Lyapunov Modes in Extended Dynamical Systems

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Theoretical Physics I – Complex Systems and Nonlinear Dynamics

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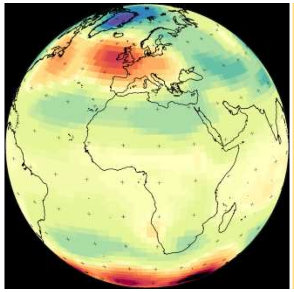

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International Workshop on

Exploring Complex Dynamics in High-Dimensional Chaotic Systems: From Weather Forecasting to Oceanic Flows

25 - 29 January 2010

Lyapunov Modes:

- **Generalization of (phonon) normal modes to chaotic systems**
- **Linearized motion in neighborhood of chaotic trajectory**
- **Hydrodynamic Lyapunov Modes (HLM): Slow, long wave-length behavior**
- **Objects of Hamiltonian nonlinear dynamics fundamental to (non-equilibrium) statistical physics?**
- **This talk: Extended dissipative systems, existence of finite number of physical, entangled modes**

2 Types of Lyapunov Vectors:

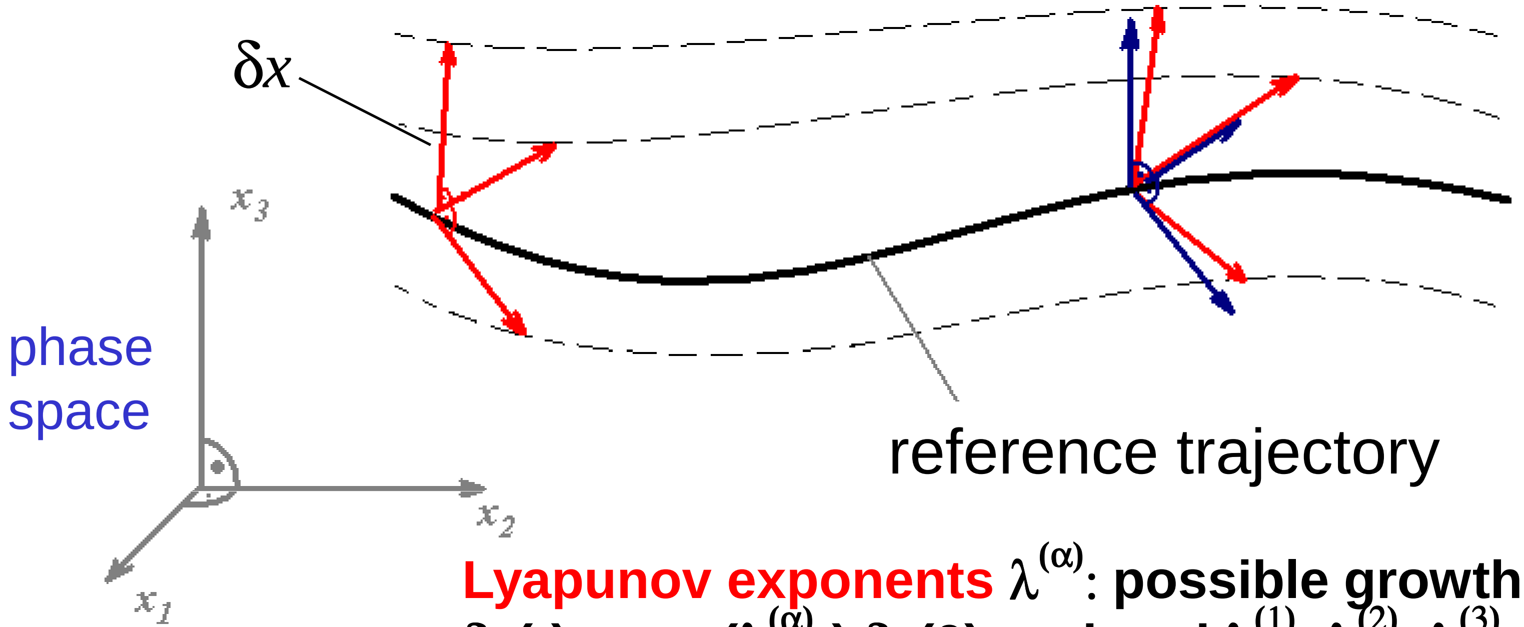
- **Orthogonal Lyapunov Vectors (OLV):**

  Studied in many extended systems (since 2000: Hard spheres, Lennard-Jones fluids, WCA fluids, Coupled map lattices, PDEs (KS equation),Dynamic XY model, FPU models, Posch, Morriss, Yang, G.R., …)

- **Covariant Lyapunov Vectors (CLV):**
  Numerically accessible since 2007*

*PRL 99,130601 (2007) Ginelli et al.

Dynamics in tangent space: OLV

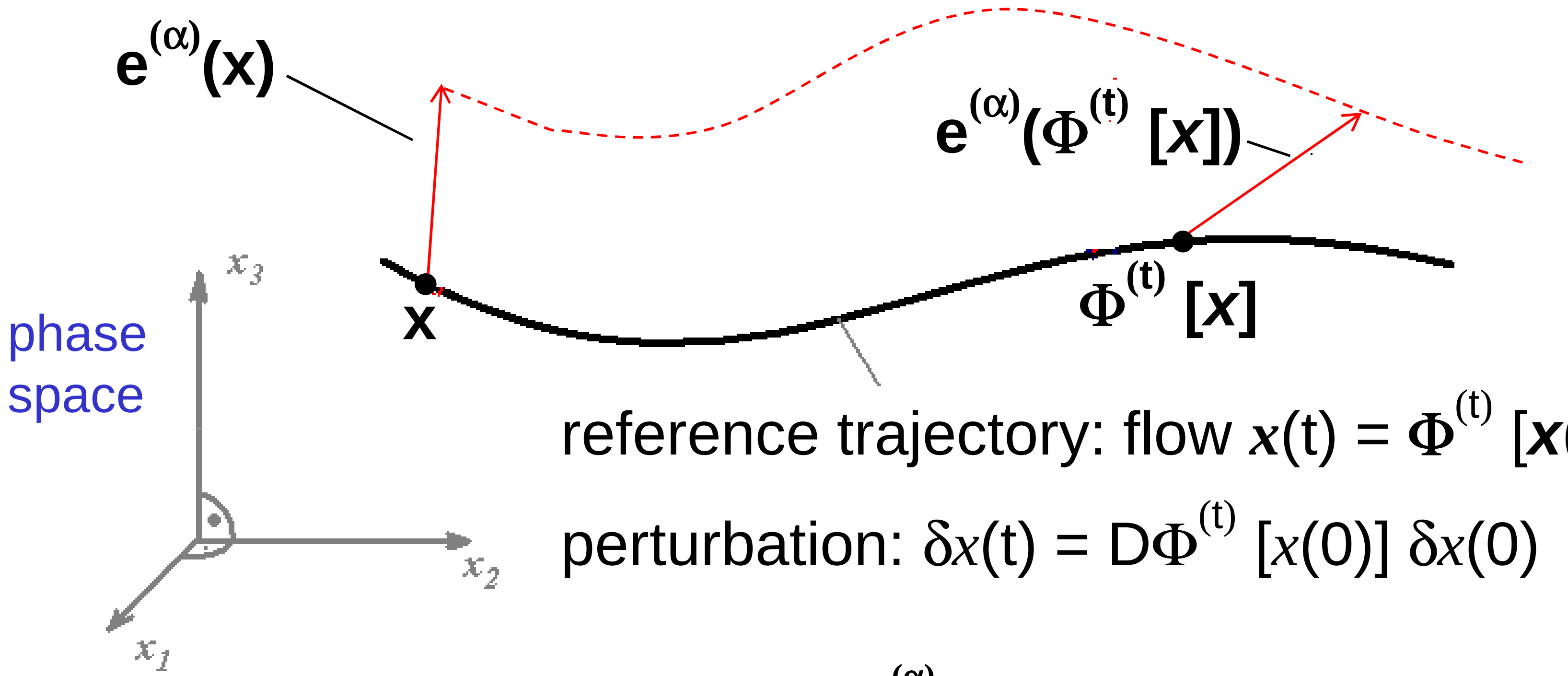


Lyapunov exponents $\lambda^{(\alpha)}$: possible growth rates
$\delta x(t) \sim \exp(\lambda^{(\alpha)} t)\, \delta x(0)$, ordered $\lambda^{(1)} > \lambda^{(2)} > \lambda^{(3)}$ …

OLV = Orthogonal Lyapunov vectors $\delta x^{(\alpha)}(t)$: dynamics of orthonormal frame $\delta x^{(\alpha)}(t)$, $\alpha = 1,\ldots,2dN$, from repeated Gram-Schmidt-reorthogonalization or QR decomposition

Property: k-dimensional parallel-epipedes align asymptotically with the space spanned by the k first Lyapunov vectors and its volume growth rate is $\lambda^{(1)}+\lambda^{(2)}+\ldots+\lambda^{(\kappa)}$

Dynamics in tangent space: CLV



reference trajectory: flow $\boldsymbol{x}(t) = \Phi^{(t)} [\boldsymbol{x}(0)]$

perturbation: $\delta x(t) = D\Phi^{(t)} [x(0)]\, \delta x(0)$

CLV = Covariant Lyapunov vectors $\mathbf{e}^{(\alpha)}(\mathbf{x})$:

$\mathbf{D\Phi^{(t)} [\boldsymbol{x}]\, e^{(\alpha)}(x)} = \Gamma^{(\alpha)}(\mathbf{x},t)\, \mathbf{e^{(\alpha)}(\Phi^{(t)} [\boldsymbol{x}])}$, stretching factor $\Gamma^{(\alpha)}(\mathbf{x},t)$

$\mathrm{Lim}_{t\to\infty}\, 1/t \log(\Gamma^{(\alpha)}(\mathbf{x},t)) = \lambda^{(\alpha)}(\mathbf{x})$ **α-th Lyapunov exponent**

$\mathbf{e}^{(\alpha)}(\mathbf{x})$ span Oseledec subspaces

Mather decomposition: CLV

CLV = Covariant Lyapunov vectors $\mathbf{e}^{(\alpha)}(\mathbf{x})$:

Decomposition of fundamental matrix (Mather spectrum):

$$\mathbf{D\Phi^{(t)}\,[x] = \Sigma_\alpha e^{(\alpha)}(\Phi^{(t)}\,[x])}\ \Gamma^{(\alpha)}(\mathbf{x},t)\ \mathbf{f^{(\alpha)}(x)^T}$$

stretching factor $\Gamma^{(\alpha)}(\mathbf{x},t)$

$\mathrm{Lim}_{t\to\infty}\ 1/t\ \log(\Gamma^{(\alpha)}(\mathbf{x},t)) = \lambda^{(\alpha)}(\mathbf{x})$ **α-th Lyapunov exponent**

$\mathbf{e}^{(\alpha)}(\mathbf{x})$ span Oseledec subspaces

$\mathbf{f}^{(\alpha)}(\mathbf{x})$ adjoint basis

Biorthogonal sets: $\mathbf{f}^{(\alpha)}(\mathbf{x})^T\,\mathbf{e}^{(\beta)}(x) = \delta_{\alpha\beta}$ and $\Sigma_\alpha \mathbf{e}^{(\alpha)}(x)\,\mathbf{f}^{(\alpha)}(\mathbf{x})^T = 1$

CLV, inertial manifolds, effective degrees of freedom of dissipative extended systems*:

Central result:

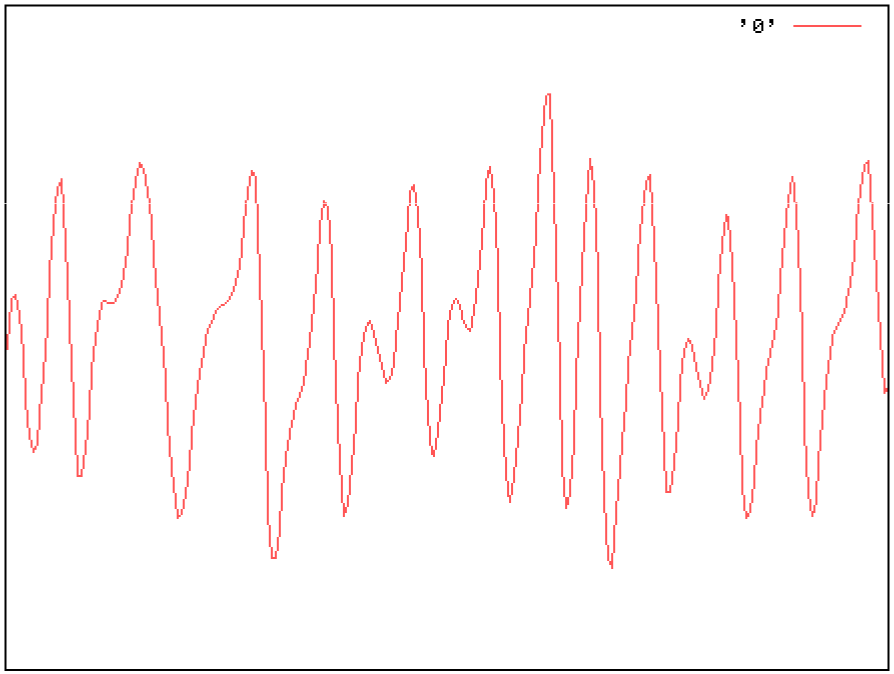
Lyapunov modes split into 2 groups:

1. **infinitely many modes with included angles bounded away from zero, associated Lyapunov exponents negative (decaying perturbations, trivial modes)**
2. **finite number of modes with repeatedly vanishing included angles, associated Lyapunov exponents positive and negative (non-decaying perturbations, physical modes), "surface" of inertial manifold (IM)**

*H. Yang, K.A. Takeuchi, F. Ginelli, H. Chaté, G.R., PRL 102, 074102 (2009)

Kuramoto-Sivashinsky (KS) Equation

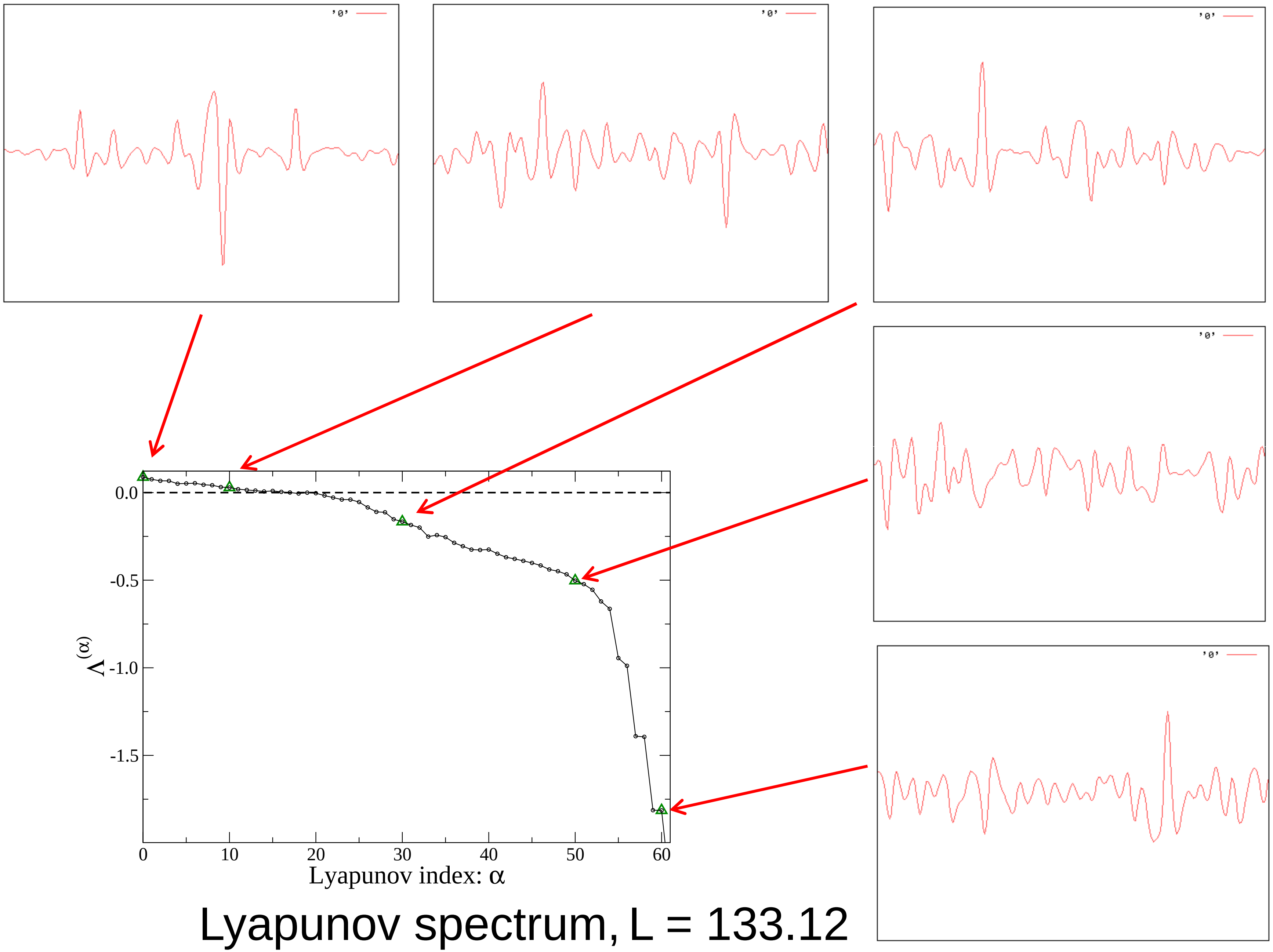
$$\partial_t u + \partial_x^2 u + \partial_x^4 u + u\partial_x u = 0$$



L = 133.12

dynamics of solution u(x,t)

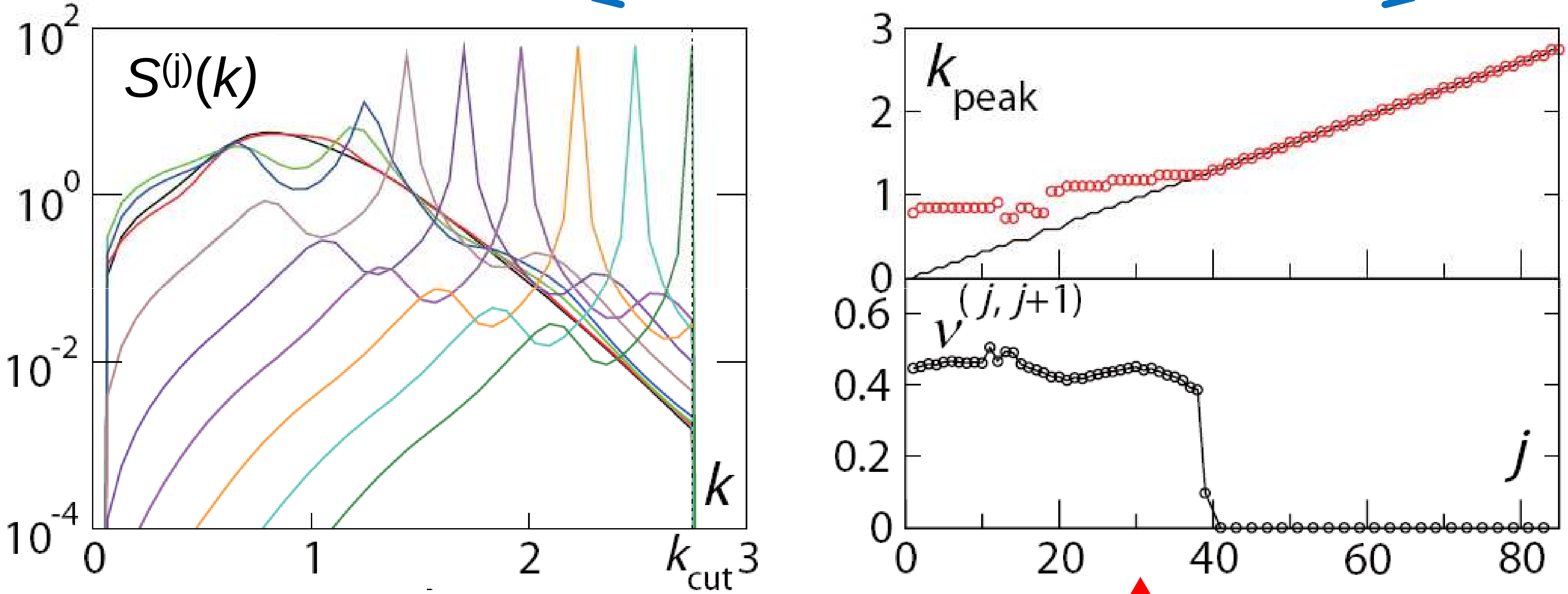
CLVs for KS system:



Lyapunov spectrum, L = 133.12

Static structure factor $S^{(j)}(k)$ of CLVs

wave-like

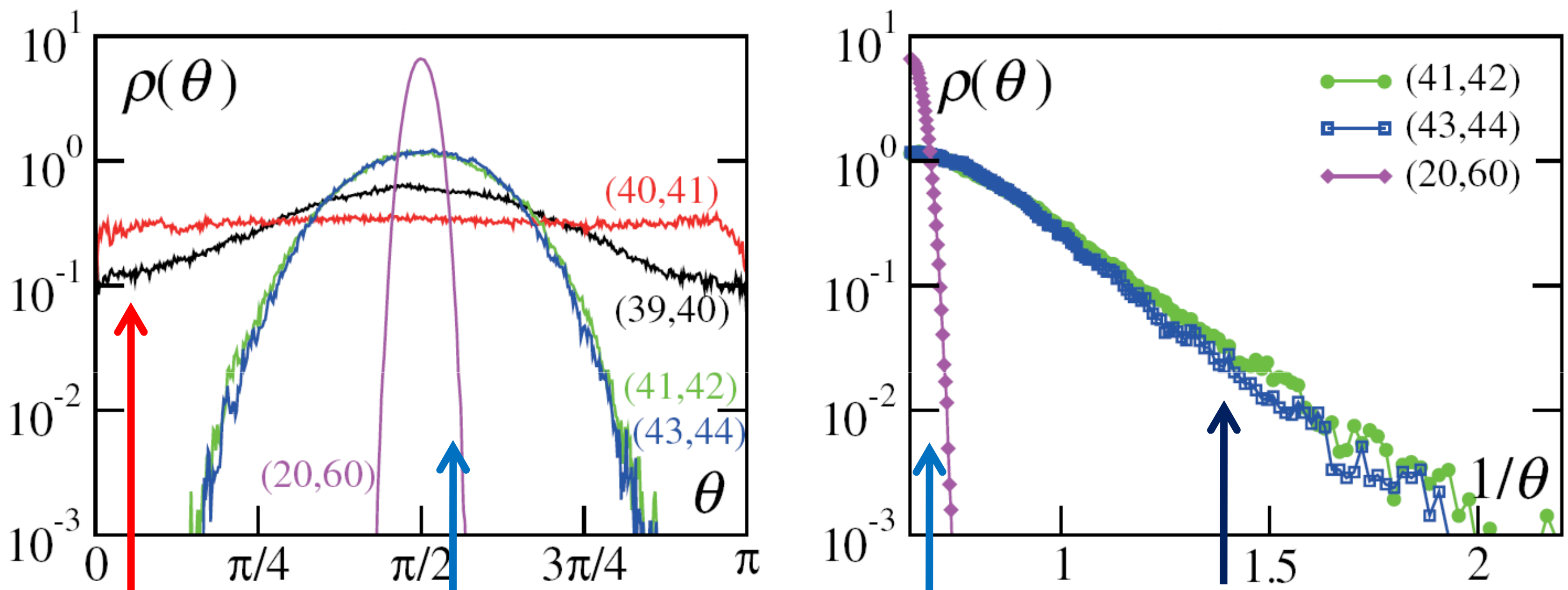


$j = 1, 16, 32, 38, 44,$
$52, 60, 68, 76, 84$

fraction of DOS violation

L = 96

Distributions of included angles of CLVs



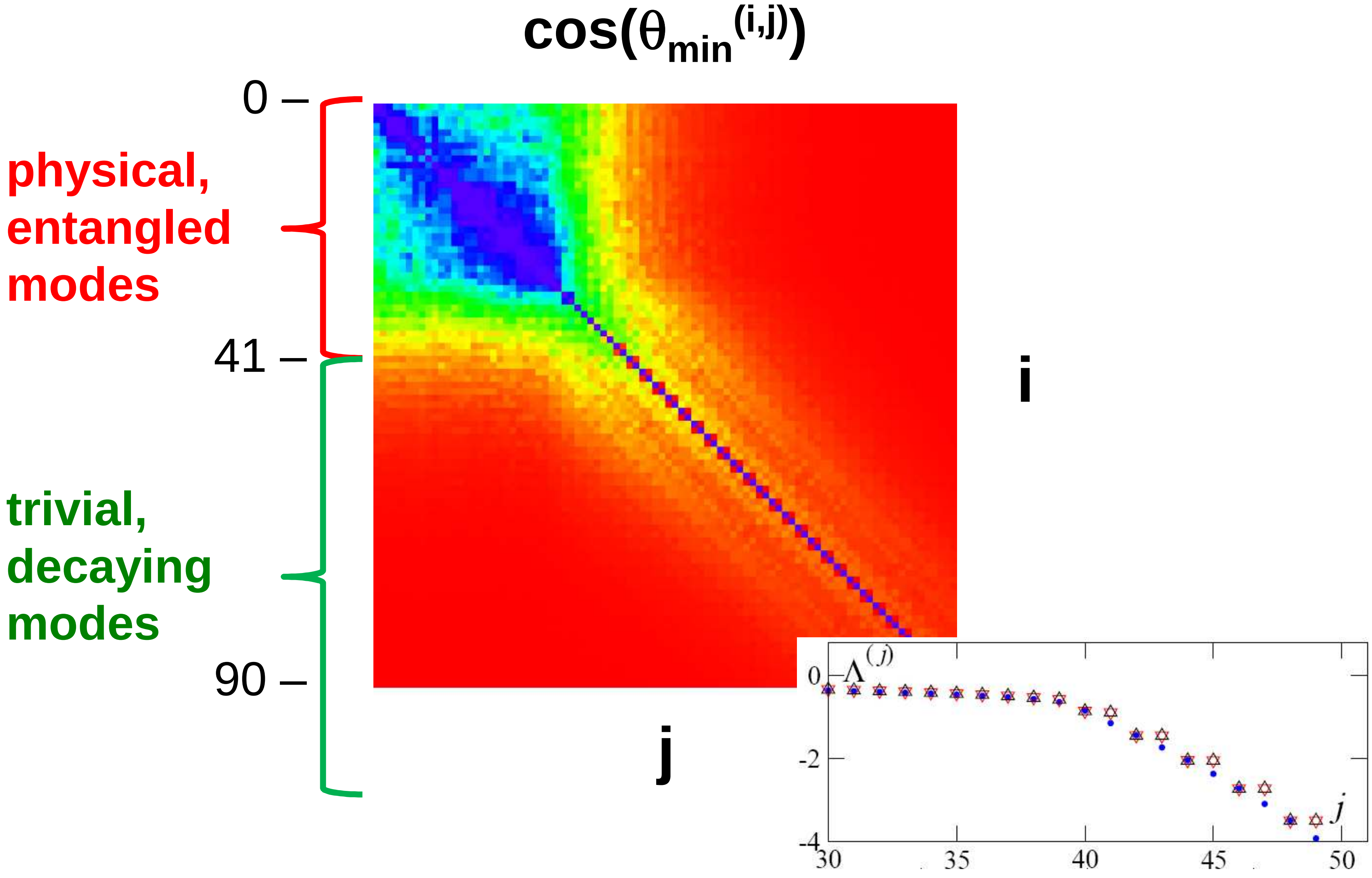
no tangencies

critical case

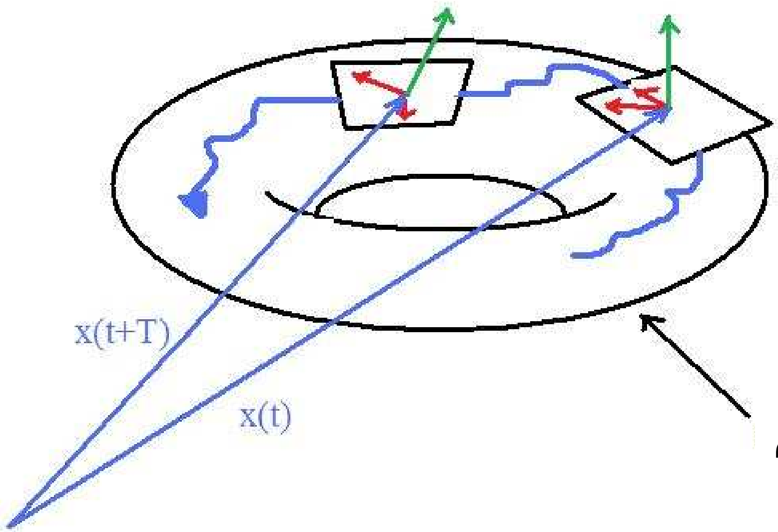
$$\rho(\theta) \sim \exp(-\text{const.}/\theta)$$

Finite probability of included angles near-zero

Matrix of minimum included angles of CLVs



Schematic picture:



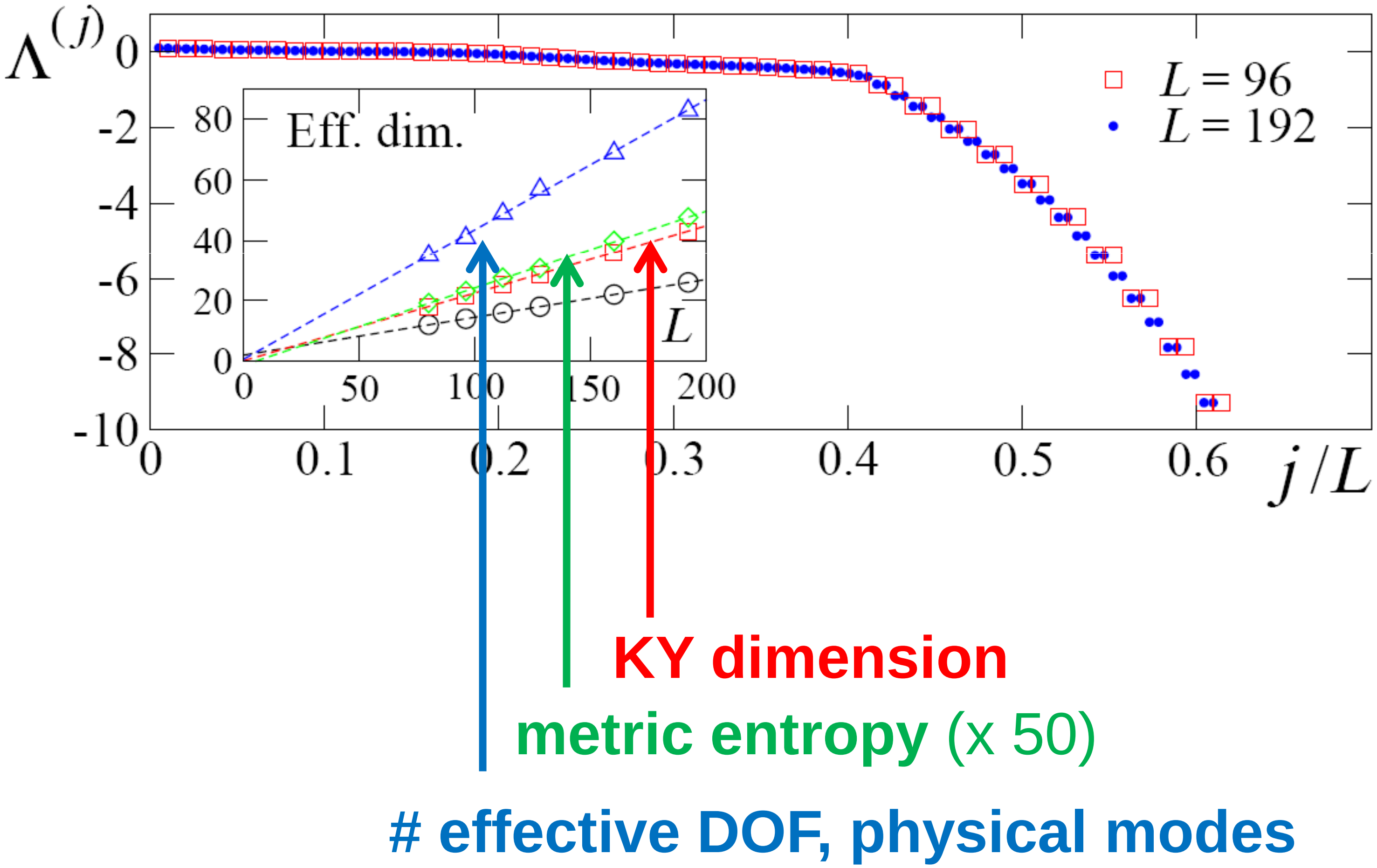
"IM", physical or entangled manifold, topology unknown

→ **trivial, decaying modes (infinitely many)**

→ **physical, entangled modes (e.g. N = 41)**

Extensivity of effective degrees of freedom:

Dependence of Lyapunov spectrum on system length L:



Domination of Oseledec Splitting (DOS)

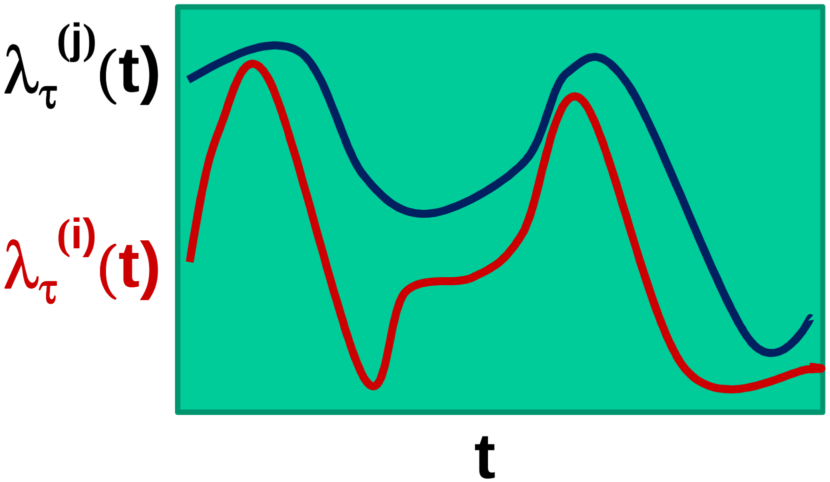
j-th finite time Lyapunov exponent:
$\lambda_\tau^{(j)}(x) = 1/\tau \log(\Gamma^{(j)}(x,\tau))$

DOS: for j < i there exists τ_0 s.t. for $\tau > \tau_0$

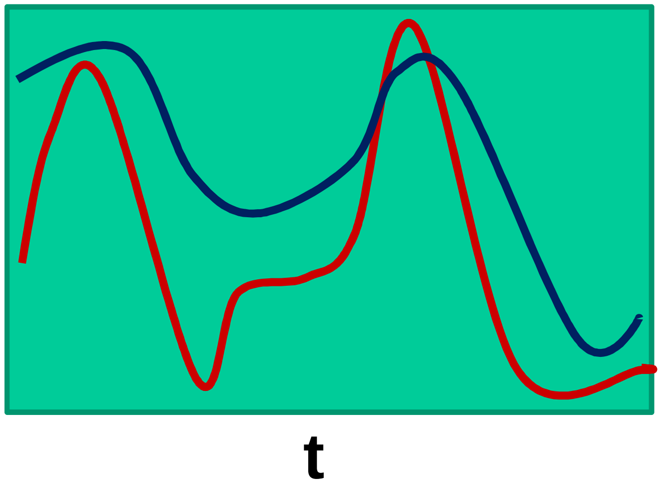
$\lambda_\tau^{(j)}(x(t)) > \lambda_\tau^{(i)}(x(t))$ **for all t**

i.e. for $\tau > \tau_0$ finite time Lyapunov exponents always in "correct" order:

DOS:



DOS violation:



Measuring how often DOS is violated (j < i):

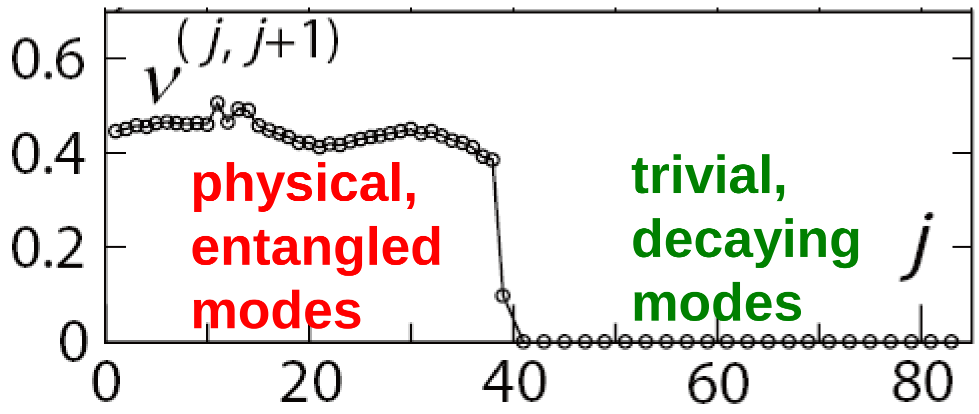
$\lambda_\tau^{(i)}(t) - \lambda_\tau^{(j)}(t) < 0$ for all t → DOS

$\lambda_\tau^{(i)}(t) - \lambda_\tau^{(j)}(t) > 0$ for some t → DOS violation

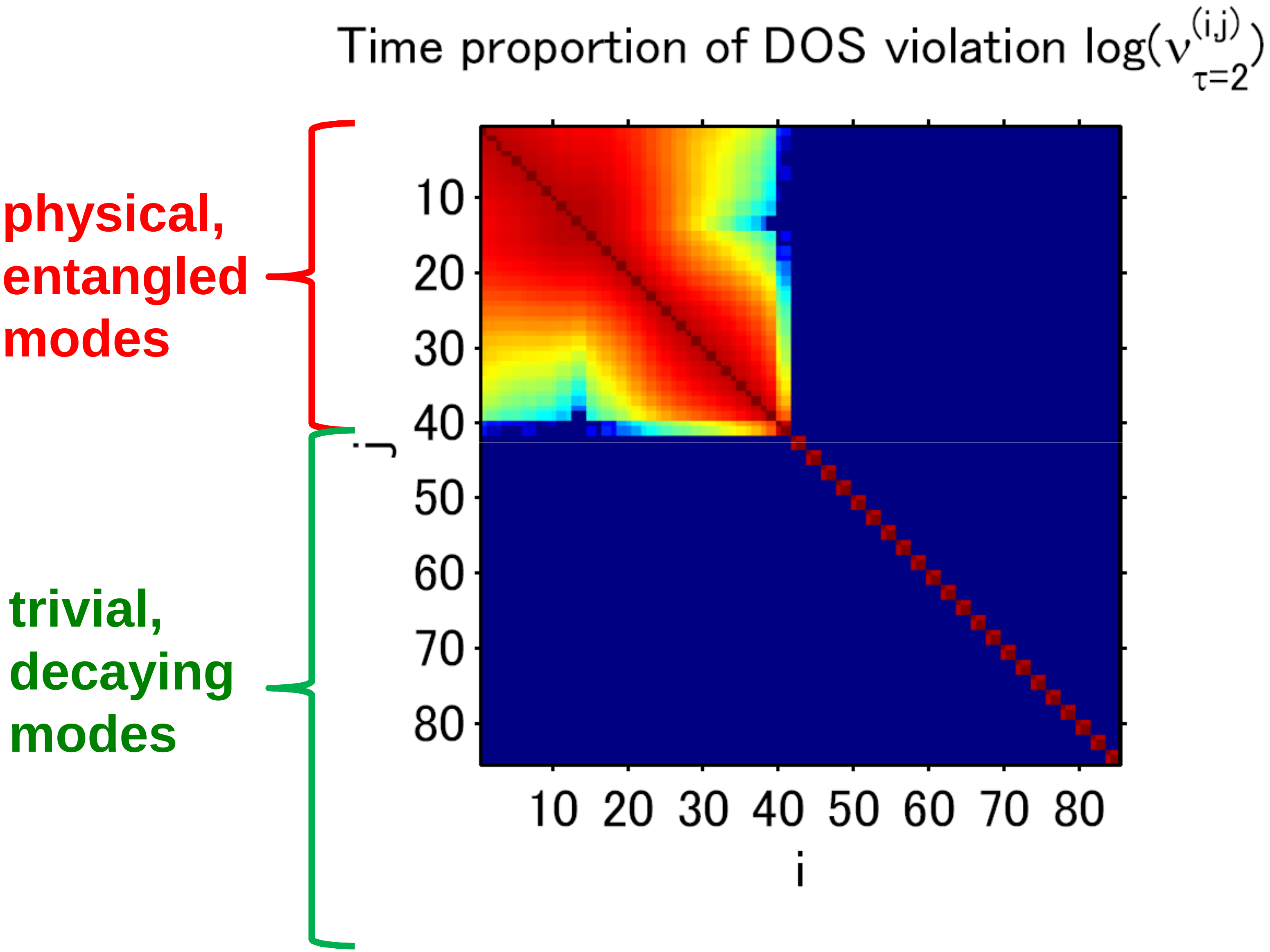
fraction of DOS violation: $\upsilon_\tau^{(j,i)} = \langle \theta(\lambda_\tau^{(i)}(t) - \lambda_\tau^{(j)}(t)) \rangle$

$\langle \dots \rangle$ = time average, $\theta(.)$ = Heaviside step function

DOS violation DOS



$\tau = 0.2$



Time proportion of DOS violation log($\nu^{(i,j)}_{\tau=2}$)
physical, entangled modes
trivial, decaying modes
10
20
30
40
50
60
70
80
j
10 20 30 40 50 60 70 80
i

How general are these findings?

Complex Ginzburg - Landau (CGL) Equation

$$\partial_t W = W - (1 + i\beta)|W|^2 W + (1 + i\alpha)\partial_x^2 W$$

$W(x,t)$ complex field in 1d space

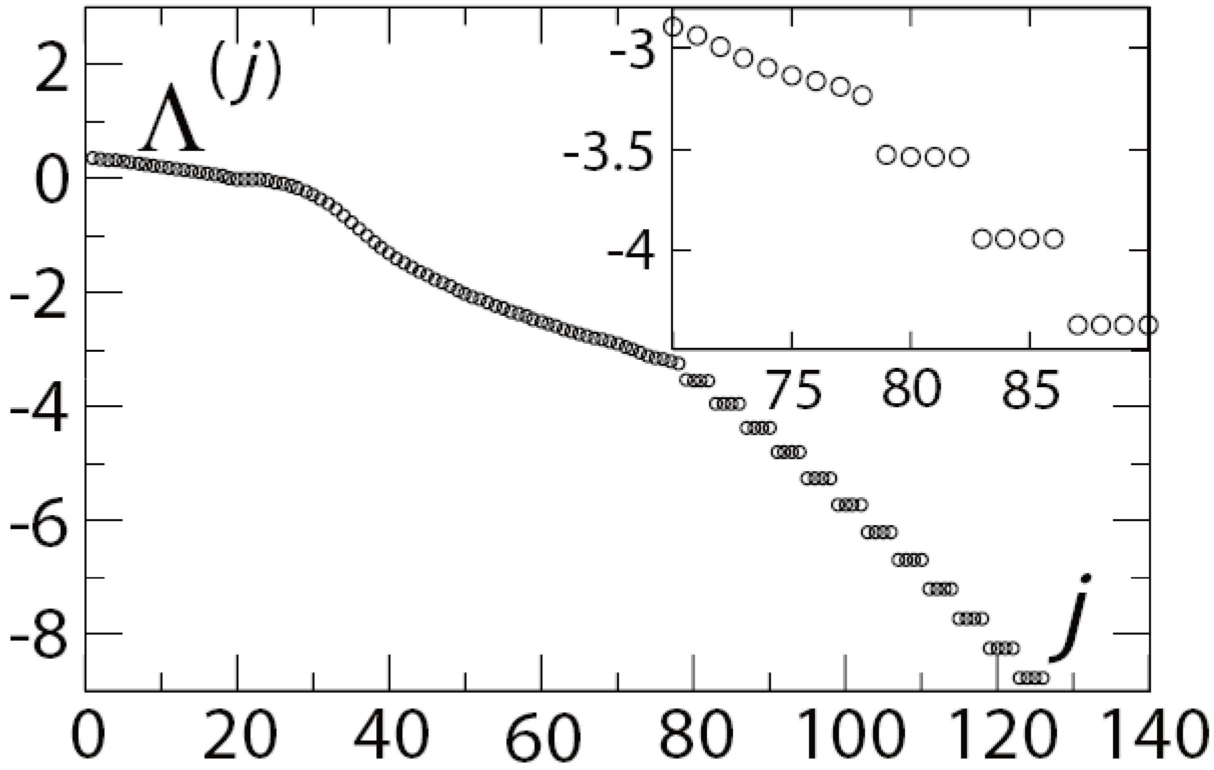
Standard model of space-time chaos

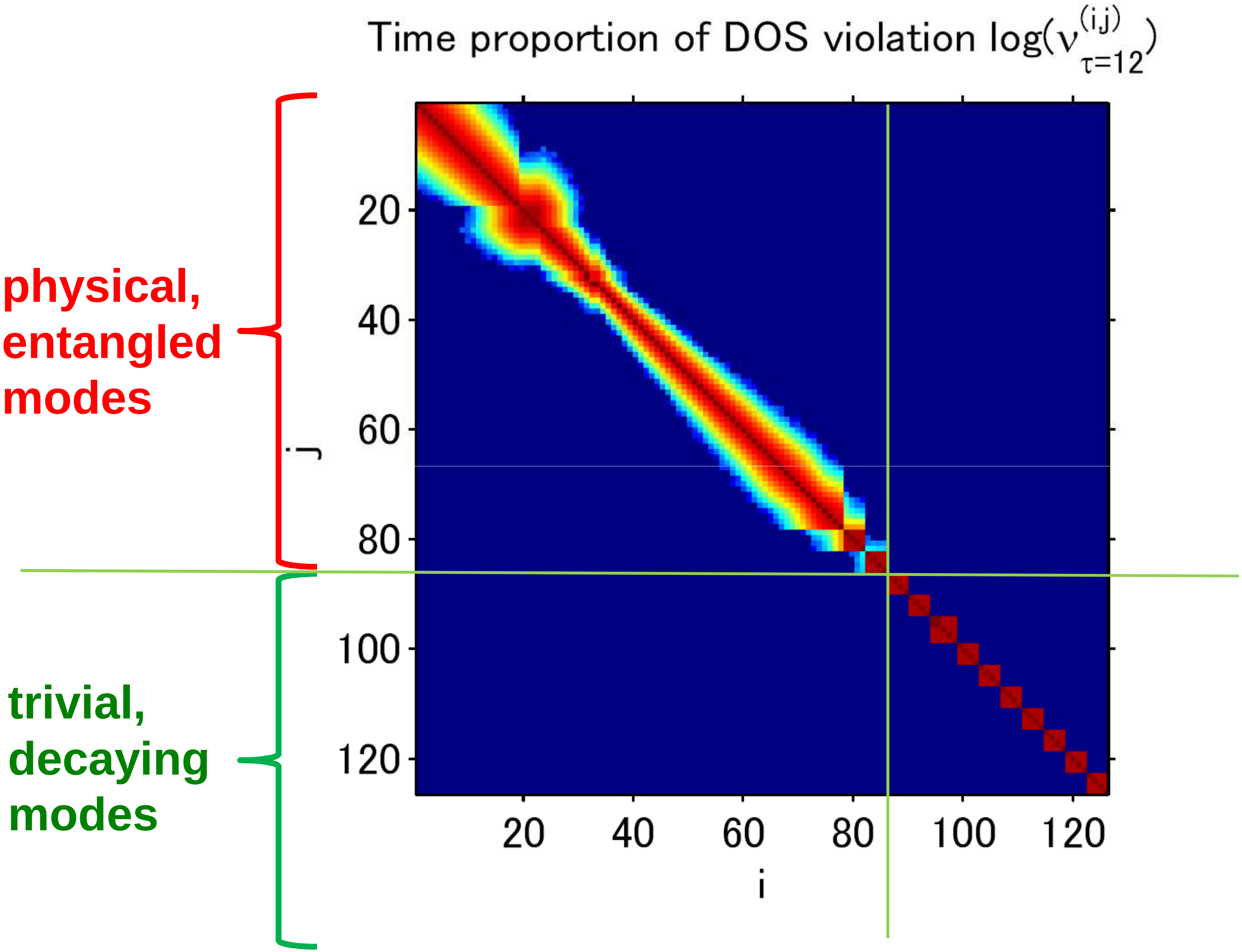
Parameters:

$\alpha = -2.0, \beta = 3.0, L = 64.$

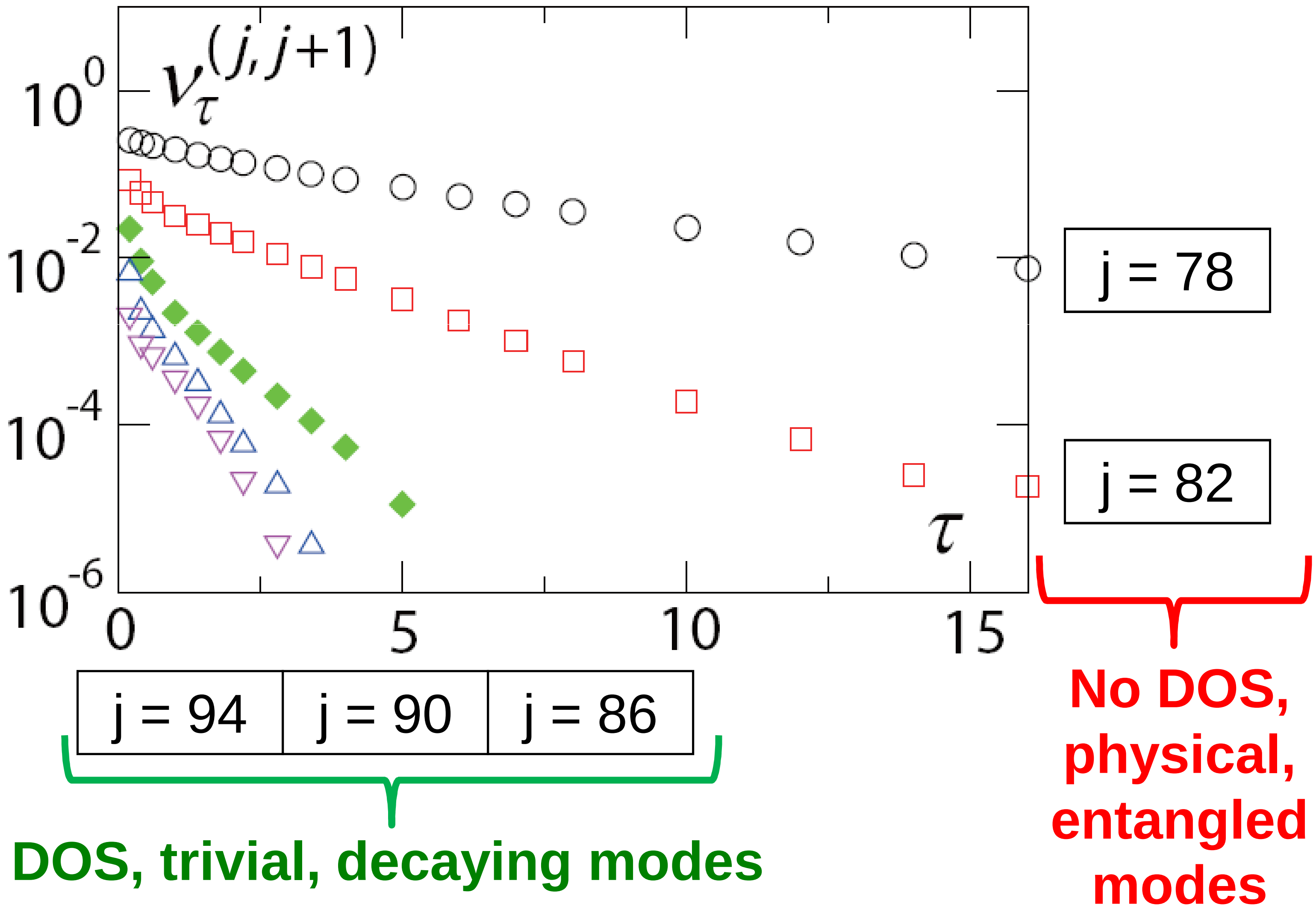
Regime of amplitude turbulence

Lyapunov spectrum:




Time proportion of DOS violation log($\nu_{\tau=12}^{(i,j)}$)
physical, entangled modes
trivial, decaying modes
j
i
20
40
60
80
100
120

τ-dependence of fraction of DOS violation (CGLE):



1. **CLVs provide a method to distinguish between physically relevant and irrelevant modes in tangent space of dissipative extended systems**

2. **relevant modes:**
   - **entangled**
   - **stable and unstable directions**
   - **finite time Lyapunov exponents strongly fluctuating (DOS violated)**
   - **trace inertial manifold → finite number of degrees of freedom**
   - **extensive**

3. **irrelevant modes:**
   - **hyperbolically isolated (DOS fulfilled)**
   - **purely decaying**
   - **infinitely many**