The Extended Kalman Filter and its Reduction to the Unstable Subspace

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Abstract

We consider the application of the Extended Kalman Filter to a chaotic system with stable, neutral and unstable manifolds of dimension given by the numbers N^+, N^0 and N^- of positive, null and negative Lyapunov exponents respectively. We show that the rank of the EKF covariance matrix, initially equal to the total number of degrees of freedom of the system, asymptotically reduces to the dimension of the unstable and neutral subspace. In a reduced form of the algorithm (Extended Kalman Filter with Assimilation in the Unstable Subspace, EKF-AUS), the assimilation is confined to the unstable and neutral directions of the tangent space. When the observations are sufficiently dense and accurate that filter divergence does not occur, the EKF and its reduced form EKF-AUS converge to the same solution.

Extended Kalman Filter

Trajectory

forecast: $\mathbf{x}^f_k = \mathcal{M}(\mathbf{x}^a_{k-1})$

analysis: $\mathbf{x}^a_k = \mathbf{x}^f_k - \mathbf{K}_k\mathcal{H}(\mathbf{x}^f_k) + \mathbf{K}_k\mathbf{y}^o_k$,

Error covariance matrices

forecast: $\mathbf{P}^f_k = \mathbf{M}_k\mathbf{P}^a_{k-1}\mathbf{M}^T_k + \mathbf{Q}_k$

analysis: $\mathbf{P}^a_k = (\mathbf{I} - \mathbf{K}_k\mathbf{H})\mathbf{P}^f_k$

where $\mathcal{M}$ is the nonlinear evolution operator and $\mathbf{M}$ is the Jacobian of $\mathcal{M}$.

$\mathbf{K}_k$ is the *gain matrix*

$$\mathbf{K}_k = \mathbf{P}^f_k\mathbf{H}^T\left(\mathbf{H}\mathbf{P}^f_k\mathbf{H}^T + \mathbf{R}\right)^{-1},$$

$\mathbf{H}$ is the Jacobian of $\mathcal{H}$, the measurement operator:

THE ALGORITHM (EKF and EKF-AUS)

We perform the assimilation in a manifold of dimension m. When m is equal to the number n of degrees of freedom of the system, the algorithm solves the standard EKF equations. When $m = N^+ + N^0$ the reduced form, with Assimilation in the Unstable Subspace (EKF-AUS) is obtained.

We set $\mathbf{Q} = 0$ (no model error). The analysis error covariance matrix is expressed as the product of two Cholesky factors

$$\mathbf{P}^a = \mathbf{X}_a\mathbf{X}^T_a$$

with $\mathbf{X}_a = [\mathbf{x}^a_1, \mathbf{x}^a_2, ..., \mathbf{x}^a_m]$ (Initially, $\mathbf{x}_i$ are random *perturbations*.)

Forecast step

The tangent linear operator $\mathbf{M}$ acts on the columns of $\mathbf{X}_a$ (the *perturbations*)

$$\mathbf{P}^f = \mathbf{M}\mathbf{X}_a(\mathbf{M}\mathbf{X}_a)^T = \mathbf{X}_f\mathbf{X}^T_f$$

The forecast perturbations are orthonormalized $\mathbf{x}^f_i \to \mathbf{e}^f_i$ by means of G.S. algorithm

$\mathbf{X}_f = [\mathbf{x}^f_1, \mathbf{x}^f_2, ..., \mathbf{x}^f_m]$, $\quad \mathbf{E}_f = [\mathbf{e}^f_1, \mathbf{e}^f_2, ..., \mathbf{e}^f_m]$

The forecast error covariance matrix is cast in the form

$$\mathbf{P}^f = \mathbf{X}_f\mathbf{X}^T_f = \mathbf{E}_f\mathbf{E}^T_f\mathbf{X}_f\mathbf{X}^T_f\mathbf{E}_f\mathbf{E}^T_f = \mathbf{E}_f\mathbf{\Gamma}_f\mathbf{E}^T_f$$

where $\mathbf{\Gamma}_f = \mathbf{E}^T_f\mathbf{X}_f\mathbf{X}^T_f\mathbf{E}_f$

Analysis step

$\mathbf{K} = \mathbf{E}_f\mathbf{\Gamma}_f\mathbf{E}^T_f\mathbf{H}^T\left(\mathbf{H}\mathbf{E}_f\mathbf{\Gamma}_f\mathbf{E}^T_f\mathbf{H}^T + \mathbf{R}\right)^{-1}$

$\mathbf{P}^a \equiv \mathbf{E}_f\mathbf{\Gamma}'_a\mathbf{E}^T_f = (\mathbf{I} - \mathbf{K}\mathbf{H})\mathbf{E}_f\mathbf{\Gamma}_f\mathbf{E}^T_f$

Collecting $\mathbf{E}_f$ and $\mathbf{E}^T_f$ from left and right

$\mathbf{\Gamma}'_a = \mathbf{\Gamma}_f - \mathbf{\Gamma}_f\mathbf{E}^T_f\mathbf{H}^T(\mathbf{R} + \mathbf{H}\mathbf{E}_f\mathbf{\Gamma}_f\mathbf{E}^T_f\mathbf{H}^T)^{-1}\mathbf{H}\mathbf{E}_f\mathbf{\Gamma}_f$

$\mathbf{P}^a \equiv \mathbf{E}_f\mathbf{\Gamma}'_a\mathbf{E}^T_f = \mathbf{E}_f\mathbf{U}\mathbf{\Gamma}_a\mathbf{U}^T\mathbf{E}^T_f$

where $\mathbf{U}$ is orthogonal and diagonalizes $\mathbf{\Gamma}'_a$ in $\mathbf{\Gamma}_a = \mathrm{diag}[\gamma^2_i]$.

$$\mathbf{E}_f\mathbf{U} = \mathbf{E}_a = [\mathbf{e}^a_1, \mathbf{e}^a_2, ..., \mathbf{e}^a_m]$$

thus we have $\mathbf{P}^a = \mathbf{E}_a\mathbf{\Gamma}_a\mathbf{E}^T_a = \mathbf{X}_a\mathbf{X}^T_a$. $\mathbf{X}_a = [\mathbf{x}^a_1, \mathbf{x}^a_2, ..., \mathbf{x}^a_m]$ where $\mathbf{x}^a_i$ are the new (orthogonal) perturbations.

It can be shown that, asymptotically, $\mathbf{x}^a_i$ and $\mathbf{x}^f_i$ span the same subspace as the leading $(N^+ + N^0)$ Lyapunov vectors (estimated errors in the stable subspace decay and only errors in the unstable and neutral subspace survive the filtering process).

Model and observations

The Lorenz 96 model equations are:

$$\dot{x}_j = (x_{j+1} - x_{j-2})x_{j-1} - x_j + F$$

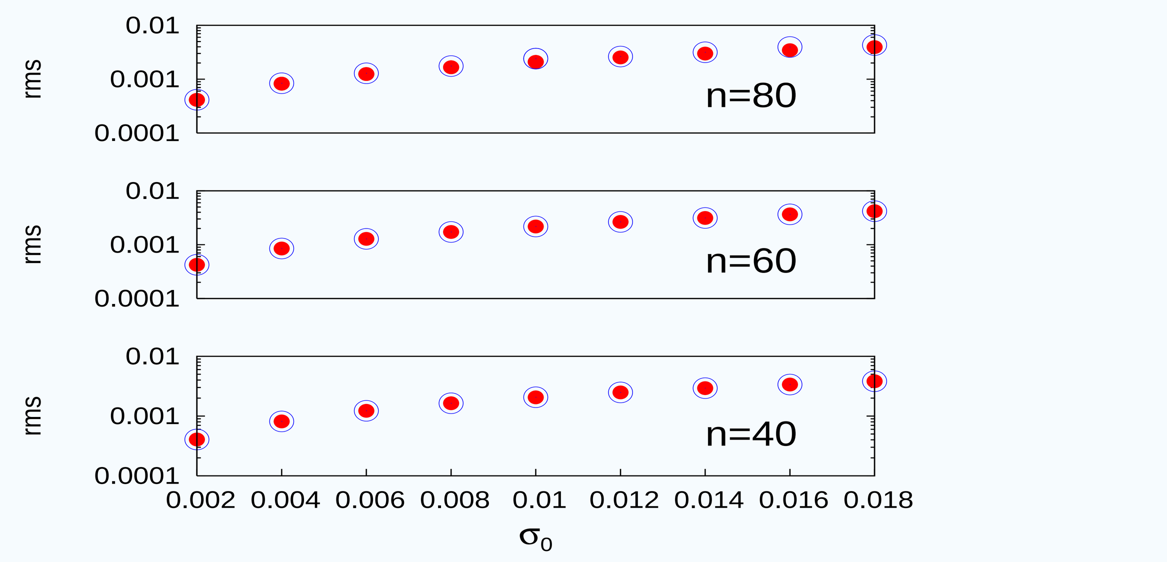
with $j = 1, ..., n$. x_j represent the values of a scalar meteorological quantity at n grid points on a periodic longitudinal domain. The model has chaotic behavior for the value of the forcing, $F = 8$. For $n = 40, 60, 80$ the model presents $13, 19, 26$ positive Lyapunov exponents and a null exponent. We perform the simulation in a perfect model scenario (null model error). We evolve a *true* trajectory, $\mathbf{x}^t_k$ and an analysis trajectory $\mathbf{x}^a_k$. The observations are obtained from the true values with the equation:

$$\mathbf{y}^0_k = \mathcal{H}(\mathbf{x}^t_k) + \sigma_0\eta_k,$$

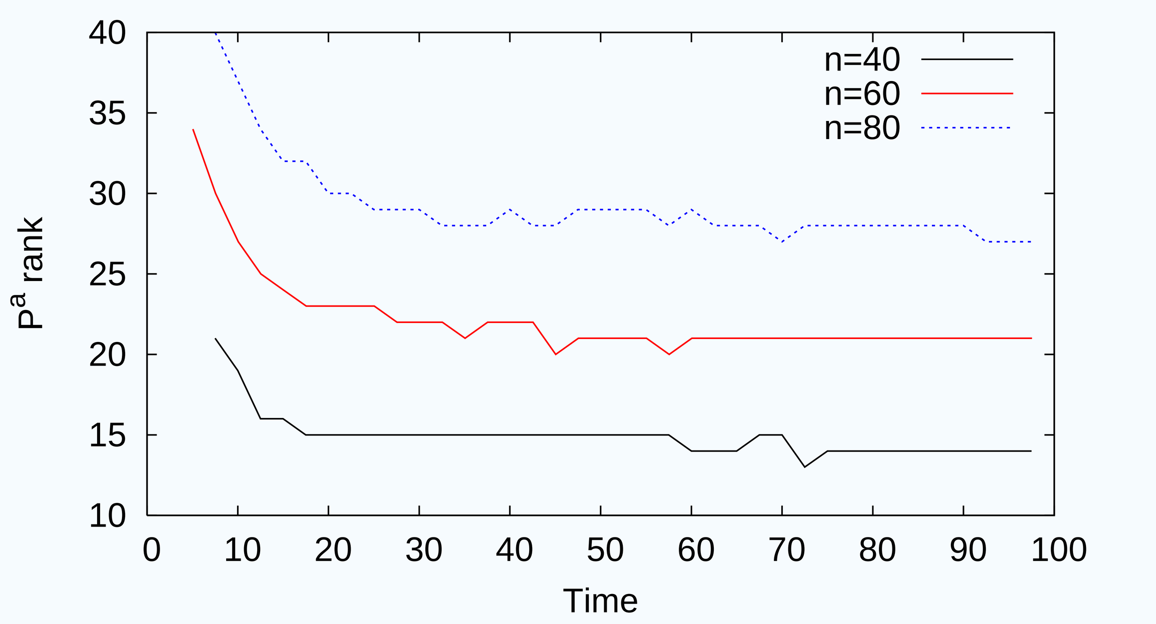
with η_k a white Gaussian noise with zero mean and variance one. Every other grid point is observed with a time interval of $t_{k+1} - t_k = 0.05$ and the observation points are shifted by one at each observation time.

Results of the application of EKF and EKF-AUS

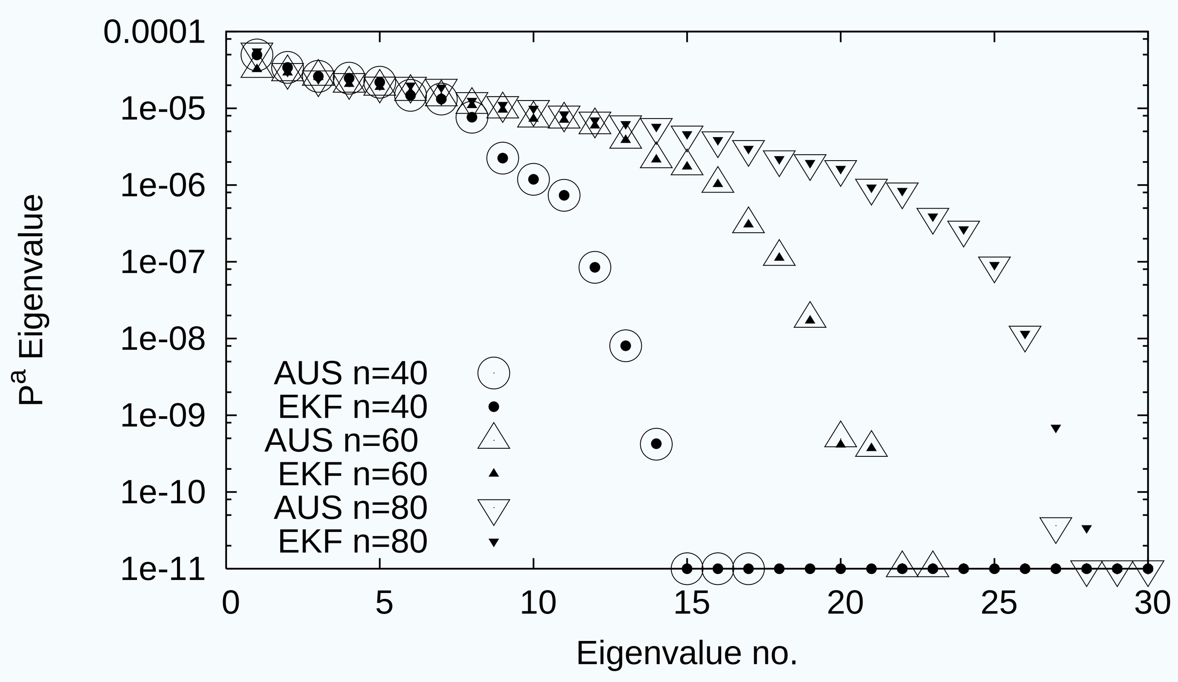
Varying the number, n, of degrees of freedom of the model, systems with different stable and unstable subspace dimension are obtained.



a) *Average (over 1000 iterations) error $\|\mathbf{x}^t_k - \mathbf{x}^a_k\|/\sqrt{n}$ for EKF (open) and EKF-AUS (full circles) as a function of the observation error, σ_0.*



b) *Rank of $\mathbf{P}^a$ in the standard EKF. The $\mathbf{P}^a$ rank decays to a value approximately equal to the dimension $N^+ + N^0$ of the unstable and neutral subspace of the three systems. With $n = 40, 60, 80$ degrees of freedom $N^+ + N^0 = 14, 20, 27$, respectively.*



c) *Asymptotic error covariance matrix for EKF and EKF-AUS. Numerical values of $\mathbf{P}^a$ and of $\mathbf{\Gamma}_a$ eigenvalues: all non-zero eigenvalues are practically identical.*

Conclusions

- We have shown that the full EKF and its reduced form EKF-AUS lead to the same results; average error (Fig. a) and $\mathbf{P}^a$ eigenvalues (Fig. c) are the same.
- The rank of $\mathbf{P}^a$ and $\mathbf{P}^f$ decays to $N^+ + N^0$ in the standard EKF calculation (Fig. b). Only errors in the unstable and neutral subspace survive the filtering process.
- An implication of the present results is that, provided errors behave linearly, $N^+ + N^0$ is the necessary and sufficient number of members needed to recover the full EKF results with an Ensemble KF.
- In systems with $N^+ + N^0 \ll n$ the reduced EKF-AUS algorithm can significantly alleviate the computational cost of the full EKF.

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