POINCARÉ RECURRENCE TIME STATISTICS IN
CHAOTIC DYNAMICAL SYSTEMS

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Figure 1: Recurrence to the region \( I \) of a trajectory in the phase space.
**Figure 1**: Recurrence to the region $I$ of a trajectory in the phase space.

Poincaré Recurrence Theorem (1890): trajectories of a confined Hamiltonian system will repeatedly return to $I$. 
Recurrences

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- Poincaré Recurrence Theorem (1890): trajectories of a confined Hamiltonian system will repeatedly return to $I$.
- Statistical Mechanics Debate:
  - Zermelo’s (1896) Recurrence Paradox.
  - Boltzmann: mean rec. time of a thermodynamical system is huge.
The mean RT is given by Kac's Lemma:

$$ h_i = \frac{1}{I} \left( I_i \right) $$

Recently the RT statistics has been connected to multi-fractal analysis, anomalous transport [Zaslavksy, Phys. Reports 2002] and time series analysis [M.S.Baptista et al., Phys. of Plasma 2001]. Special interest to the distribution of the $r$th recurrence times $T_i$: 

$$ P(T_i; r; \delta) $$
Recurrence Time (RT) Statistics

The mean RT is given by Kac's Lemma:

\[ \langle T \rangle = \frac{1}{I} \sum T_i \]

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To characterize the time properties of non-linear dynamical systems we explore the Poincaré recurrence time (RT) distribution. The informations obtained are compared with other time quantifiers.
Purpose of this work

To characterize the time properties of non-linear dynamical systems we explore the Poincaré recurrence time (RT) distribution. The informations obtained are compared with other time quantifiers.

**Dynamical Systems:**

- Random Process.
- Logistic Map.
- Chaotic Attractors.
- Pomeau-Manneville intermittency.
- Crisis-induced intermittency.
- Hamiltonian intermittency.
Suppose a memoryless dynamical process with an invariant measure. Since the events are completely independent each recurrence \( \Box \) has a fixed probability \( \mu \).

\( \Box \square \Box \square \ldots \Box \)

\( T \) observations \( \square \) and \( r \) events \( X \)
Suppose a memoryless dynamical process with an invariant measure. Since the events are completely independent each recurrence has a fixed probability \( \mu \).

\[ P(T; r, \mu) = \frac{(T - 1)!}{(T - r)!(r - 1)!} \mu^r (1 - \mu)^{T-r} \]

This is a kind of Bernoulli-Trial problem and the solution is a Binomial-like distribution:
The Poisson-like Distribution

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The first RT ($r = 1$) is given by the Poissonian distribution:

$$P(T; 1, \mu) = \mu (1 - \mu)^{T-1} \rightarrow \mu e^{-\mu T} \text{ when } (\mu \to 0)$$
The Binomial distribution is valid for random systems and deterministic systems with strong mixing properties (such as Anosov system). It is illustrated here by a Gaussian random walk.
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The invariant measure.
The Logistic map with $b=4$

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The RT distribution
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The RT distribution

Short Time memory effect \( n^* \approx 10 \gg \frac{1}{\lambda} \approx 1.4 \)
The Logistic map with $b=4$

How about the fully chaotic logistic map? $[x_{n+1} = 4x_n(1 - x_n)]$

The RT distribution

For details see: [Altmann et al. nlin.CD/030427]
\[ \begin{align*}
\dot{x} &= \alpha \cdot (y - x - k(x)), \\
\dot{y} &= x - y + z, \\
\dot{z} &= -\beta \cdot y,
\end{align*} \]

\[ k(x) = \begin{cases} 
bx + (a - b) & \text{if } x \geq 1 \\
bx - (a - b) & \text{if } x \leq -1 \\
ax & \text{if } |x| < 1
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\[ \alpha = 11.30 > \alpha_c = 11.2834 \]
RT Statistics in the Rössler attractor

For $\alpha < \alpha_c$:

- Short time memory effect $T \in [5, 15]$.
- Effect of the autocorrelation $T \in [20, 30]$.
RT statistics in the Double Scroll attractor

For $\alpha \gtrapprox \alpha_c$:

- Double exponential decay associated to the movement in each “Scroll”.
- Characteristic time $T^*$. 

$T^*$
This is the same relation deduced in [Grebogi et al. PRA (87)] for the residence time in crisis induced intermittency.
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\[ T^* = \frac{N}{(\alpha - \alpha_c)^\lambda} \]
Manneville map, a type-II intermittency system:

\[ x_{n+1} = (1 + \varepsilon)x_n + (1 - \varepsilon)x_n^2 \ (mod \ 1) \]
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$$x_{n+1} = (1 + \varepsilon)x_n + (1 - \varepsilon)x_n^2 \pmod{1}$$

**Power-law** ($\gamma \approx 2$) until $T \approx 1/\varepsilon$.

**Exponential** ($b \approx \varepsilon$)

Equivalent to the statistics of laminar phases.
Standard Map (on the torus), a typical near-integrable Hamiltonian system:

\[
\begin{align*}
y_{n+1} &= y_n - K \sin(2\pi x_n) \mod(1) \\
x_{n+1} &= x_n + y_{n+1} \mod(1),
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\]
Hamiltonian Systems intermittency

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Why Intermittency?
Stochastic sea → near island → St. sea . . .
↓  ↓
Chaos → ~ Regular → Chaos → . . .
Hamiltonian Systems RT statistics

- Short time memory effect.
- Exponential decay.
- Power-law tail.
- see [G. Zaslavsky Phys. Reports (2002)]
All the information about the deterministic dynamics of a system is found in deviations of the Binomial-like RT distribution.
Conclusions

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III The recurrence time (RT) statistics captures the essential temporal aspects of intermittent systems and generalize the “laminar” phase statistics.
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THANK YOU