LYAPUNOV-LIKE VECTORS FOR ENSEMBLE FORECAST (SPATIO-TEMPORAL MVL ANALYSIS)

Diego Pazó
Instituto de Física de Cantabria (IFCA), CSIC-UC, Santander (Spain)

ENSEMBLE FORECASTING

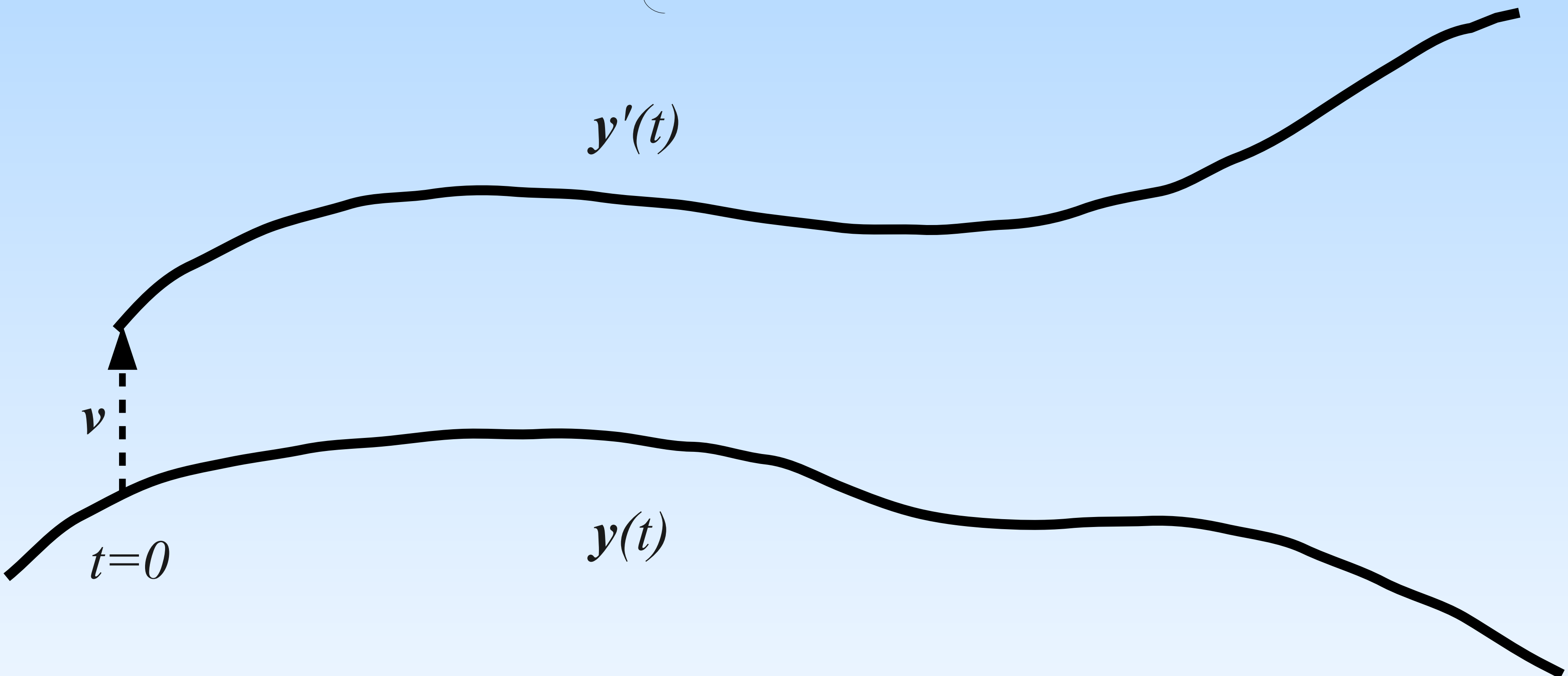
Leith (1974): Ensemble mean outperforms deterministic forecasting.

Hypothesis:
- Imperfect model (Canada)
- Perfect model: **dynamic vectors**
  - Bred vectors (USA)
  - Singular vectors (Europe)

- What do we know about the evolution of perturbations initialized along dynamic vectors in **simple** spatio-temporal systems?
- How do their spatial structures evolve?

SPATIO-TEMPORAL EVOLUTION OF AN ENSEMBLE MEMBER

Initialization
$$\begin{cases} \boldsymbol{y}'(t=0) = \boldsymbol{y}(t=0) + \boldsymbol{v} \\ \boldsymbol{v} \propto Lyapunov\text{-}like\ vector \\ \|\boldsymbol{v}\| \ll 1 \end{cases}$$



Spatio-temporal evolution of the finite perturbation ***v(t)*=*y'(t)*-*y(t)*** ????

"LYAPUNOV-LIKE" (BRED, LYAPUNOV, SINGULAR) VECTORS FUNDAMENTALS

Finite vectors:

Bred vectors

$$t_m = mT$$

$$\boldsymbol{l}(t_m) = \boldsymbol{y}'(t_m) - \boldsymbol{y}(t_m)$$

$$\boldsymbol{y}'(t_m) = \boldsymbol{y}(t_m) + \epsilon_0 \frac{\boldsymbol{l}(t_m)}{\|\boldsymbol{l}(t_m)\|} \quad ; \quad \|\boldsymbol{l}\| = \left| \prod_{x=1}^{L} l_x \right|^{1/L}$$

Geometric mean "Log-BVs"

Infinitesimal vectors:

Infinitesimal perturbations evolve linearly: $\boldsymbol{\delta}\,\mathbf{u}(\mathrm{t}+\tau) = \mathbf{A}(\mathrm{t}+\tau,\mathrm{t})\,\boldsymbol{\delta}\,\mathbf{u}(\mathrm{t})$

- Backward Lyapunov vectors: byproduct of Gram-Schmidt method (past growth).
- Forward Lyapunov vectors: future growth.
- Characteristic Lyapunov vectors: covariant with the linear dynamics.

- Singular vectors: perturbation that will exhibit maximal amplification after τ.

"LYAPUNOV-LIKE" (BRED, LYAPUNOV, SINGULAR) VECTORS FUNDAMENTALS

	Vector type	Notation	Magnitude	Computation interval	Eigenvector of	Control parameter(s)
Bred	Log-BV	$\mathbf{l}_{M_0}$	Finite	$(-\infty, 0)$	—	$M_0 = \ln(\epsilon_0)$
Lyapunov	B-LV	$\mathbf{b}_n$	Infinitesimal	$(-\infty, 0)$	$\mathbf{A}(0, -\infty)\mathbf{A}^*(0, -\infty)$	n ($\leftrightarrow$ n-th LE)
Lyapunov	F-LV	$\mathbf{f}_n$	Infinitesimal	$(0, \infty)$	$\mathbf{A}^*(\infty, 0)\mathbf{A}(\infty, 0)$	n ($\leftrightarrow$ n-th LE)
Lyapunov	C-LV	$\mathbf{g}_n$	Infinitesimal	$(-\infty, 0) \cup (0, \infty)$	—	n ($\leftrightarrow$ n-th LE)
Singular	SV	$\mathbf{s}_\tau$	Infinitesimal	$(0, \tau)$	$\mathbf{A}^*(\tau, 0)\mathbf{A}(\tau, 0)$	τ (here $n = 1$ only)

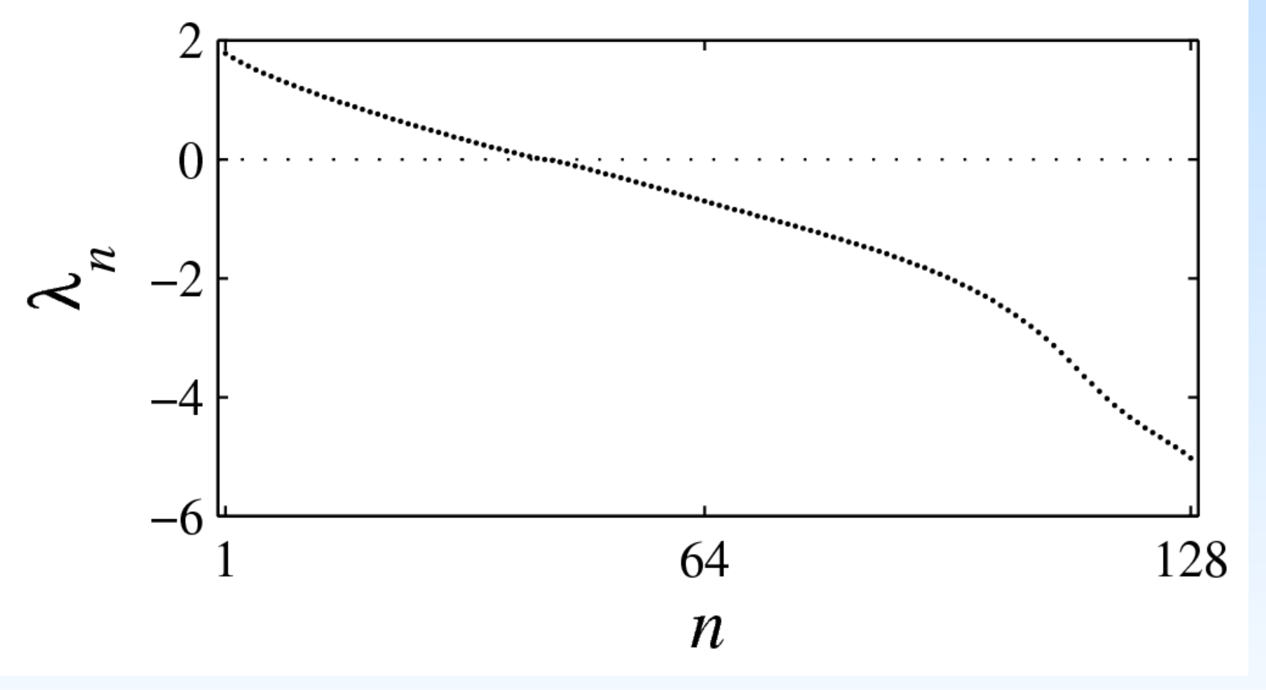
Characteristic-LVs are:

- Independent of the scalar product.
- Covariant with the *linear* dynamics: $\mathbf{g}_n(t+\tau) = \mathbf{A}(t+\tau, t)\mathbf{g}_n(t)$
- Expanding/shrinking with corresponding LE: $\lambda_n = \lim_{\tau \to \pm\infty} \frac{1}{|\tau|} \|\boldsymbol{A}(\tau, t)\boldsymbol{g}_n(t)\|$

SPATIO-TEMPORAL CHAOTIC SYSTEM: LORENZ '96 MODEL

$$\frac{d\,y_x}{dt} = -y_x - y_{x-1}\left(y_{x-2} - y_{x+1}\right) + F \qquad x = 1, \ldots, L = 128$$

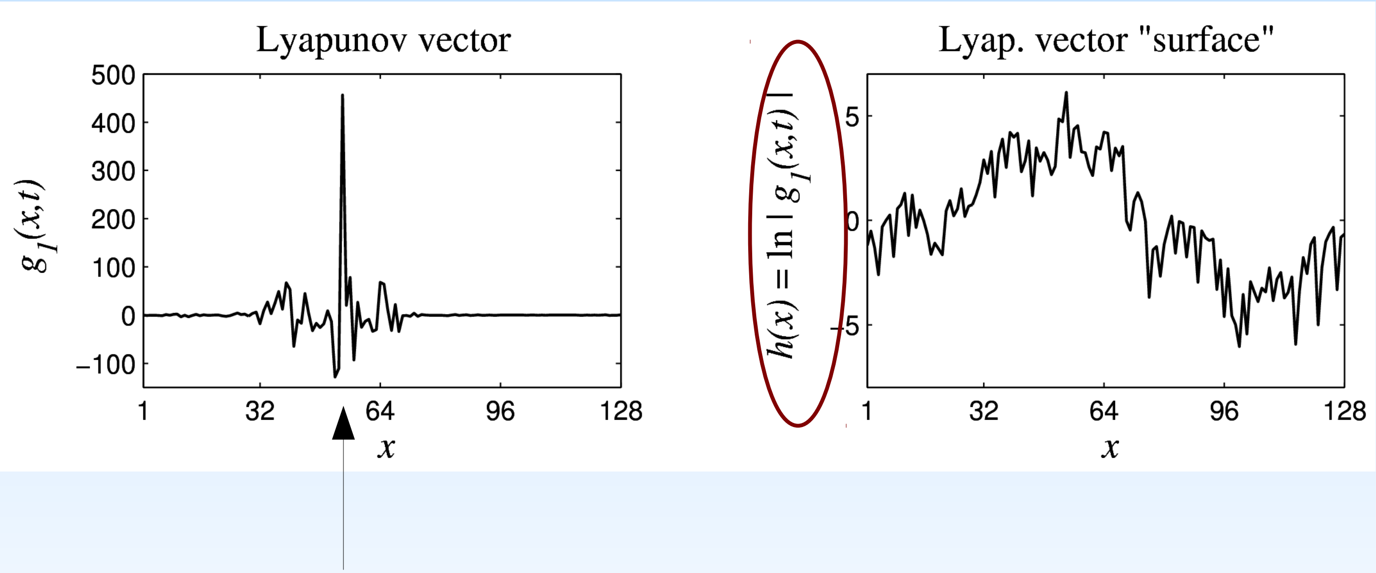
Lyapunov spectrum ($F=8$):



THEORETICAL FRAMEWORK I (Pikovsky and Politi, 1998) : INFINITESIMAL PERTURBATION

- An infinitesimal perturbation exhibits dynamical localization.

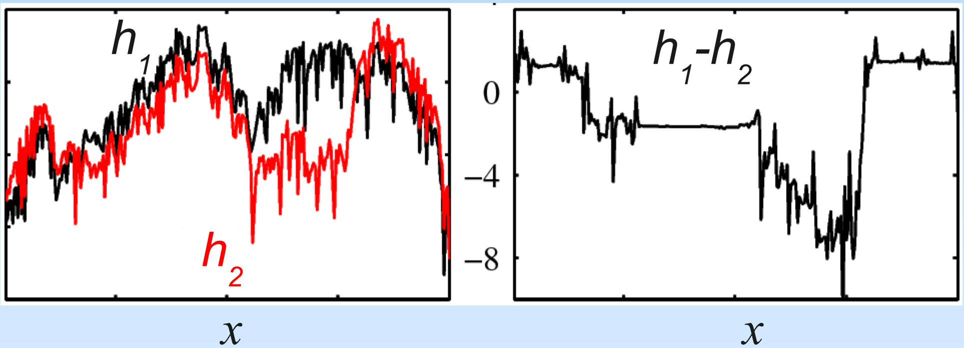
- **In log scale** it belongs to the universality class of the Kardar-Parisi-Zhang (KPZ) equation [$\partial_t h = \xi + \nabla^2 h + (\nabla h)^2$].



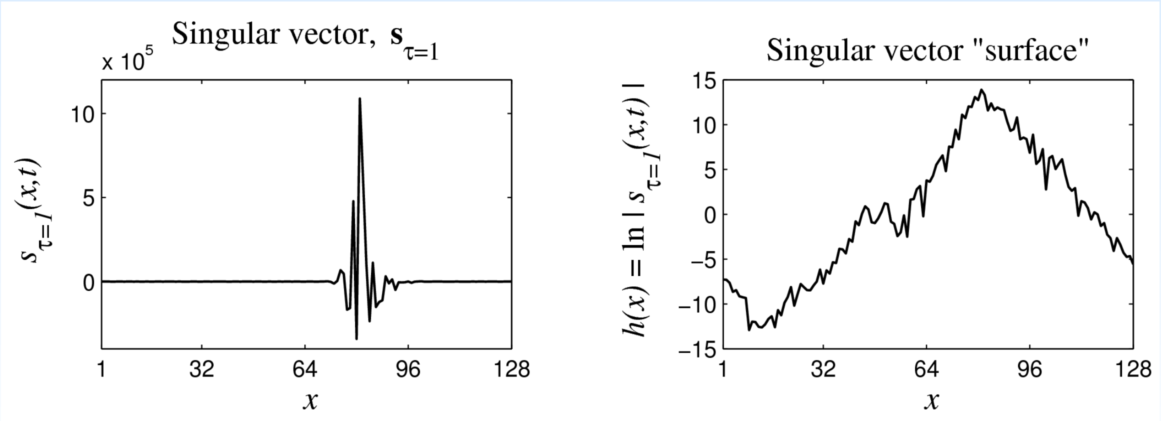
streched-exponential localization: $|g_1(x)| \sim e^{-k\sqrt{|x-x_0|}}$

THEORETICAL FRAMEWORK II: LYAPUNOV-LIKE VECTORS

- López et al (2004), Primo et al. (2005,2006): Finite perturbations and **bred vectors** (uncorrelation at long scales).
- Szendro et al. (2007), Pazó et al. (2008): **Lyapunov vectors** are "piecewise" copies of the first Lyapunov vector.

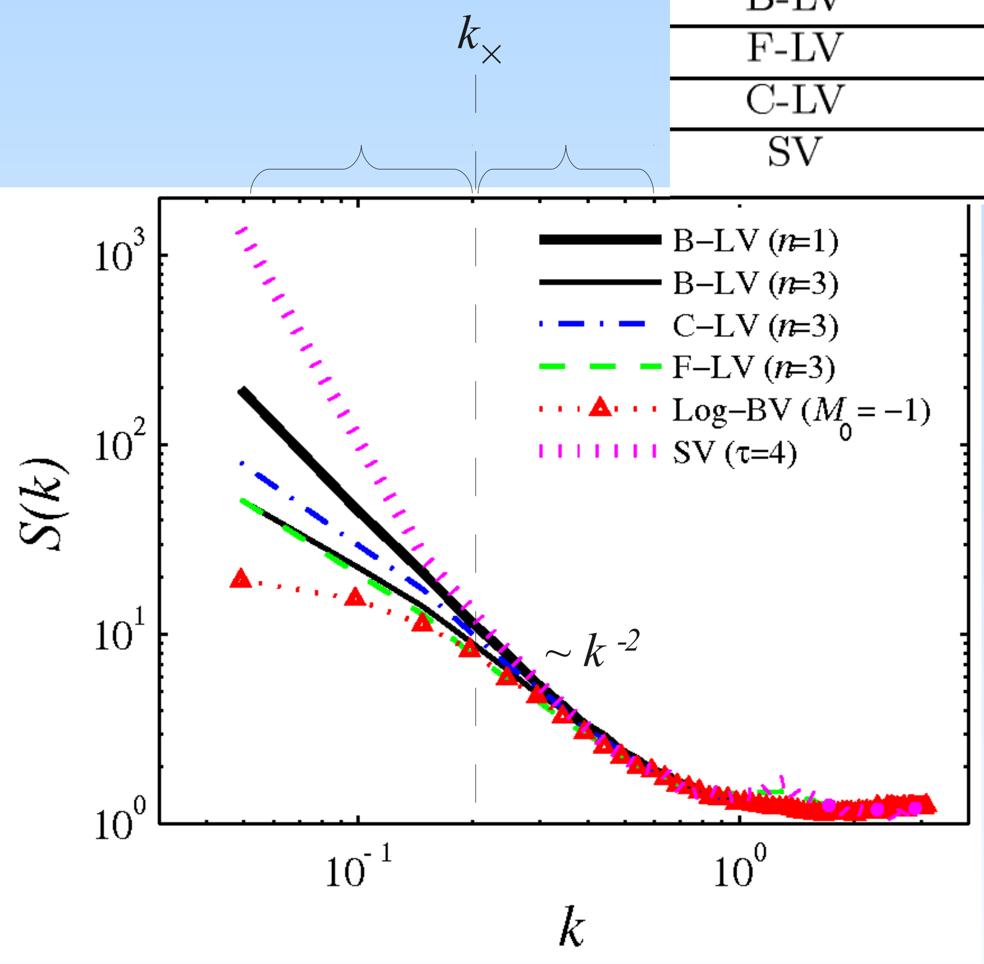


- Pazó et al. (2009): **Singular vectors** are exponentially localized and scale as KPZ with periodic/quenched noise.



SPATIAL POWER SPECTRAL DENSITY

Vector type	Cross. length $l_\times = \frac{2\pi}{k_\times}$	$S(k < k_\times)$	$S(k > k_\times)$
Log-BV	$\sim M_0^2$	Constant	k^{-2}
B-LV	$\approx (L/n)^\theta \ (\theta \approx 1)$	k^{-1}	
F-LV	$\approx (L/n)^\theta \ (\theta \approx 1)$	k^{-1}	
C-LV	$\approx (L/n)^\theta \ (\theta \approx 1)$	$k^{-1.15}$	
SV	$\sim \tau^{\gamma/2} \ (\gamma \simeq 0.78)$	k^{-4}	



Parameters control $k_\times$!!!

QUANTIFICATION OF LOCALIZATION STRENGTH AND CROSSOVER LENGTH: the variance V

Surface picture: $h(x,t) = \ln|v(x,t)|$

$$V(t) = \frac{1}{L}\sum_{x=1}^{L} h(x,t)^2 - \left(\frac{1}{L}\sum_{x=1}^{L} h(x,t)\right)^2 \Longleftrightarrow$$

Parceval's identity $V = \int S(k)\,dk$

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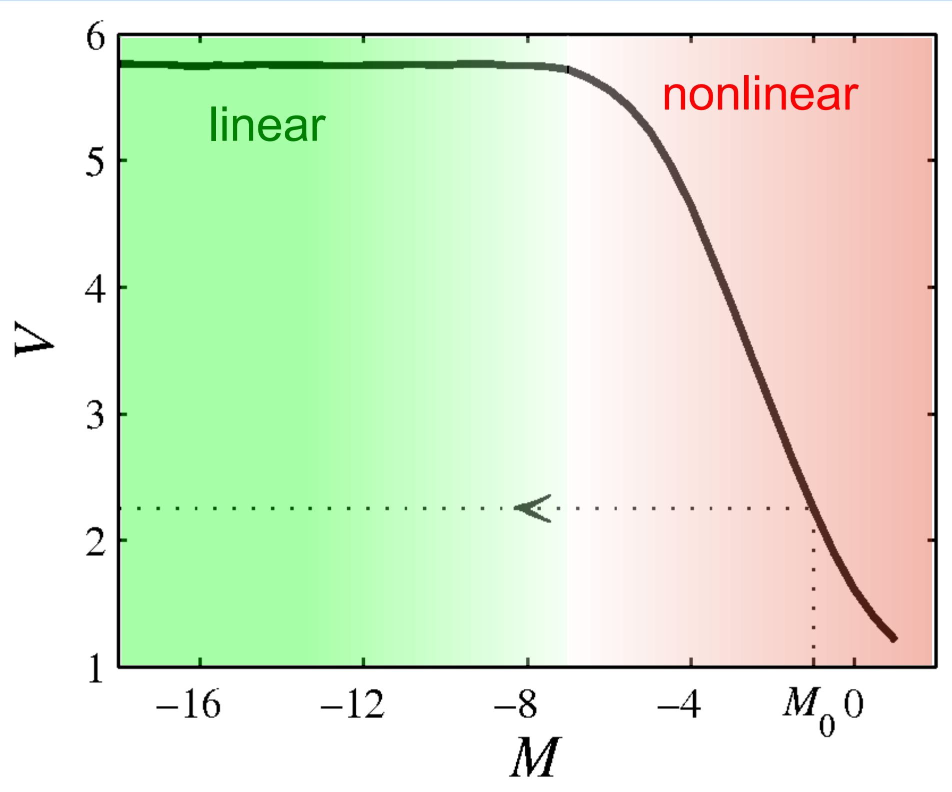
QUANTIFICATION OF AMPLITUDE (finite perturbations): the mean M

$$M(t) = \frac{1}{L}\sum_{x=1}^{L} h(x,t)$$ (logarithm of the geometric **norm**)

Fixing the perturbation's norm, say $M=M_0$, one obtains the logarithmic bred vectors.

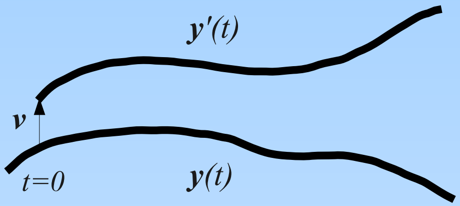
$$\mathbf{l}_{M_0 \to -\infty} \propto \mathbf{b}_1 = \mathbf{g}_1$$



GROWTH OF FINITE PERTURBATIONS

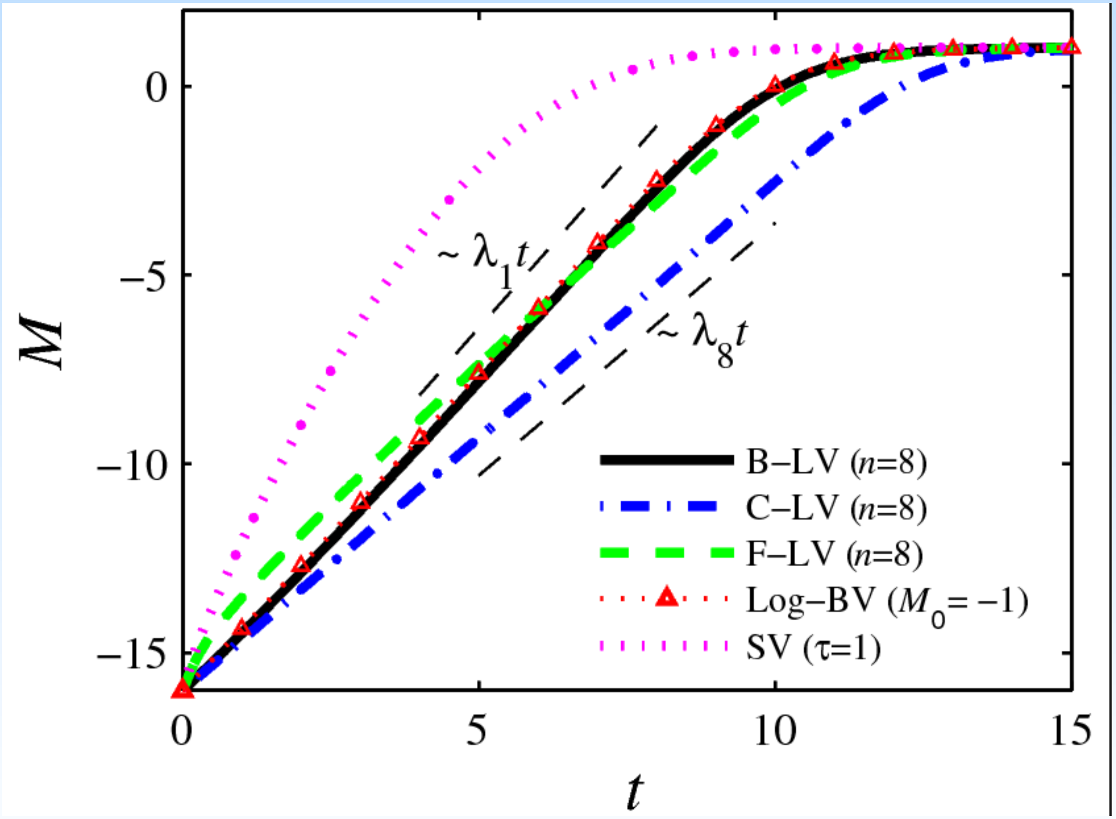
Initialization

$$\begin{cases} \boldsymbol{y}'(t=0) = \boldsymbol{y}(t=0) + \boldsymbol{v} \\ \boldsymbol{v} \propto \{\boldsymbol{l}_{M_0}, \boldsymbol{b}_n, \boldsymbol{f}_n, \boldsymbol{g}_n, \boldsymbol{s}_\tau\} \\ \ln\|\boldsymbol{v}\| = M(t=0) = -16 \end{cases}$$



$$\boldsymbol{v}(t) = \boldsymbol{y}'(t) - \boldsymbol{y}(t)$$

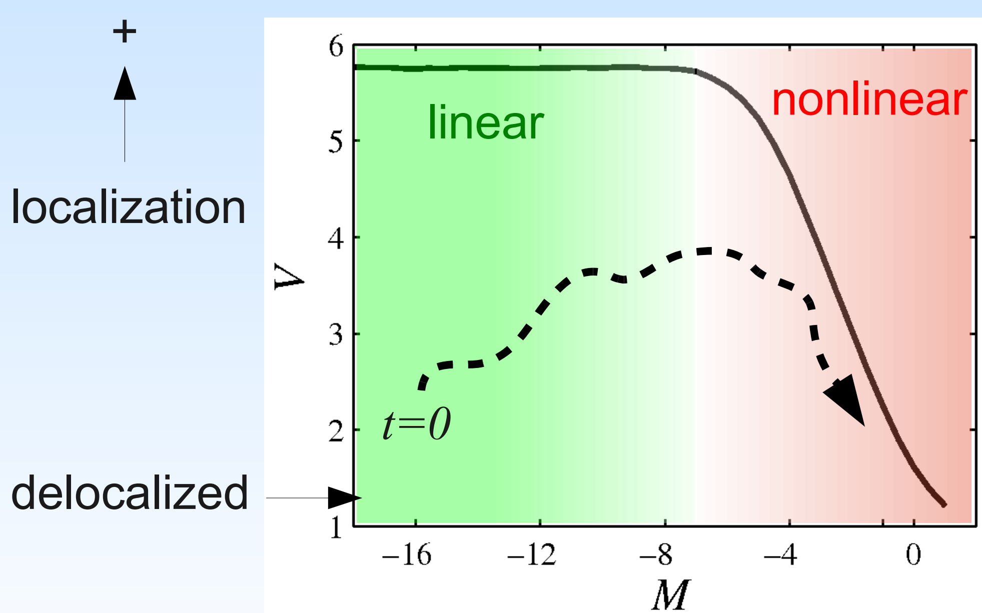
$$M(t) = \frac{1}{L}\sum_{x=1}^{L} \ln|v(x,t)|$$



SPATIO-TEMPORAL EVOLUTION OF PERTURBATIONS

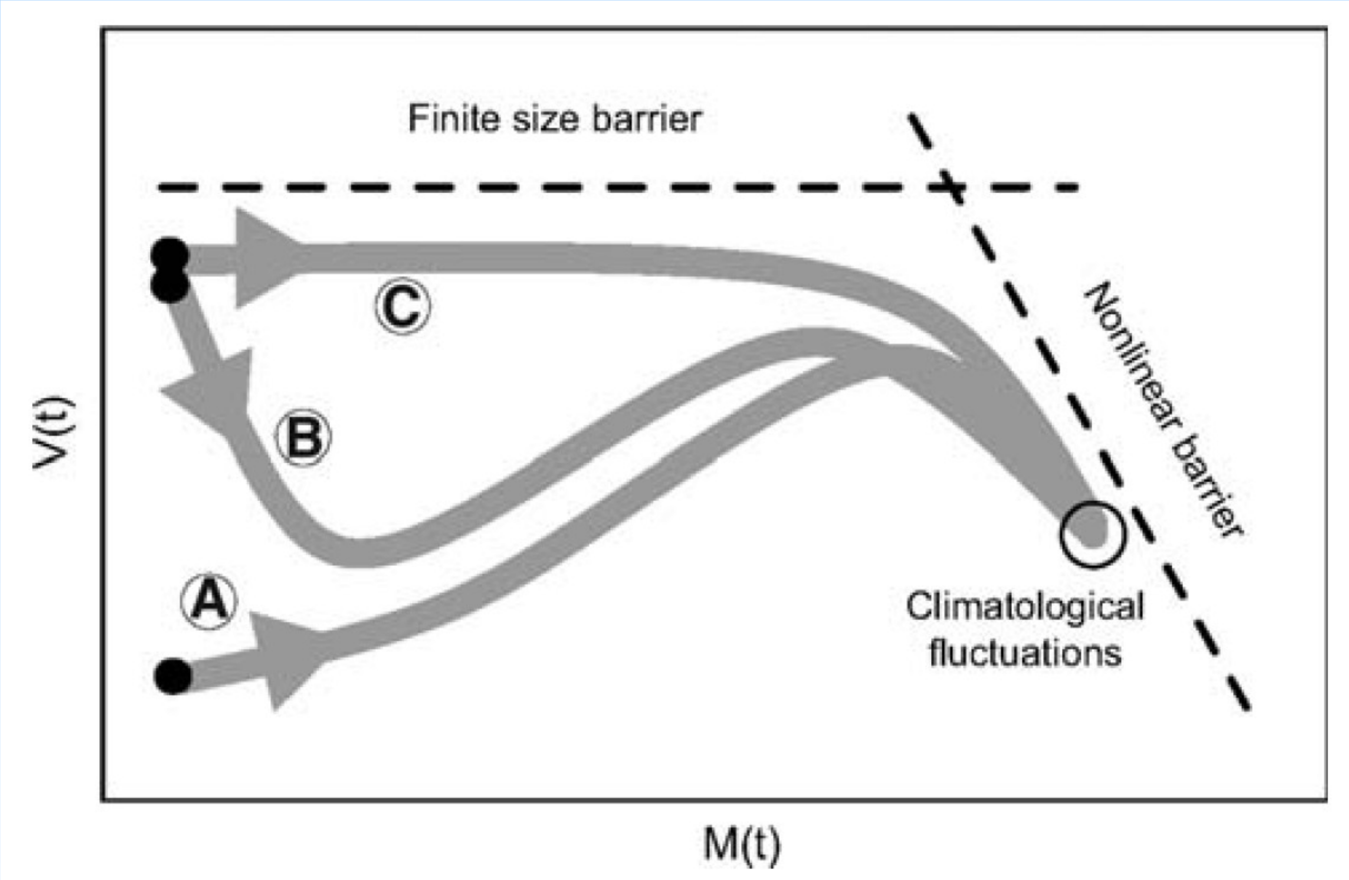
Initialization
$$\begin{cases} \boldsymbol{y}'(t=0)=\boldsymbol{y}(t=0)+\boldsymbol{v} \\ \boldsymbol{v}\propto\{\boldsymbol{l}_{M_0},\boldsymbol{b}_n,\boldsymbol{f}_n,\boldsymbol{g}_n,\boldsymbol{s}_\tau\} \\ \ln\|\boldsymbol{v}\|=M(t=0)=-16 \\ V(t=0)\text{ is intrinsic to the selected vector.} \end{cases}$$

We track $\boldsymbol{v}(t)=\boldsymbol{y}'(t)-\boldsymbol{y}(t)$ in M,V coordinates



MVL (Mean-Variance of Logarithms) DIAGRAM

Introduced by Primo et al. in Phys. Rev. E (2005) and Phys. Rev. Lett. (2007)

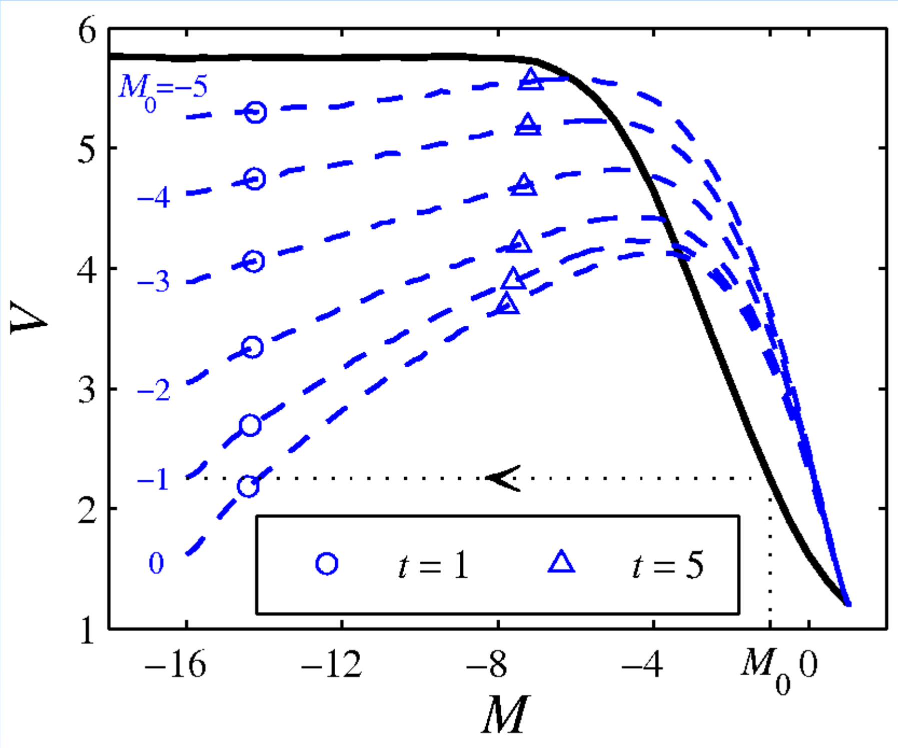


- J.M. Gutiérrez et al., Spatiotemporal characterization of ensemble prediction systems: the mean-variance of logarithms (MVL) diagram, Nonlin. Processes Geophys. (2008).
- J. Fernández et al., MVL spatiotemporal analysis for model intercomparison in EPS: application to the DEMETER multimodel ensemble. Clim. Dyn. (2009).

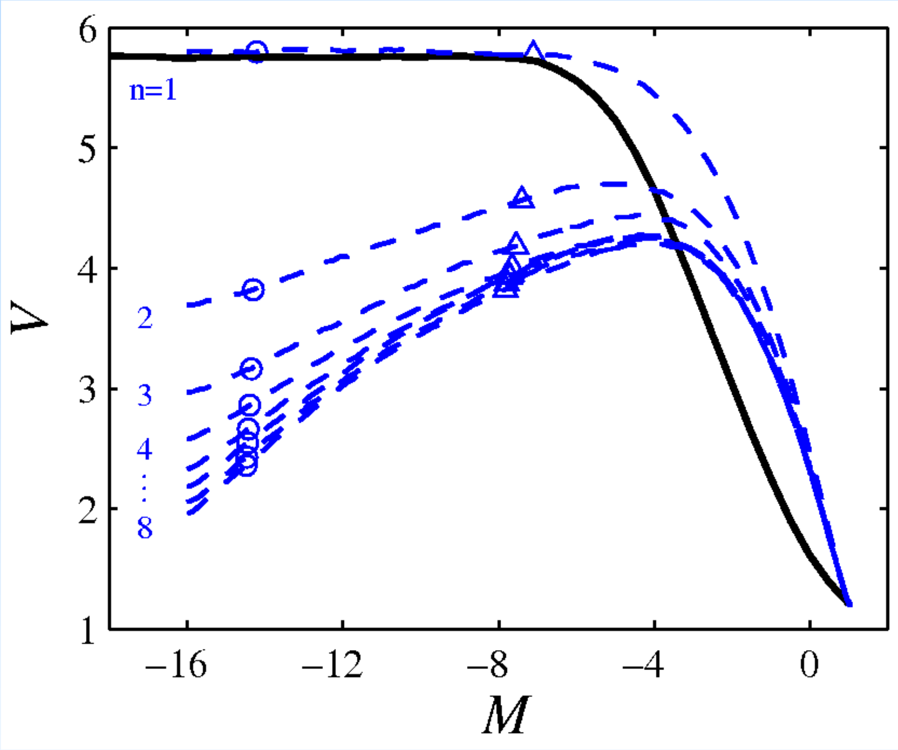
RESULTS

1. Vectors generated from the past are well adapted (and exhibit similar behaviour). Exponential growth rate ≈ λ_1

v(0) ∝ Logarithmic bred vector



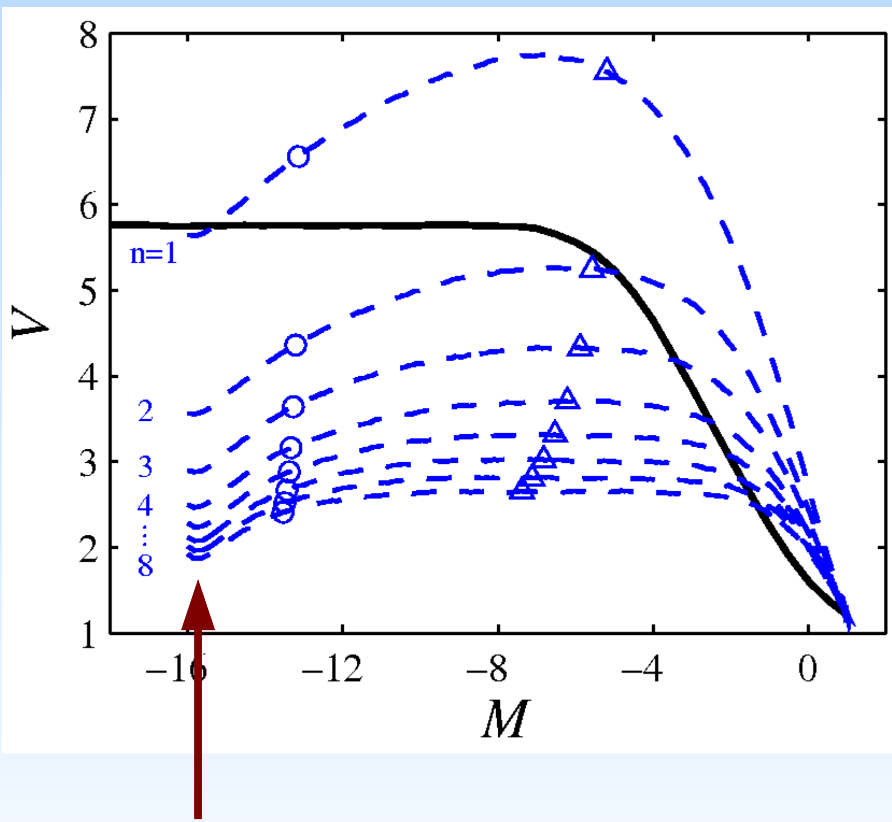
v(0) ∝ Backward Lyapunov vector



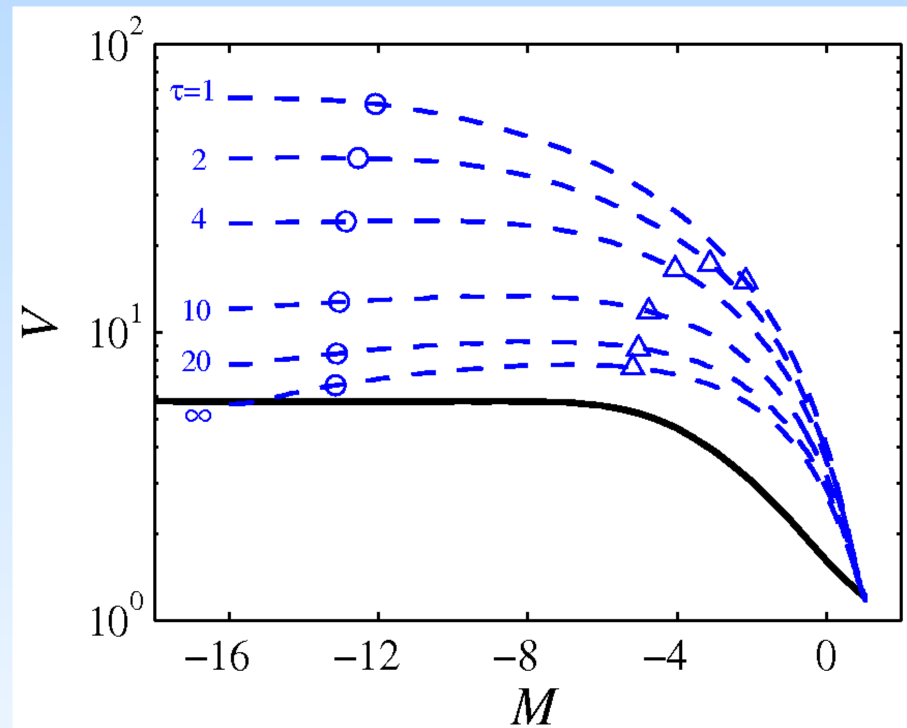
RESULTS

2. Vectors generated from the future are not in the attractor (and exhibit severe transients). Exponential growth rate ≠ λ_1

v(0) ∝Forward Lyapunov vector



v(0) ∝Singular vector



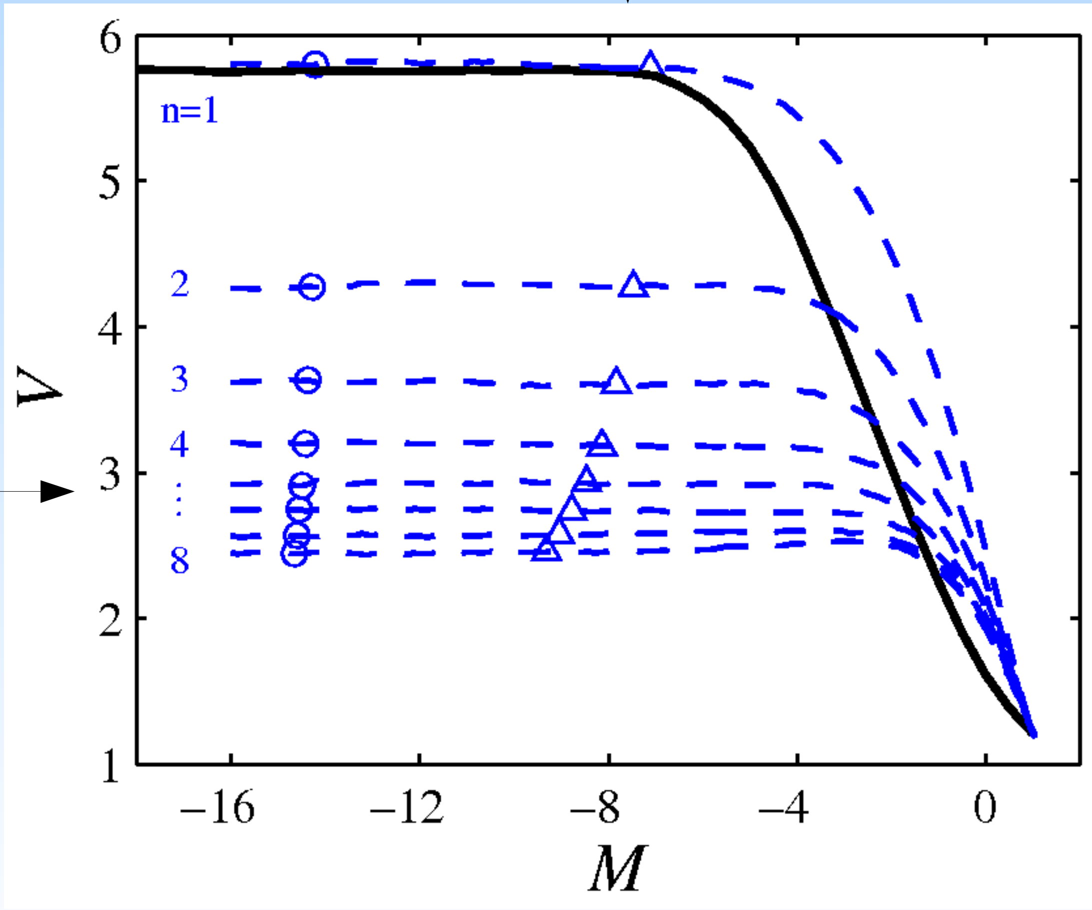
Dip due to bad adaptation to the flow

RESULTS

3. Characteristic Lyapunov vectors permit to control growth rate and structure.

different growth rates

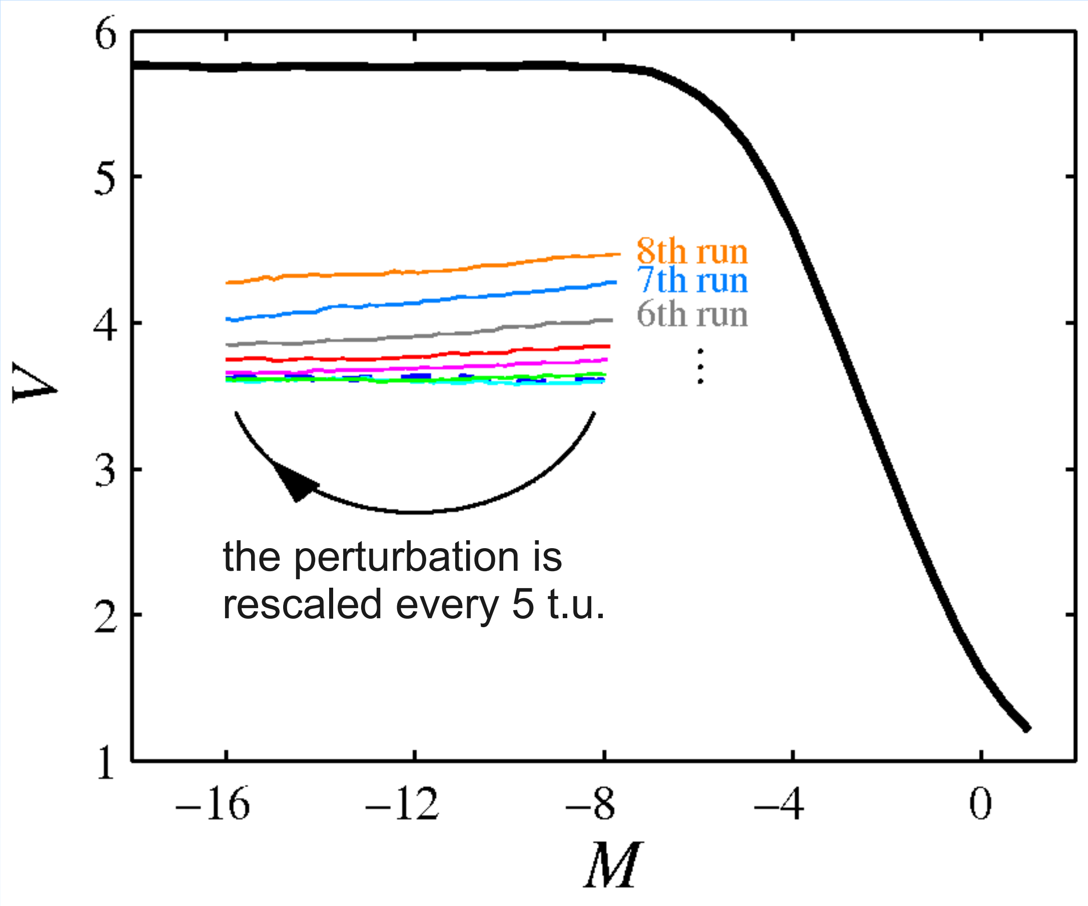
constant structure



RESULTS

4. Characteristic Lyapunov vectors. Robustness!

$\mathbf{v}(0) \propto$ 3-rd Characteristic Lyapunov vector



CONCLUSIONS

- Vectors evolved from the past: well adapted, but common growth rate ≈λ_1
- Vectors with information from the future: controllable growth, but badly adpated (strong transients).
- Characteristic LV:
  - Well adapted and different growth rates $\{\lambda_n\}$.
  - Very robust!

THANKS!!

[Miguel A. Rodríguez, Juan M. López, Ivan G. Szendro and Sarah Hallerberg]

Main reference: Pazó et al., Tellus **62A**, 10-23 (2010)
[http://www.ifca.unican.es/~pazo]