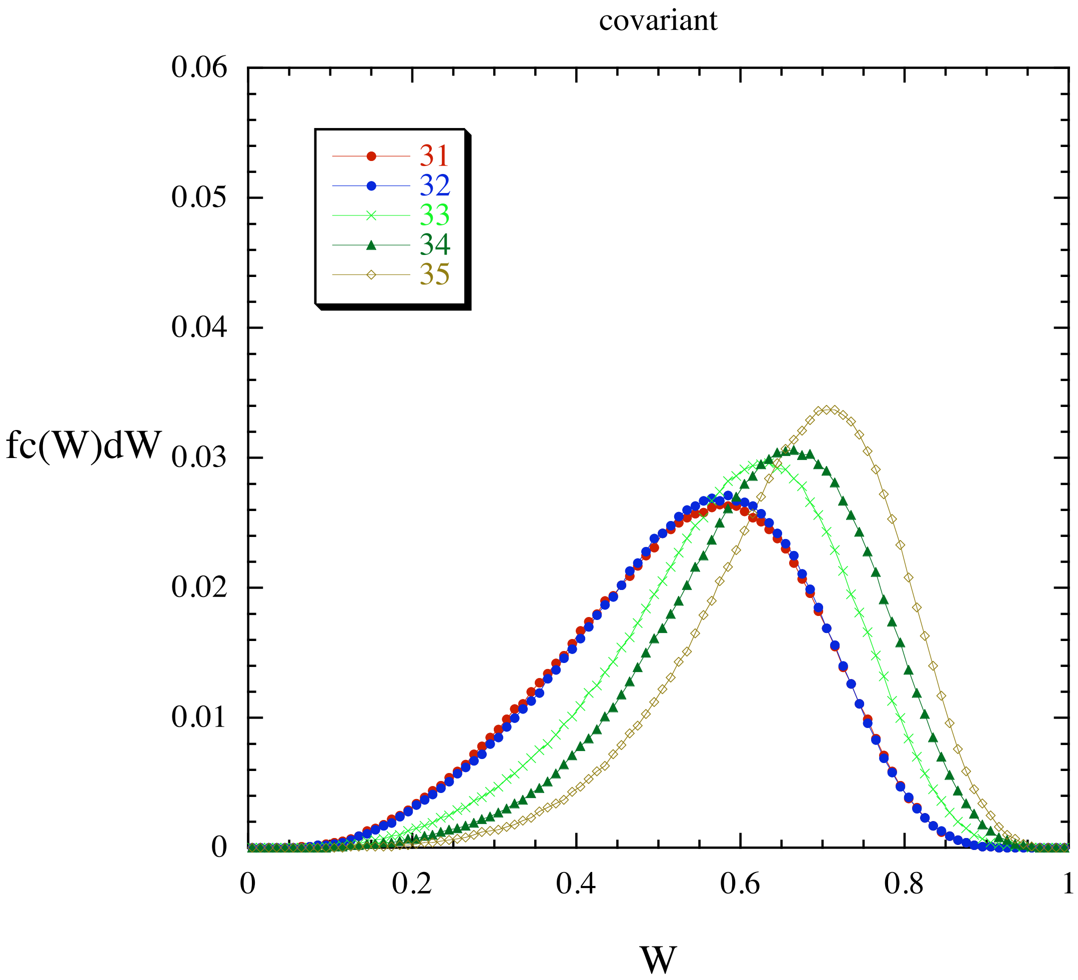
covariant
fc(W)dW
W
11
12
13
14
15
0
0.01
0.02
0.03
0.04
0.05
0.06
0
0.2
0.4
0.6
0.8
1



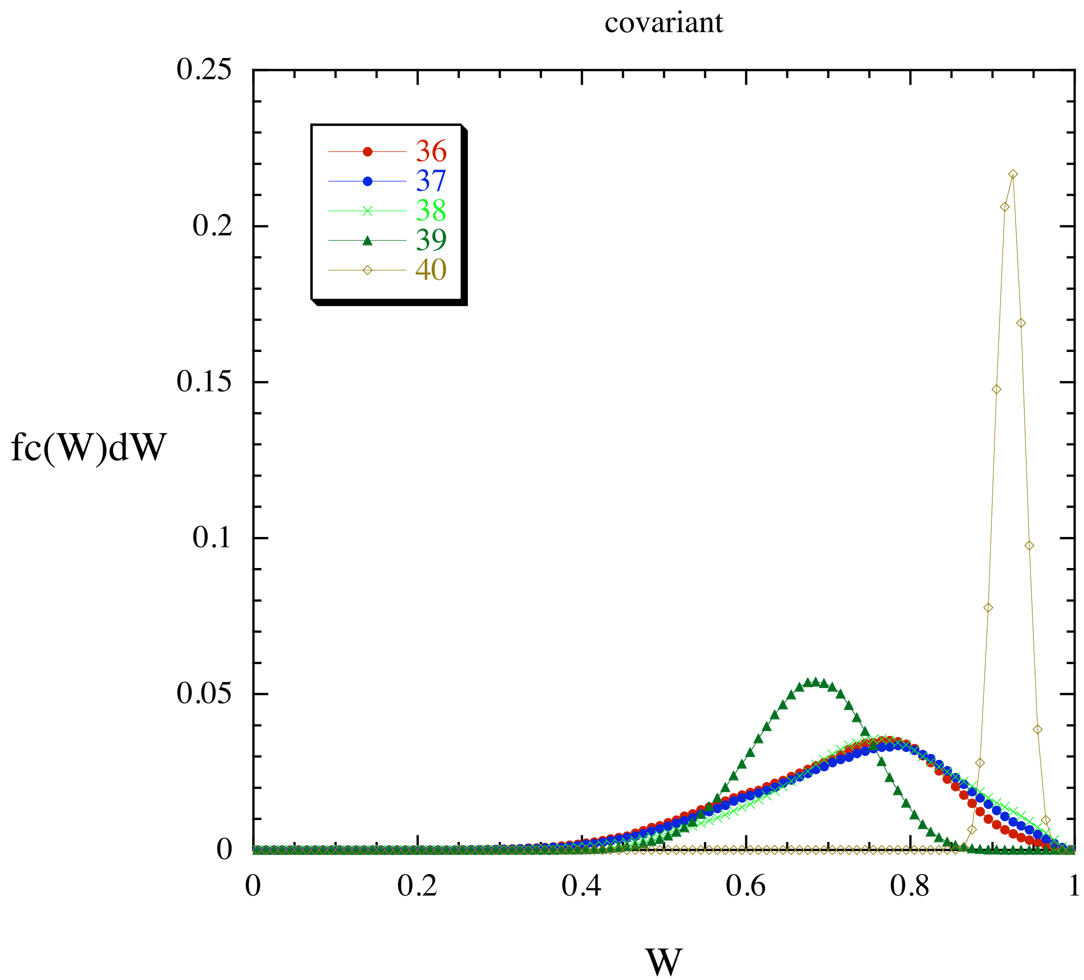
covariant
fc(W)dW
W
0.06
0.05
0.04
0.03
0.02
0.01
0
0.2
0.4
0.6
0.8
1
16
17
18
19
20


covariant
fc(W)dW
W
0.06
0.05
0.04
0.03
0.02
0.01
0
0
0.2
0.4
0.6
0.8
1
21
22
23
24
25

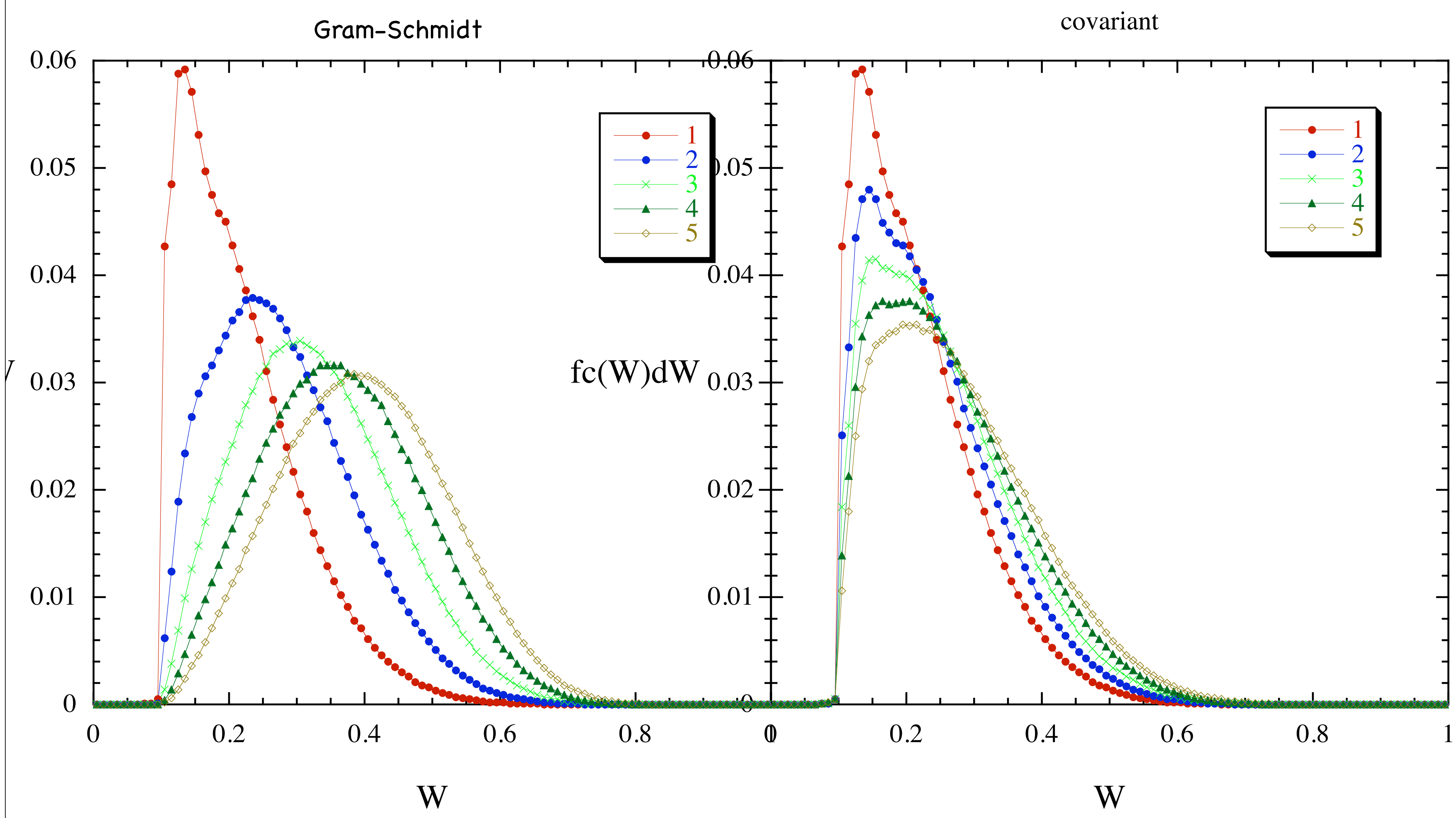

covariant
0.06
0.05
0.04
0.03
0.02
0.01
0
fc(W)dW
0
0.2
0.4
0.6
0.8
1
W
26
27
28
29
30



covariant
fc(W)dW
W
0.06
0.05
0.04
0.03
0.02
0.01
0
0.2
0.4
0.6
0.8
1
31
32
33
34
35

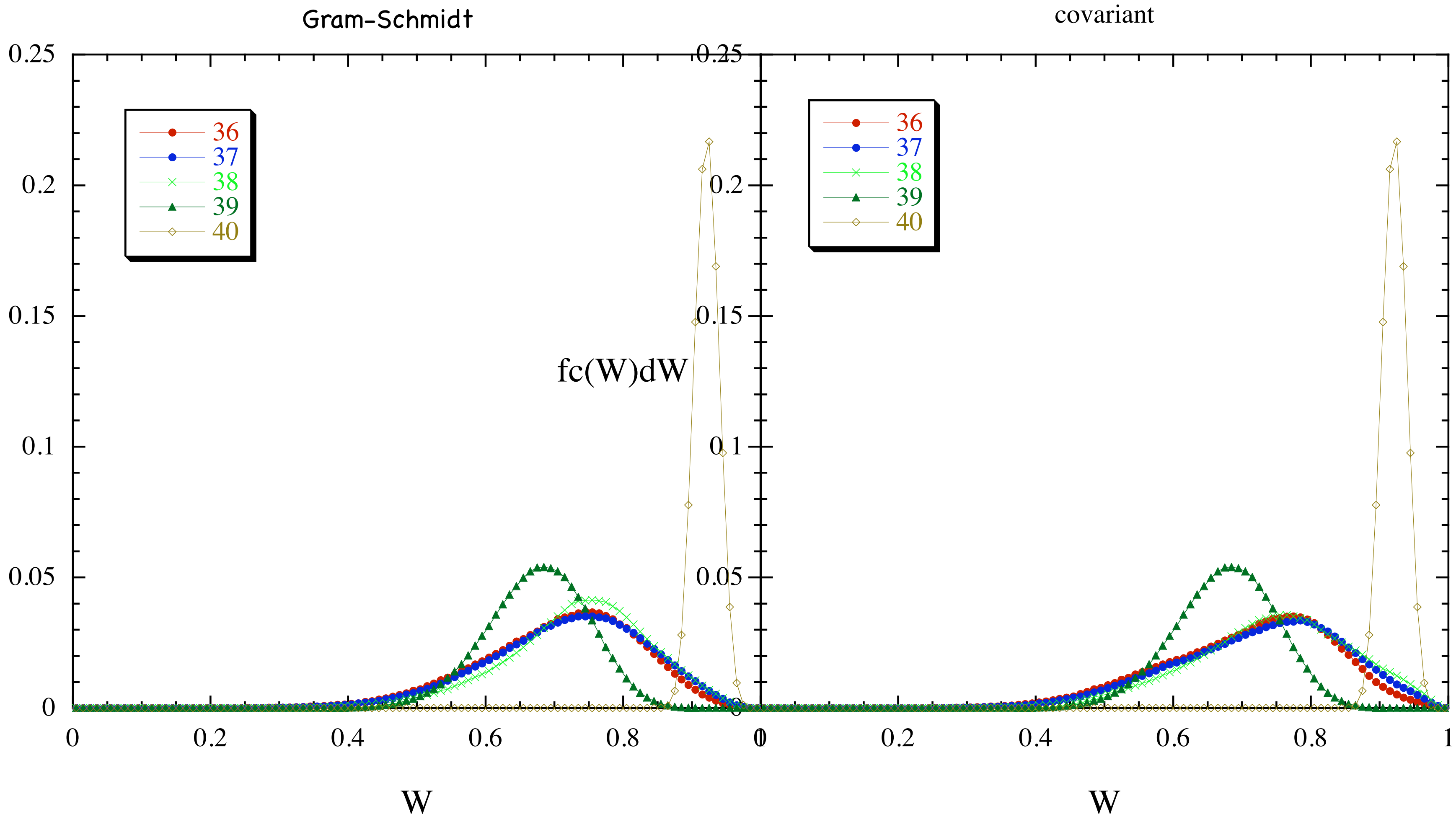

covariant
0.25
0.2
0.15
0.1
0.05
0
fc(W)dW
0
0.2
0.4
0.6
0.8
1
W
36
37
38
39
40

Mode Localization Distributions



Density = 0.3

Mode Localization Distributions



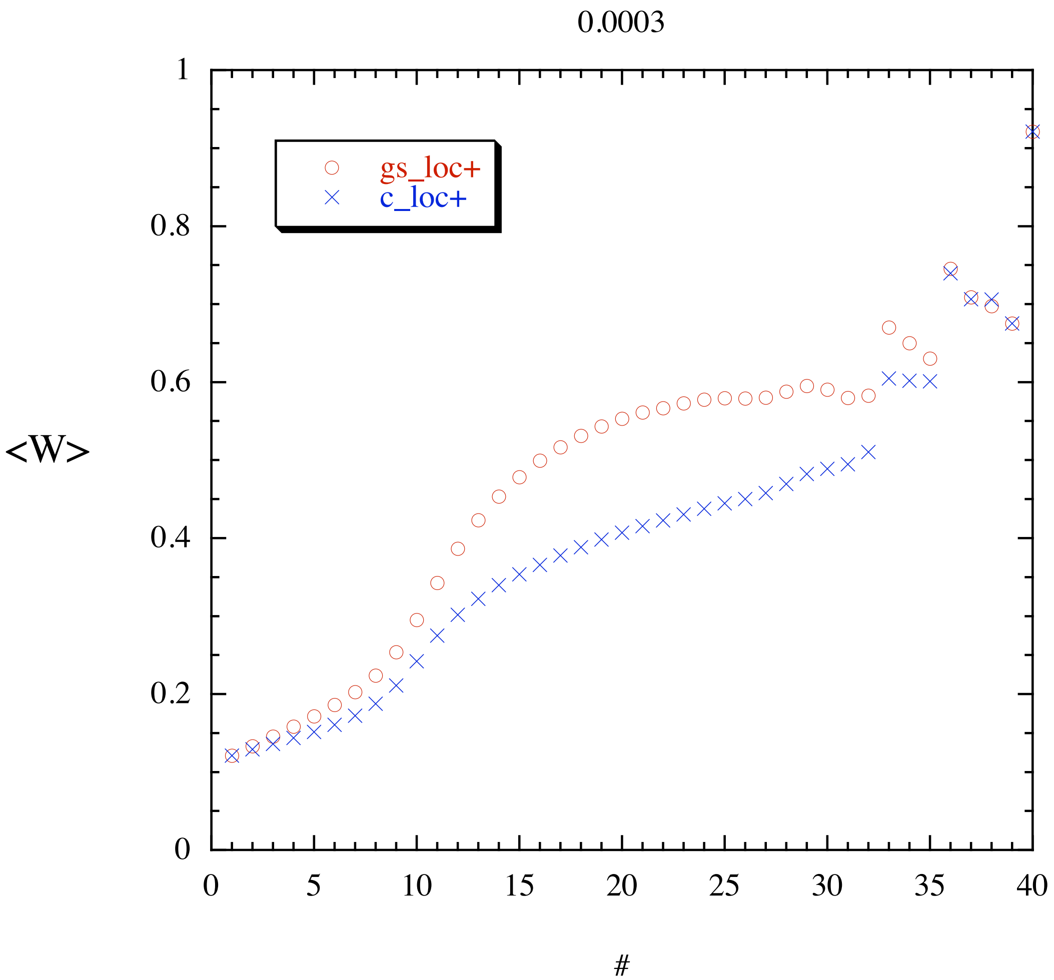
Density = 0.3

Localization Distributions

GS & Covariant vectors

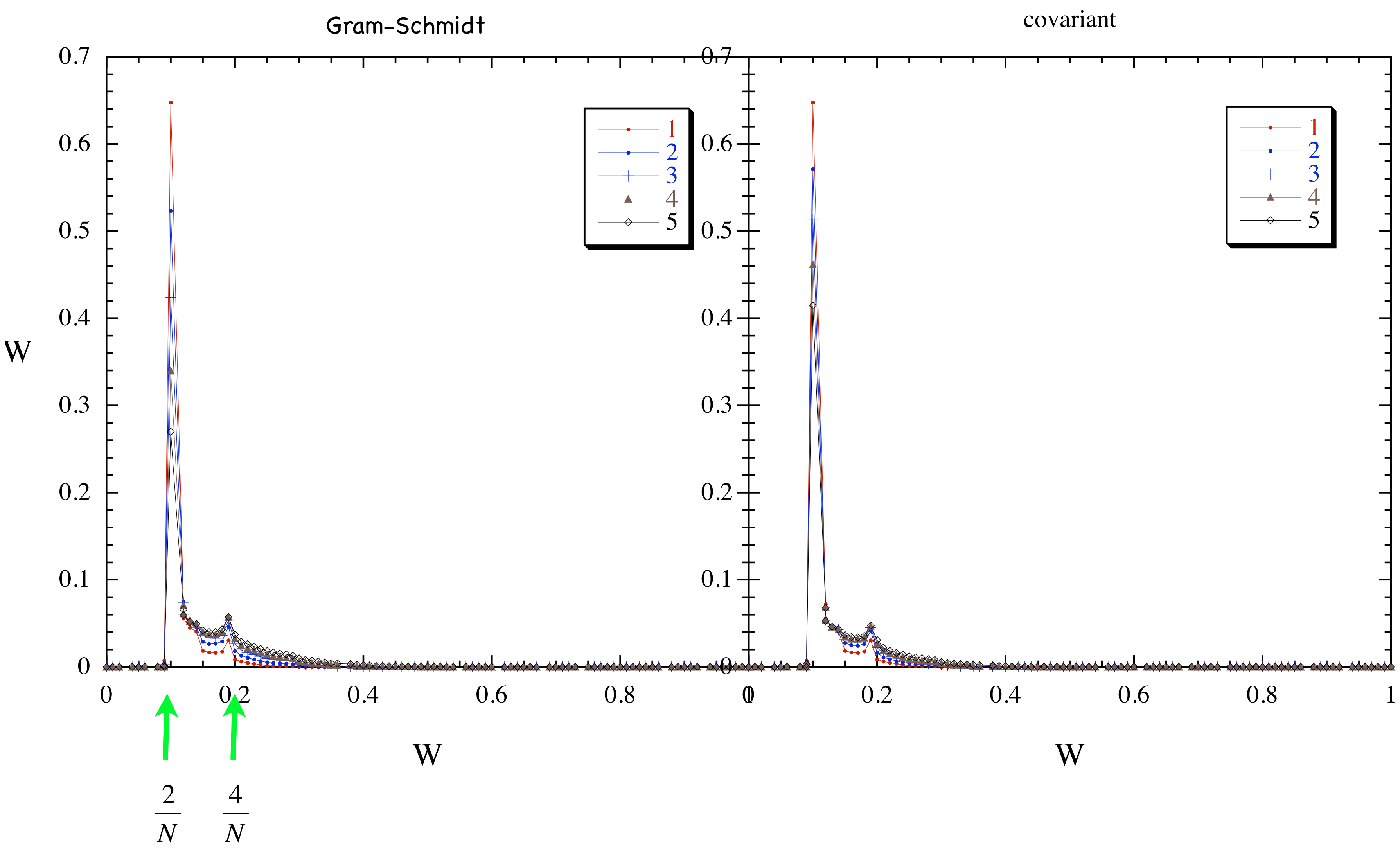
Density = 0.0003

Average Localization

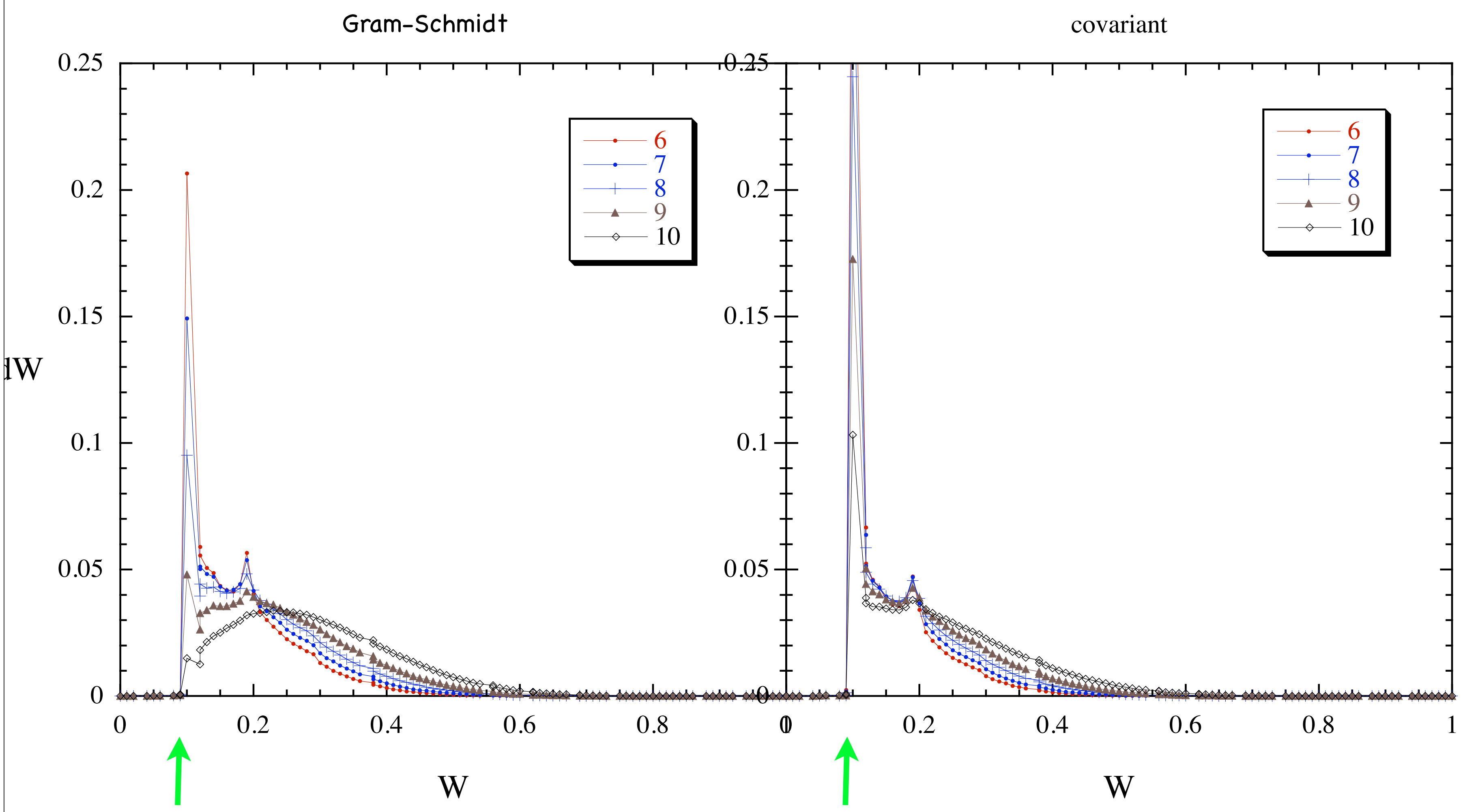


Density=0.0003

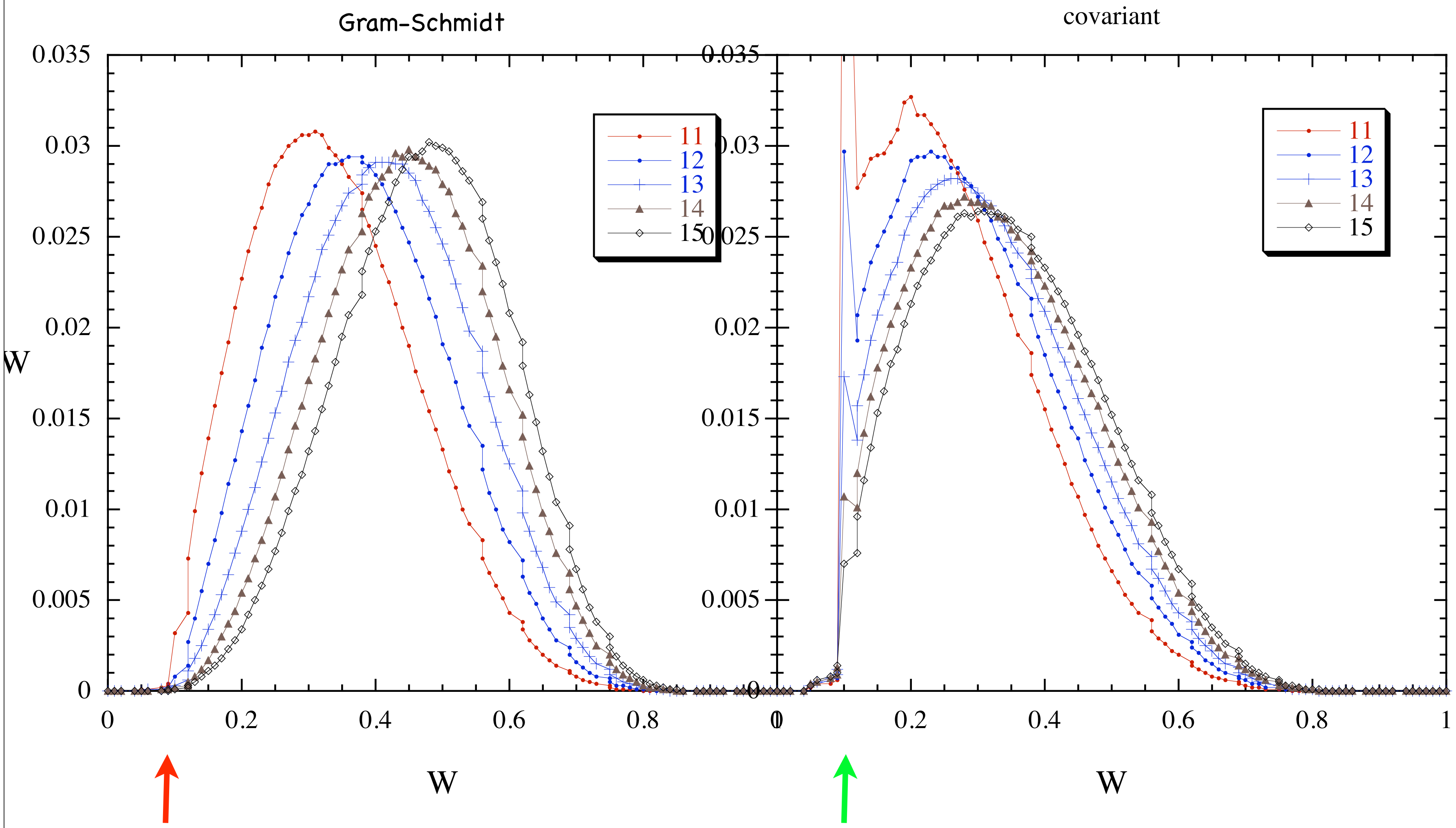
Localization Distributions



Localization Distributions



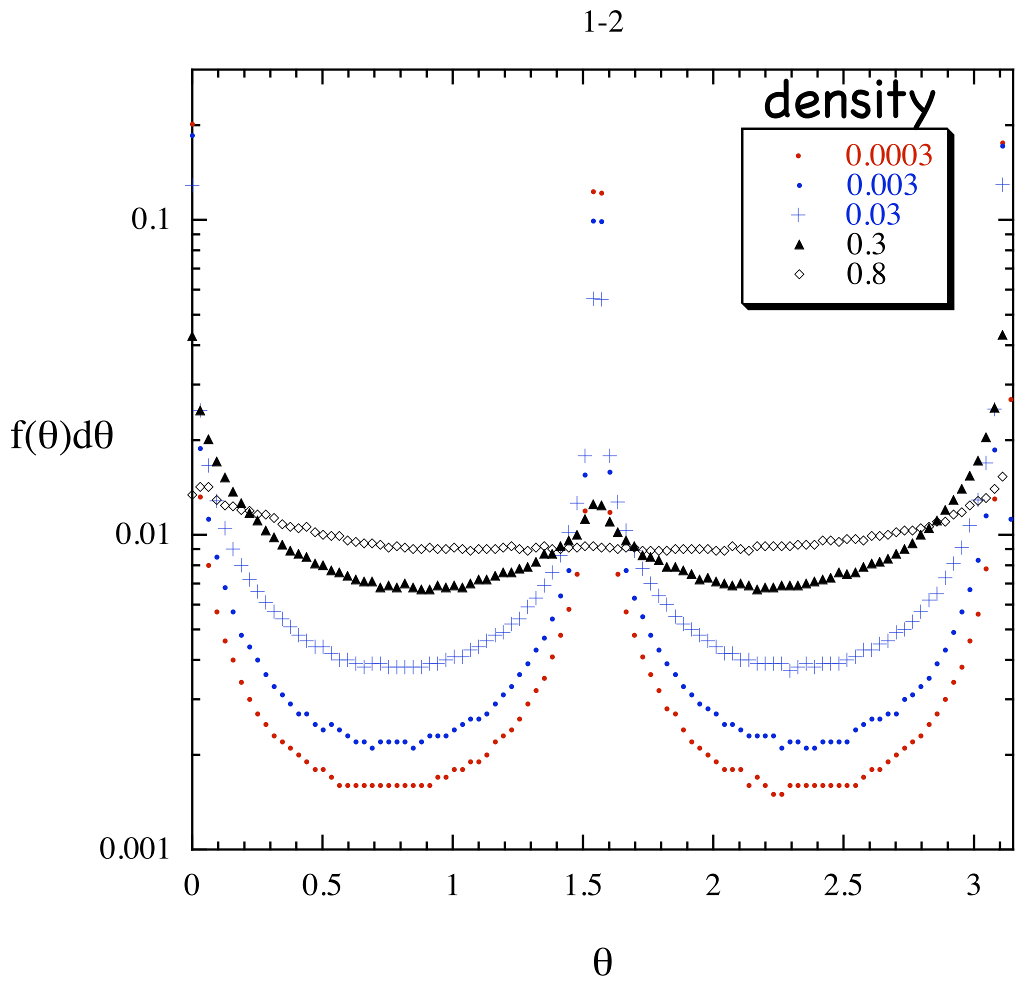
Localization Distributions



no most localized

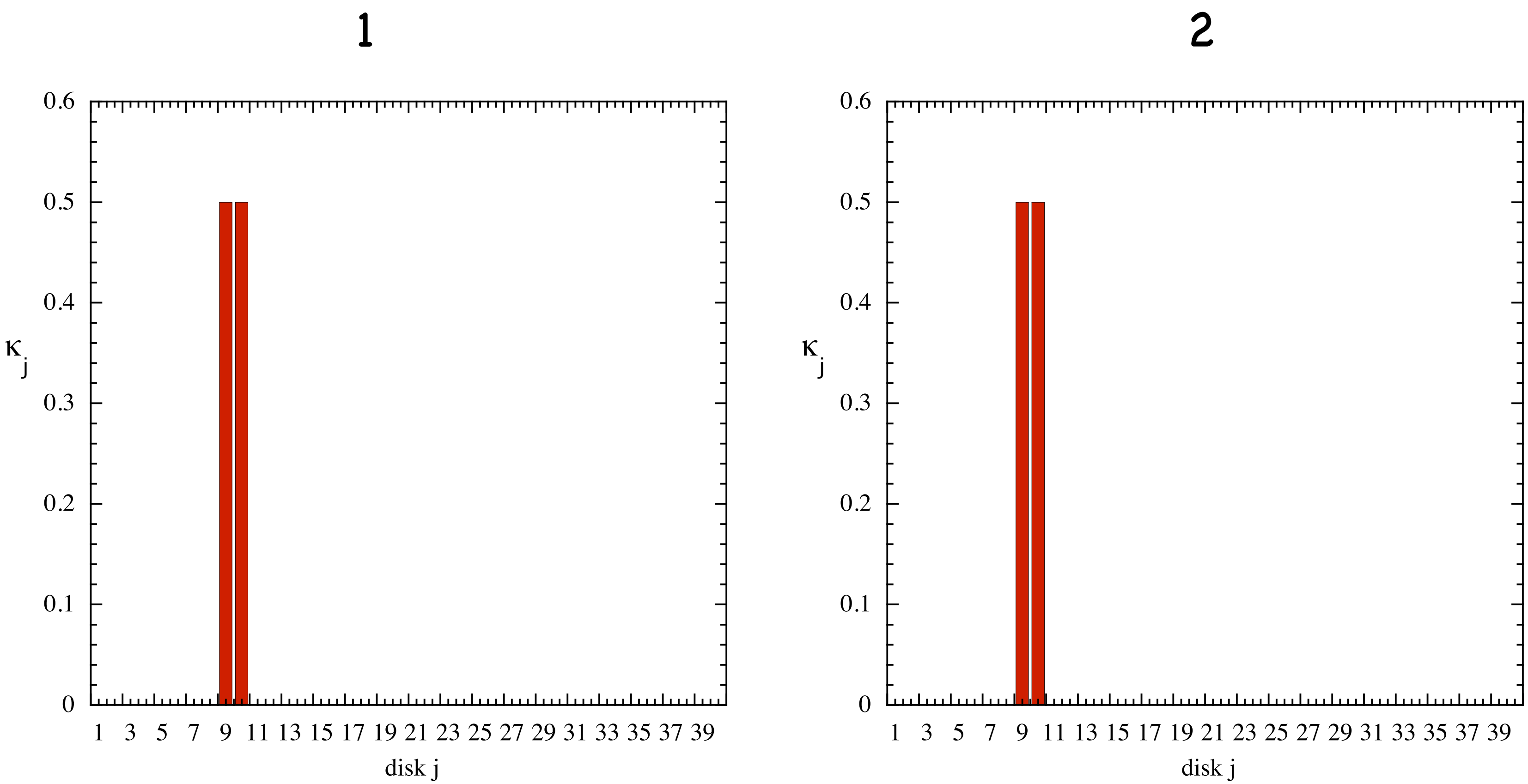
some most localized

Angle Distributions



Low density

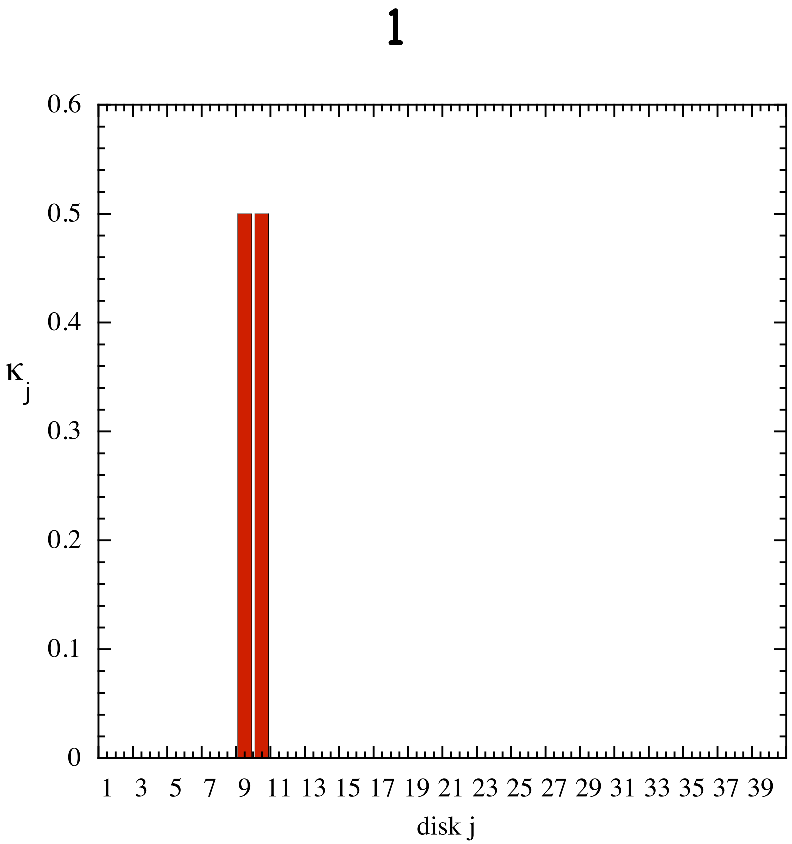
Tangency

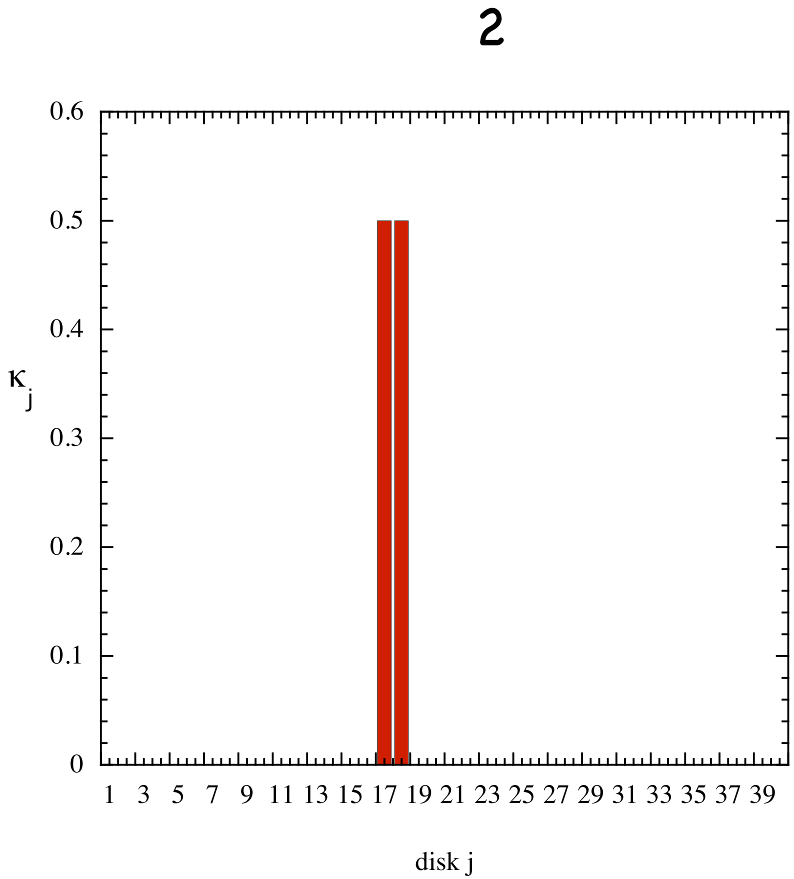


angle = 0 (or π)

Low density

Orthogonality





angle = π/2

Conclusions

- Equilibrium:- Steps in the spectrum of exponents
- Transverse (T) modes & time dependent longitudinal (L) and momentum proportional (P) modes
- Localization: Gram Schmidt & covariant
- Low density Localization
- Angle distributions

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