Lyapunov Modes in Extended Dynamical Systems

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Lyapunov Modes:

• Generalization of (phonon) normal modes to chaotic systems
• Linearized motion in neighborhood of chaotic trajectory
• Hydrodynamic Lyapunov Modes (HLM): Slow, long wave-length behavior
• Objects of Hamiltonian nonlinear dynamics fundamental to (non-equilibrium) statistical physics?
• This talk: Extended dissipative systems, existence of finite number of physical, entangled modes
2 Types of Lyapunov Vectors:

• **Orthogonal Lyapunov Vectors (OLV):**
  Studied in many extended systems (since 2000: Hard spheres, Lennard-Jones fluids, WCA fluids, Coupled map lattices, PDEs (KS equation), Dynamic XY model, FPU models, Posch, Morriss, Yang, G.R., …)

• **Covariant Lyapunov Vectors (CLV):**
  Numerically accessible since 2007*

Dynamics in tangent space: OLV

Lyapunov exponents $\lambda^{(\alpha)}$: possible growth rates
$\delta x(t) \sim \exp(\lambda^{(\alpha)} t) \delta x(0)$, ordered $\lambda^{(1)} > \lambda^{(2)} > \lambda^{(3)}$ ...

OLV = Orthogonal Lyapunov vectors $\delta x^{(\alpha)}(t)$: dynamics of orthonormal frame $\delta x^{(\alpha)}(t)$, $\alpha = 1,\ldots,2dN$, from repeated Gram-Schmidt-reorthogonalization or QR decomposition

Property: $k$-dimensional parallel-epipeds align asymptotically with the space spanned by the $k$ first Lyapunov vectors and its volume growth rate is $\lambda^{(1)} + \lambda^{(2)} + \ldots + \lambda^{(k)}$
Dynamics in tangent space: CLV

reference trajectory: flow \( x(t) = \Phi(t)[x(0)] \)

perturbation: \( \delta x(t) = D\Phi(t)[x(0)] \delta x(0) \)

CLV = Covariant Lyapunov vectors \( e^{(\alpha)}(x) \):

\[
D\Phi(t)[x] e^{(\alpha)}(x) = \Gamma^{(\alpha)}(x,t) e^{(\alpha)}(\Phi(t)[x]),
\]

stretching factor \( \Gamma^{(\alpha)}(x,t) \)

\[
\lim_{t \to \infty} \frac{1}{t} \log(\Gamma^{(\alpha)}(x,t)) = \lambda^{(\alpha)}(x) \quad \alpha\text{-th Lyapunov exponent}
\]

\( e^{(\alpha)}(x) \) span Oseledec subspaces
Mather decomposition: CLV

CLV = Covariant Lyapunov vectors $e^{(\alpha)}(x)$:

Decomposition of fundamental matrix (Mather spectrum):

$D\Phi^{(t)}[x] = \sum_{\alpha} e^{(\alpha)}(\Phi^{(t)}[x]) \Gamma^{(\alpha)}(x,t) f^{(\alpha)}(x)^T$

stretching factor $\Gamma^{(\alpha)}(x,t)$

$\lim_{t \to \infty} \frac{1}{t} \log(\Gamma^{(\alpha)}(x,t)) = \lambda^{(\alpha)}(x)$ \(\alpha\)-th Lyapunov exponent

e^{(\alpha)}(x) span Oseledec subspaces

$f^{(\alpha)}(x)$ adjoint basis

Biorthogonal sets: $f^{(\alpha)}(x)^T e^{(\beta)}(x) = \delta_{\alpha\beta}$ and $\sum_{\alpha} e^{(\alpha)}(x) f^{(\alpha)}(x)^T = 1$
CLV, inertial manifolds, effective degrees of freedom of dissipative extended systems:

Central result:
Lyapunov modes split into 2 groups:
1. infinitely many modes with included angles bounded away from zero, associated Lyapunov exponents negative (decaying perturbations, trivial modes)
2. finite number of modes with repeatedly vanishing included angles, associated Lyapunov exponents positive and negative (non-decaying perturbations, physical modes), “surface” of inertial manifold (IM)

Kuramoto-Sivashinsky (KS) Equation

$$\partial_t u + \partial^2_x u + \partial^4_x u + u \partial_x u = 0$$

L = 133.12

dynamics of solution $u(x,t)$
CLVs for KS system:

Lyapunov spectrum, $L = 133.12$
Static structure factor $S^{(j)}(k)$ of CLVs

$S^{(j)}(k)$

$\frac{k_{\text{peak}}}{L} = 96$

fraction of DOS violation

$j = 1, 16, 32, 38, 44, 52, 60, 68, 76, 84$

$L = 96$
Distributions of **included angles** of CLVs

Finite probability of included angles near-zero

No tangencies

Critical case

$$\rho(\theta) \sim \exp(-\text{const.} / \theta)$$
Matrix of \textbf{minimum included angles} of CLVs

\[ \cos(\theta_{\min}^{(i,j)}) \]

\begin{itemize}
\item physical, entangled modes
\item trivial, decaying modes
\end{itemize}
Schematic picture:

- trivial, decaying modes (infinitely many)
- physical, entangled modes (e.g. N = 41)

“IM”, physical or entangled manifold, topology unknown
Extensivity of effective degrees of freedom:

Dependence of Lyapunov spectrum on system length $L$:

- $\Lambda^{(j)}$: Lyapunov exponent
- $L$: system length
- KY dimension
- Metric entropy (x 50)
- # effective DOF, physical modes
Domination of Oseledec Splitting (DOS)

\(j\)-th finite time Lyapunov exponent:
\[
\lambda_\tau^{(j)}(x) = \frac{1}{\tau} \log(\Gamma^{(j)}(x,\tau))
\]

DOS: for \(j < i\) there exists \(\tau_0\) s.t. for \(\tau > \tau_0\)
\[
\lambda_\tau^{(j)}(x(t)) > \lambda_\tau^{(i)}(x(t)) \text{ for all } t
\]

i.e. for \(\tau > \tau_0\) finite time Lyapunov exponents always in “correct” order:

DOS:

\[
\lambda_\tau^{(j)}(t) \quad \lambda_\tau^{(i)}(t)
\]

DOS violation:

\[
\lambda_\tau^{(j)}(t) \quad \lambda_\tau^{(i)}(t)
\]
Measuring how often DOS is violated \((j < i)\):

\[
\lambda^{(i)}_{\tau}(t) - \lambda^{(j)}_{\tau}(t) < 0 \quad \text{for all } t \rightarrow \text{DOS}
\]

\[
\lambda^{(i)}_{\tau}(t) - \lambda^{(j)}_{\tau}(t) > 0 \quad \text{for some } t \rightarrow \text{DOS violation}
\]

fraction of DOS violation: \(\nu^{(j,i)}_{\tau} = \langle \theta(\lambda^{(i)}_{\tau}(t) - \lambda^{(j)}_{\tau}(t))\rangle\)

\(\langle \ldots \rangle = \text{time average}, \theta(.) = \text{Heaviside step function}\)
physical, entangled modes

trivial, decaying modes
How general are these findings?

Complex Ginzburg - Landau (CGL) Equation

$$\partial_t W = W - (1 + i\beta)|W|^2W + (1 + i\alpha)\partial_x^2 W$$

$W(x, t)$ complex field in 1d space

Standard model of space-time chaos

Parameters:

$$\alpha = -2.0, \beta = 3.0, L = 64$$

Regime of amplitude turbulence
Lyapunov spectrum:
physical, entangled modes

trivial, decaying modes
\( \tau \)-dependence of fraction of DOS violation (CGLE):

\[ V_{\tau}(j, j+1) \]

- For \( j = 78 \):
  - No DOS, physical, entangled modes

- For \( j = 82 \):
  - No DOS, physical, entangled modes

- For \( j = 90 \), \( j = 94 \), \( j = 86 \):
  - DOS, trivial, decaying modes

**Diagram:**
- Axes: \( \tau \) (horizontal) and \( V_{\tau}(j, j+1) \) (vertical)
- Data points for different values of \( j \)
- Legend indicating different modes
1. CLVs provide a method to distinguish between physically **relevant** and **irrelevant** modes in tangent space of dissipative extended systems

2. **relevant** modes:
   - entangled
   - stable and unstable directions
   - finite time Lyapunov exponents strongly fluctuating (DOS violated)
   - trace inertial manifold $\rightarrow$ finite number of degrees of freedom
   - extensive

3. **irrelevant** modes:
   - hyperbolically isolated (DOS fulfilled)
   - purely decaying
   - infinitely many