Entanglement spectrum of the 2D Bose-Hubbard model

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Outline

- Introduction.
- Entanglement spectrum (ES).
- Entanglement spectrum in the 2D Bose-Hubbard.
- In the Mott insulator:
  - ES is **boundary local**.
  - **Perturbative** structure.
  - Entanglement **dispersions**.
  - Effective boundary **entanglement Hamiltonian**.
- In the superfluid phase:
  - **Tower of states** structure (symmetry breaking).
  - Entanglement **gap**.
- Entanglement **entropies** and the **gap**.

Introduction

- Consider a quantum system in d dimensions in the ground state $|\Psi\rangle$.

$$\rho \equiv |\Psi\rangle\langle\Psi|$$

- If the system is bipartite:

$$H = H_A \otimes H_B \rightarrow \rho_A = Tr_B \rho$$

- How to quantify the entanglement (quantum correlations) between A and B?
  - von Neumann entropy $S_A = -Tr\rho_A \log \rho_A = -\sum_i \lambda_i \log \lambda_i$
- Area law: away from criticality the entanglement is proportional to the surface area of subsystem A

$$\boxed{S_A \approx L^{d-1}} \qquad (1D \quad S \rightarrow const)$$

- For critical systems violations of the area law [Vidal et al,2003][Calabrese, Cardy,2004]

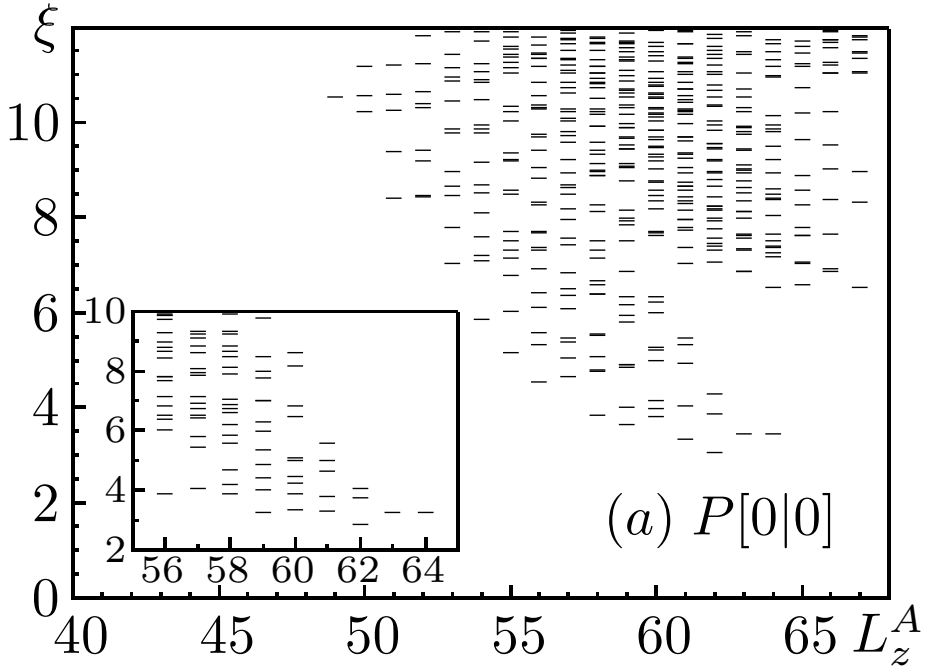
$$1\text{D} \rightarrow \boxed{S_A = \frac{c}{3} \log L}$$

Entanglement spectrum (ES)

- Given the reduced density matrix ρ_A:

$$\{\lambda\} = \sigma(\rho_A) \quad \Rightarrow \quad \rho_A = e^{-\mathcal{H}_A} \quad \Rightarrow \quad \boxed{\sigma(\mathcal{H}_A) \equiv -\log(\{\lambda\})}$$

- Quantum Hall: the ES retains all the features of the critical edge modes (ES$\rightarrow$ edge spectra correspondence) [Li,Haldane, 2008]

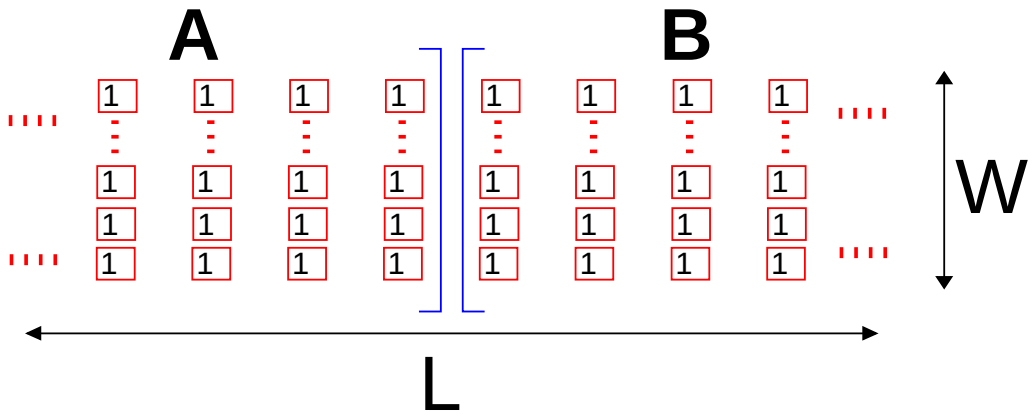


- More general: all the relevant information to describe the physics of a system is encoded in the entanglement spectrum.

2D Bose-Hubbard on a cylinder

- $2D$ Bose-Hubbard at unit filling on a cylinder.

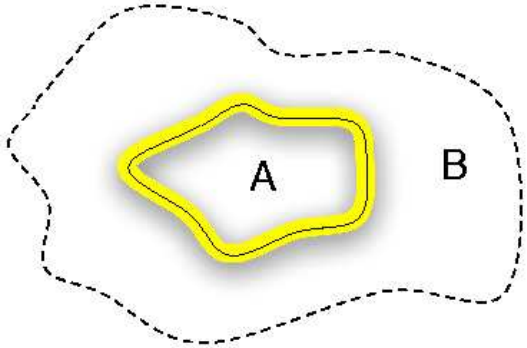
$$\mathcal{H} = -\sum_{\langle ij \rangle} (b_i^\dagger b_j + h.c.) + \frac{U}{2} \sum_i n_i (n_i - 1)$$



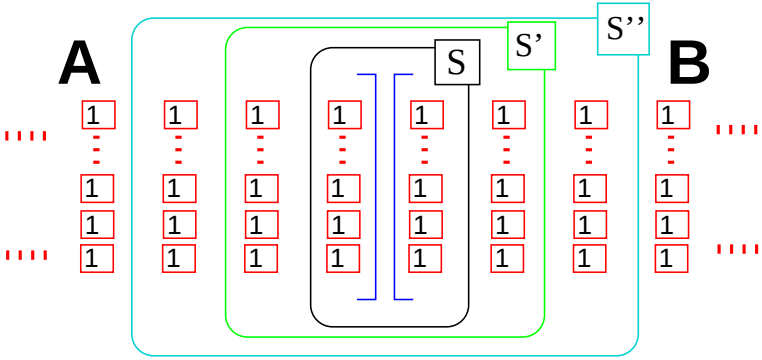
[Fisher,Weichman,Grinstein,Fisher,1989]
[Jaksch,Bruder,Cirac,Gardiner,1998]

Boundary locality of the ES in gapped systems

- Gapped systems → area law (in $1D$ [Hastings,2007]).



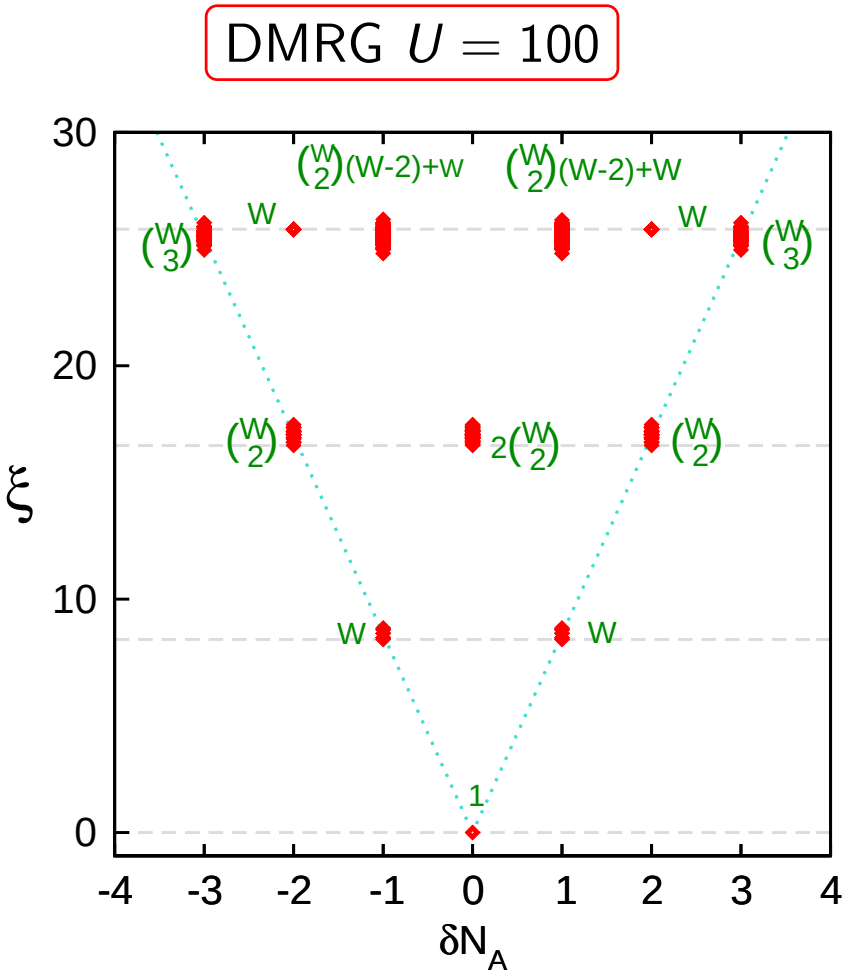
- The ES is a boundary local quantity. [V.A.,M. Haque,A.M. Läuchli,2012]

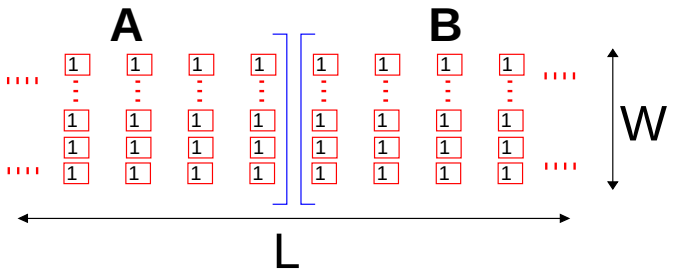


Lower part of ES is dominated by the physics at the boundary between A and B.

Higher levels in the ES involve degrees of freedom deeper into the subsystems bulk.

Mott insulator: ES structure





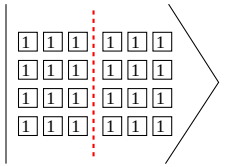
- Linear **"envelope"** (incompressibility of Mott phase).
- **Multiplet** structure (entanglement **dispersions**)
- Simple **multiplicity** counting only function of W (**boundary** locality).

Boundary-local perturbation theory for ES

- At $U = \infty$ the ground state is a product (Mott) state Ψ_0:

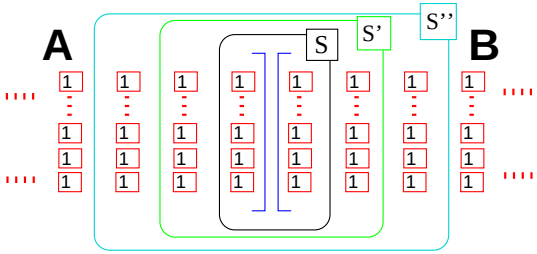
$$\mathcal{H} = H_0 + H_p \qquad H_0 \equiv \sum_i n_i(n_i - 1)$$

$$\boxed{H_p \equiv \frac{1}{U} \sum_{\langle i,j \rangle} (b_i^\dagger b_j + h.c.)}$$



- At large U **perturbative expansion** $|\Psi\rangle \sim \sum_{\alpha \geq 0} U^{-\alpha} |\Psi_\alpha\rangle$.

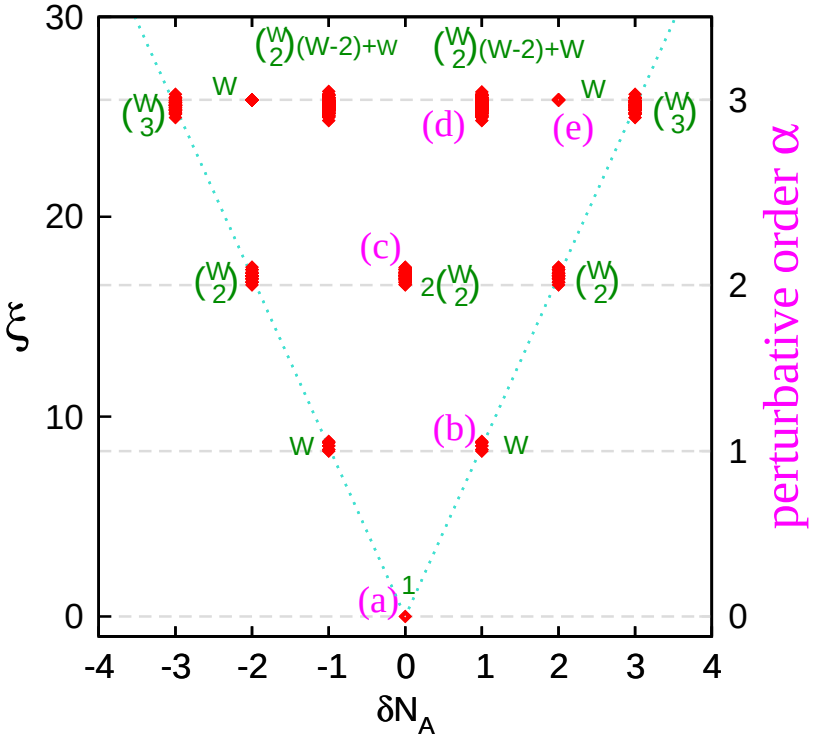
$$|\Psi_\alpha\rangle \sim (H_p)^\alpha |\Psi_0\rangle$$



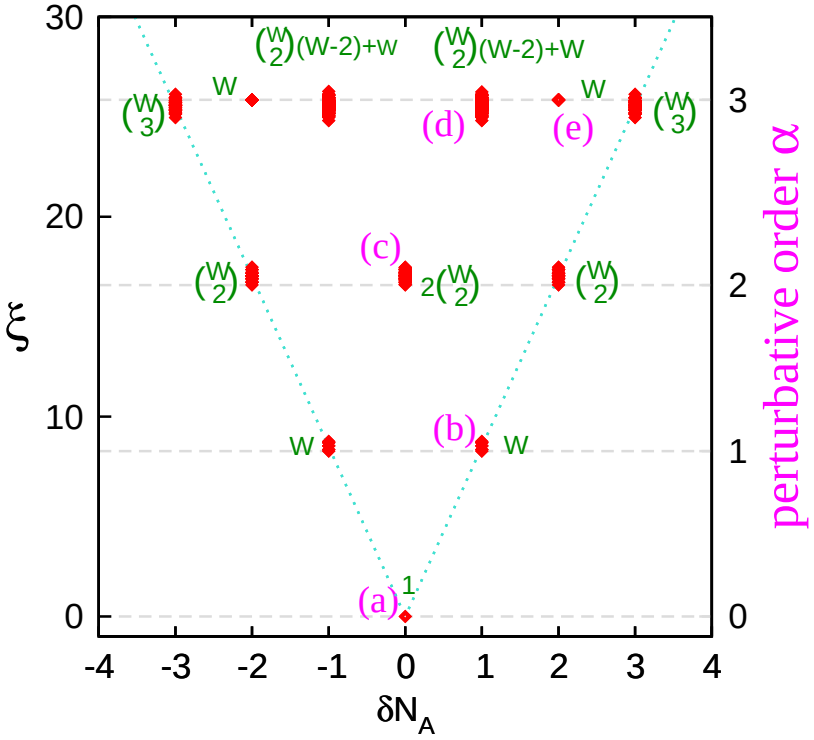
- **Boundary locality** and perturbation theory [V.A.,M. Haque,A.M. Läuchli,2012]:

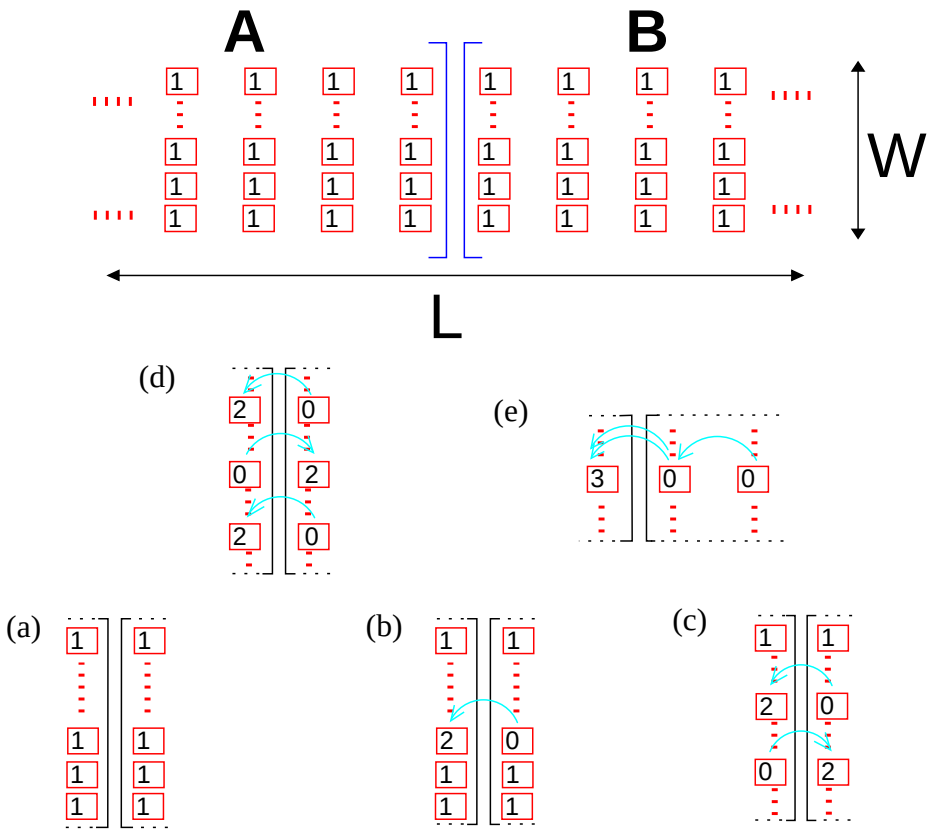
> To have the ES accurate up to order α it is sufficient to consider **only** the perturbative terms living in a box $S^{(\alpha)}$

Boundary-local perturbation theory at work



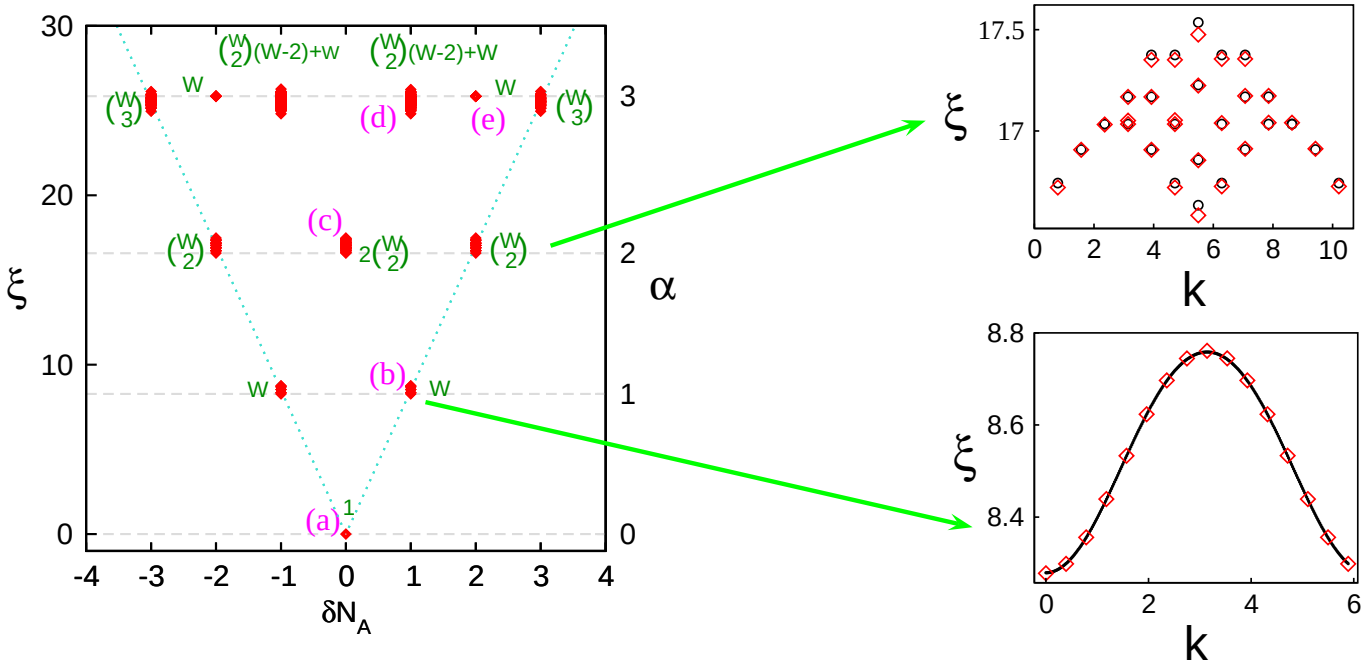
Boundary-local perturbation theory at work





- **Boundary locality**: The ES structure (multiplicities, envelope behavior, ...) is understood in terms of **boundary perturbative** processes.

Entanglement dispersions

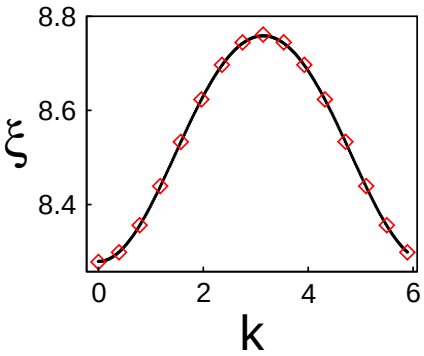


k boundary momentum

- The envelope ES multiplets show "simple" **entanglement dispersions**.

Boundary entanglement Hamiltonians

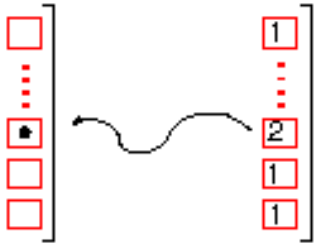
- Let us focus on the sector $\delta N_A = 1$:



$$\xi_k = \underbrace{\log(U^2/2) + 2(2W+10)/U^2}_{A_0} - \underbrace{24\left[1/U + (2W-25)/U^3\right]}_{A_1} \cos k$$

$$- \underbrace{68/U^2}_{A_2} \cos 2k + \underbrace{552/U^3}_{A_3} \cos 3k$$

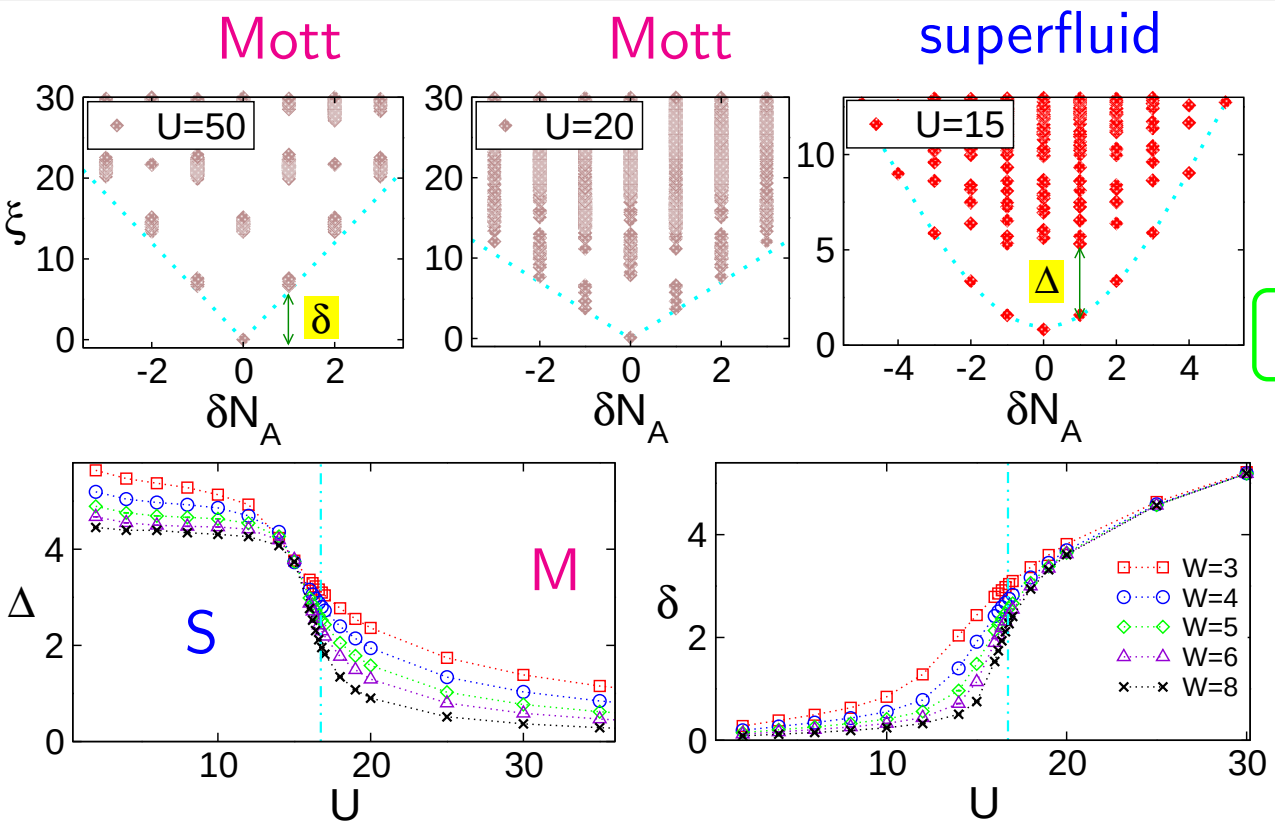
- Single particle **1D** tight binding **entanglement Hamiltonian**:



$$\mathcal{H}_E = \sum_{ir} A_r (b_i^\dagger b_{i+r} + h.c.)$$

- Effective degrees of freedom localized at the boundary (**boundary locality**).
- Gapped phase $\Rightarrow \mathcal{H}_E$ is **short range** ($A_r \approx U^{-r}$).[Cirac et al.,2011]

ES across the Mott superfluid transition



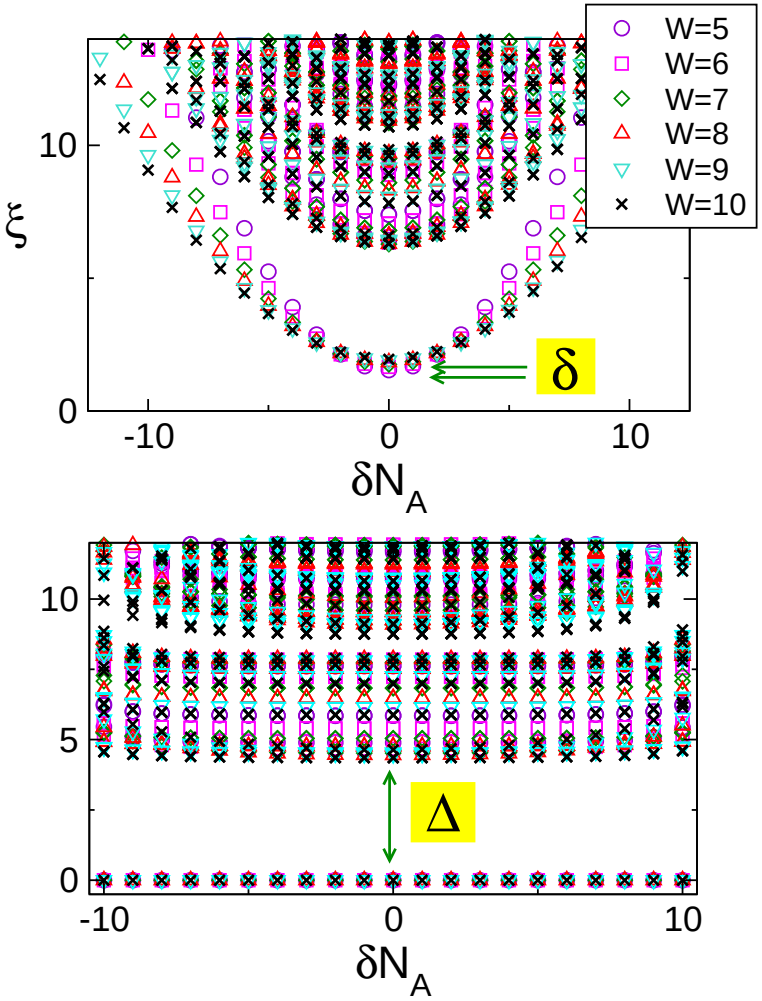
$$\delta \equiv \xi_{(\delta N_A=1,k=0)} - \xi_0$$

$$\Delta \equiv \xi_{(\delta N_A=1,k_{min})} - \xi_{(\delta N_A=1,k=0)}$$

- Dramatic change in the ES structure across the transition ($U_c \approx 17$).
- Dual behavior of the **gap** Δ and the **level spacing** δ in the two phases.

ES structure in the superfluid phase

DMRG $U = 2, L = 2W$



- Drastic differences with the ES in the Mott phase.
- In the superfluid **parabolic envelope**.
- As $W \to \infty$ the envelope "curvature" vanishes (i.e. $\delta \to 0$).
- **Entanglement gap** Δ
- Δ signature of finite **condensate** fraction.

Tower of states mechanism

$$\mathcal{H} = -\sum_{\langle ij \rangle} (b_i^\dagger b_j + h.c.) + \frac{U}{2} \sum_i n_i (n_i - 1)$$

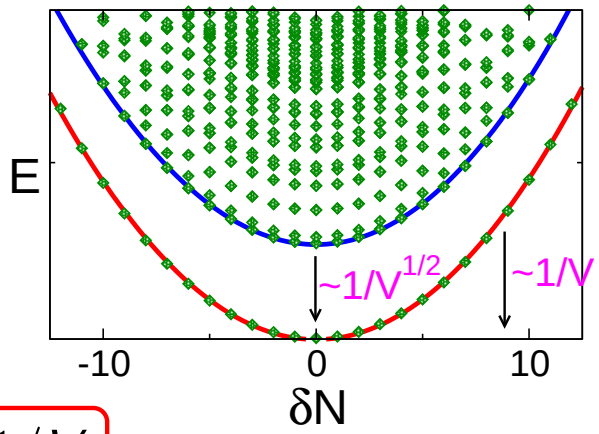
- In the superfluid $\rightarrow$ **Bose condensate** $\rightarrow$ $U(1)$ **(continuous) symmetry breaking**.
- Problem: the system is in a **finite volume** V (no symmetry breaking).

$b_i = e^{i\phi_i}\sqrt{n_i}$ $\qquad$ $n_i = n_0 + \delta n_i$

$[n_i, \phi_j] = -i\delta_{ij}$ $\qquad$ $Jn_0 \gg U$ $\qquad$ $\mathcal{H} \approx \sum_i (\delta n_i)^2 + \frac{U}{2} \sum_{\langle ij \rangle} (\phi_i - \phi_j)^2$

- **Anderson tower of states** structure:

$$\mathcal{H} \approx \underbrace{\boxed{\frac{U}{2}\frac{(\delta N_{tot})^2}{V}}}_{\mathcal{H}_{tos}} + \underbrace{\boxed{\sum_{k\neq 0} \left(\frac{U}{2}|n_k|^2 + \frac{\epsilon_k}{2}|\phi_k|^2 \right)}}_{\mathcal{H}_{sw}}$$



- $\epsilon_k \sim v_s|k| \rightarrow \boxed{(\Delta E)_{sw} \sim 1/\sqrt{V}}$ while $\boxed{(\Delta E)_{tos} \sim 1/V}$.

Tower of states structure and ES envelope

- The lower part of the ES (ES **envelope**) is described by the **tower of states** Hamiltonian ($\mathcal{H}_{tos}$) restricted to the subsystem.

[Grover,Metlitski,(2011)]

$$\mathcal{H}_E \approx \mathcal{H}_{tos}^{(A)} / T_E$$

$$T_E \sim v_s / V^{1/2}$$

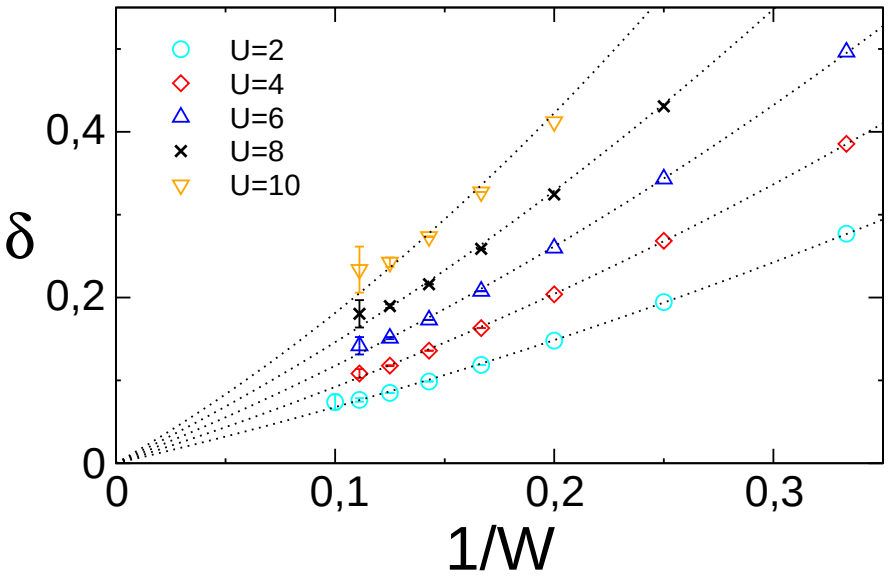
- For the Bose-Hubbard:

$$\mathcal{H}_E \approx \sqrt{\frac{U}{V}} (\delta N_A)^2$$

- $\Rightarrow$ the ES envelope is **parabolic** and the **level spacing** $\delta \sim \sqrt{U/V}$.

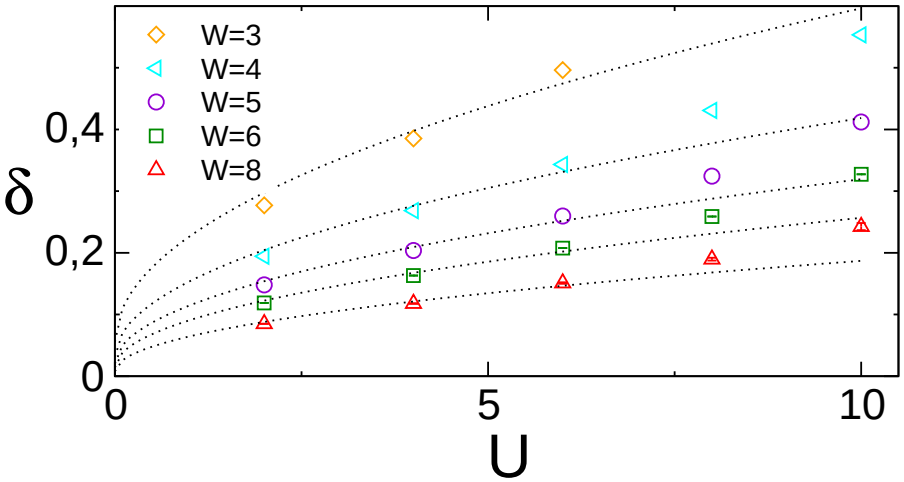
Tower of states scenario: numerical checks

- DMRG data at fixed aspect ratio $W/L = 1/2$.



Behavior of the **level spacing** δ:

- fixed $U \to \delta \sim A/W + B/W^2$.
- fixed $W/L \to \delta \sim \alpha + \beta\sqrt{U}$

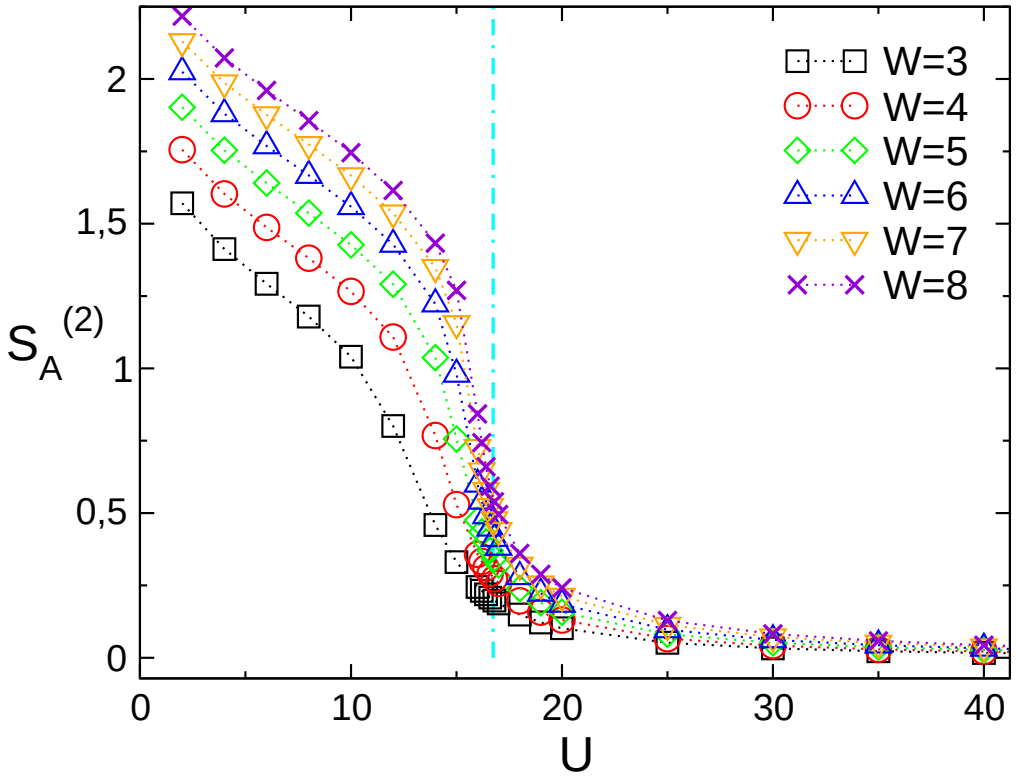


Theoretical prediction $\delta \sim \sqrt{U}/W(\delta N_A)^2$ fully confirmed.

Entanglement entropies across the phase transition

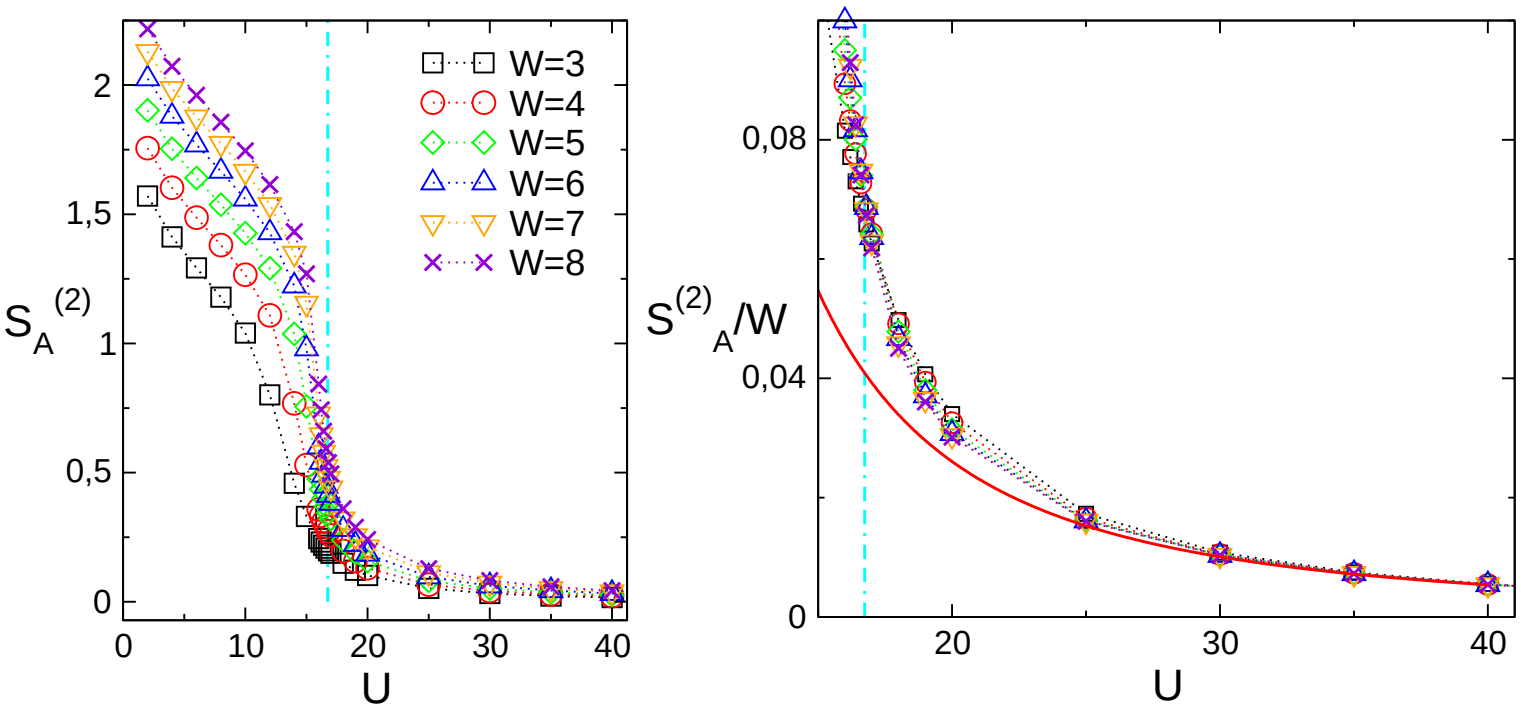
- Consider the Renyi $S_A^{(2)}$

$$S_A^{(2)} = -\log \mathrm{Tr}\rho_A^2$$



- Sharp change of $S_A^{(2)}$ at the transition.

Entanglement entropies: Mott phase

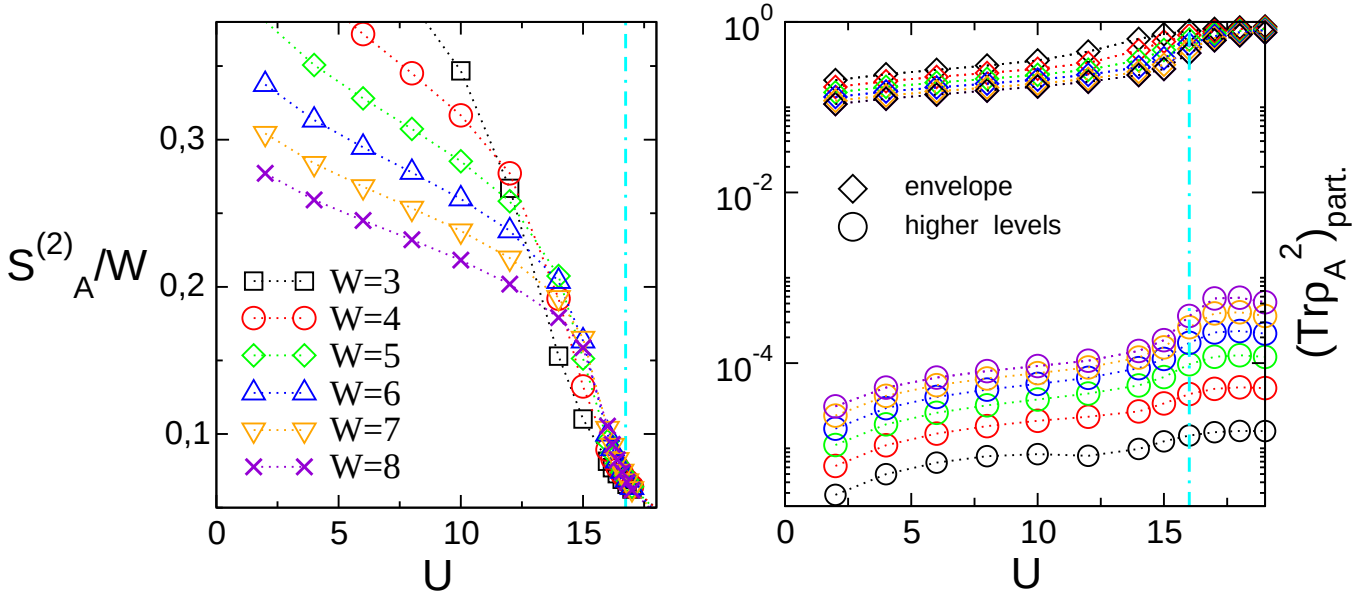


- In the Mott phase $S^{(2)} \sim c_1 W$ (**area law**) in agreement with boundary perturbation theory:

$$S_A^{(2)}/W = c_1 = \frac{8}{U^2} + \frac{968}{U^4} + \mathcal{O}(U^{-6})$$

Entanglement entropies: the superfluid

$$S_A^{(2)} = -\log \mathrm{Tr}\rho_A^2$$



- **Finite** gap $\Delta \Rightarrow$ only the **envelope** contributes to $S_A^{(2)}$.
- ES envelope gives a subleading **logarithmic** correction

$$S_A^{(2)} = c_1 W + c_2 \log W$$

[Metlitski,Grover,2011]

- No enough ES levels above the gap to build the area law.

Summary

We have studied the ES of the **2D** Bose-Hubbard in both the Mott insulating and the superfluid phase.

- Mott insulator:
  - **Boundary locality** of the ES.
  - ES shows a **perturbative** structure.
  - Entanglement **dispersions**.
  - Boundary local 1D **entanglement Hamiltonians**.
- Superfluid:
  - Signatures of the **symmetry breaking**.
  - Lower part of the ES (ES **envelope**) related to **tower of states**.
  - Finite **gap**.
- Entropies:
  - **Area law** behavior in the Mott phase.
  - **No** area law (entropies are dominated by the envelope levels).

Thanks!