Entanglement, thermal area laws,
and a "Wick's theorem" for matrix-product states

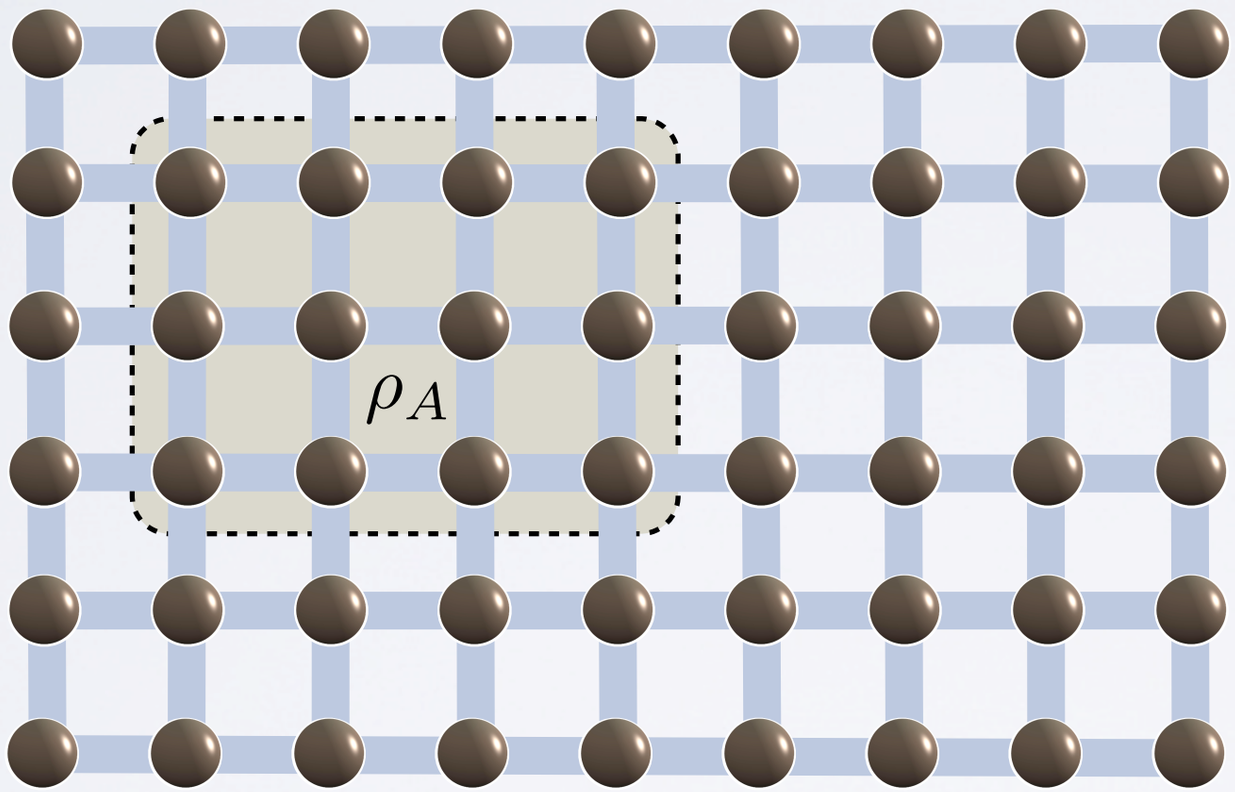
Jens Eisert

Freie Universität Berlin

Entanglement spectra in complex quantum wave functions

Mentions joint work with H. Bernigau, M. Kastoryano, R. Huebener, A. Mari

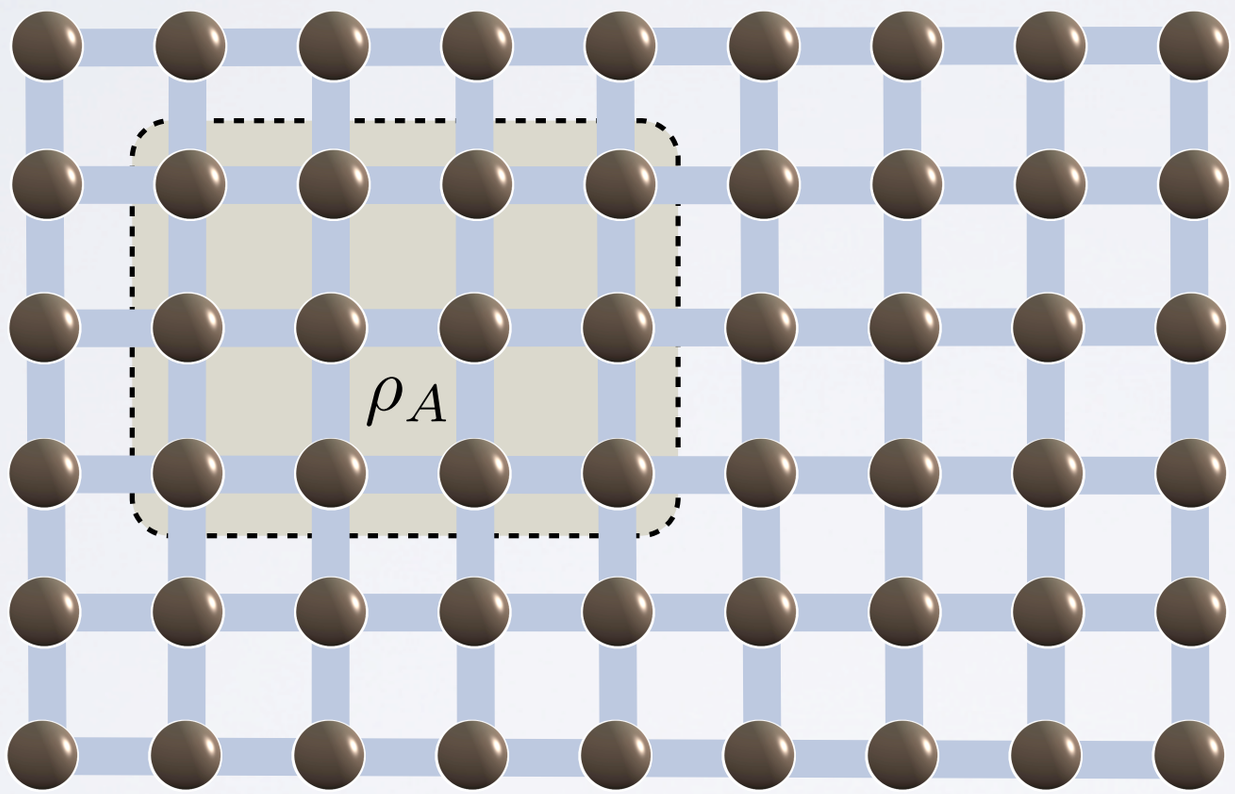
Entanglement spectra and area laws



- Reduced density operator ρ_A
- Renyi entropies $S_\alpha(\rho_A) = \dfrac{1}{1-\alpha} \log_2 \mathrm{tr}(\rho_A^\alpha)$
- Collection of all: entanglement spectrum

Chung, Peschel, *Phys Rev B* **62**, 4191 (2000)
Bernevig, Haldane, *Phys Rev Lett* **100**, 246802 (2008)
Li, Haldane, *Phys Rev Lett* **101**, 010504 (2008)
Calabrese, Lefevre, *Phys Rev A* **78**, 03239 (2008)
Eisert, Cramer, Plenio, *Rev Mod Phys* **82**, 277 (2010)
Cirac, Poilblanc, Schuch, Verstraete, *Phys Rev B* **83**, 245134 (2011)
Alba, Haque, Laeuchli, *J Stat Mech* P08011 (2012)

Entanglement spectra and area laws



- Topological order
- Boundary theories
- Approximability with tensor network states

Chung, Peschel, *Phys Rev B* **62**, 4191 (2000)
Bernevig, Haldane, *Phys Rev Lett* **100**, 246802 (2008)
Li, Haldane, *Phys Rev Lett* **101**, 010504 (2008)
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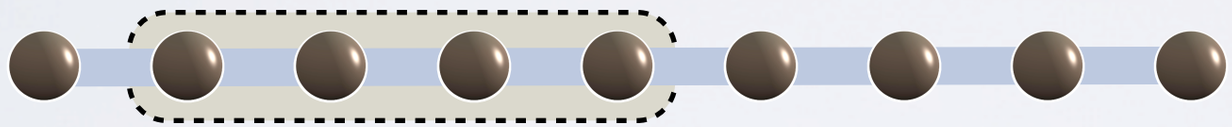
Alba, Haque, Laeuchli, *J Stat Mech* P08011 (2012)

One-dimensional systems



- **Situation specifically clear in 1D:**
  - **Gapped models satisfy area laws:** $S_\alpha(\rho_A) \leq C$

    - Lieb-Robinson bounds
    - Detectability lemma
    - Decay of correlations

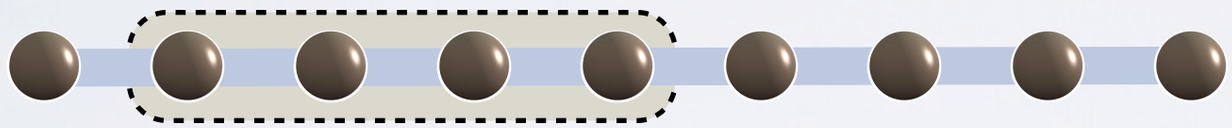
Hastings, *J Stat Mech* P08024 (2007)
Audenaert, Eisert, Plenio, Werner, *Phys Rev A* **66**, 042327 (2002)
Arad, Landau, Vazitani, *Phys Rev B* **85**, 195145 (2012)
Brandao, Horodecki, arXiv:1206.2947

  - **Critical models:** Conformal field theory

$$S_\alpha(\rho_A) = \frac{c}{6}\left(1 + \frac{1}{\alpha}\right)\log(l/a) + o(l)$$

Calabrese, Cardy *J Stat Mech* P06002 (2004)
Vidal, Latorre, Rico, Kitaev, *Phys Rev Lett* **90**, 227902 (2003)
Holzhey, Larsen, Wilczek, *Nucl Phys B* **424**, 443 (1994)

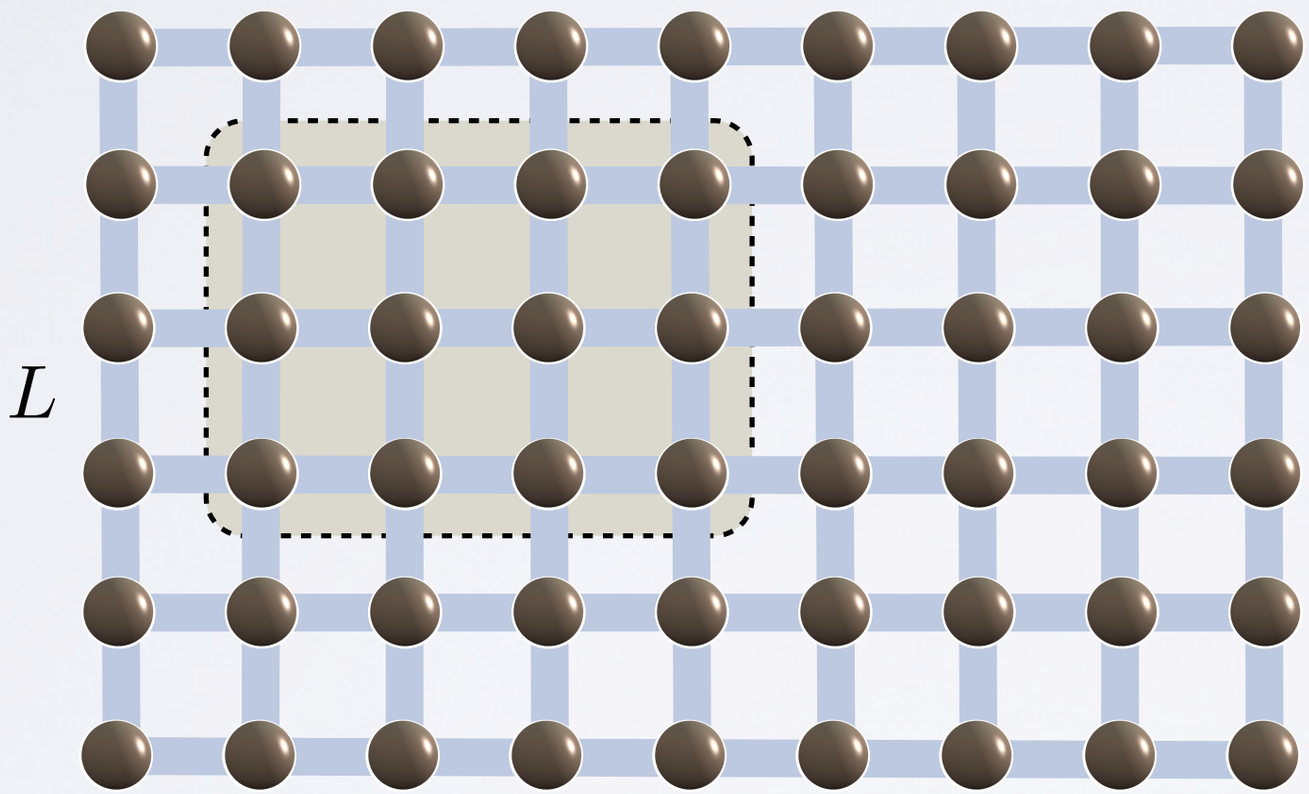
Matrix-product states



- **Situation specifically clear in 1D:**
    - **Matrix-product states** satisfy area law, converse is also true:
    - States satisfying (Renyi) area laws can be **efficiently approximated**

Schuch, Wolf, Verstraete, Cirac, *Phys Rev Lett* **100**, 030504 (2008)
Schollwoeck, *Rev Mod Phys* **77**, 259 (2005)

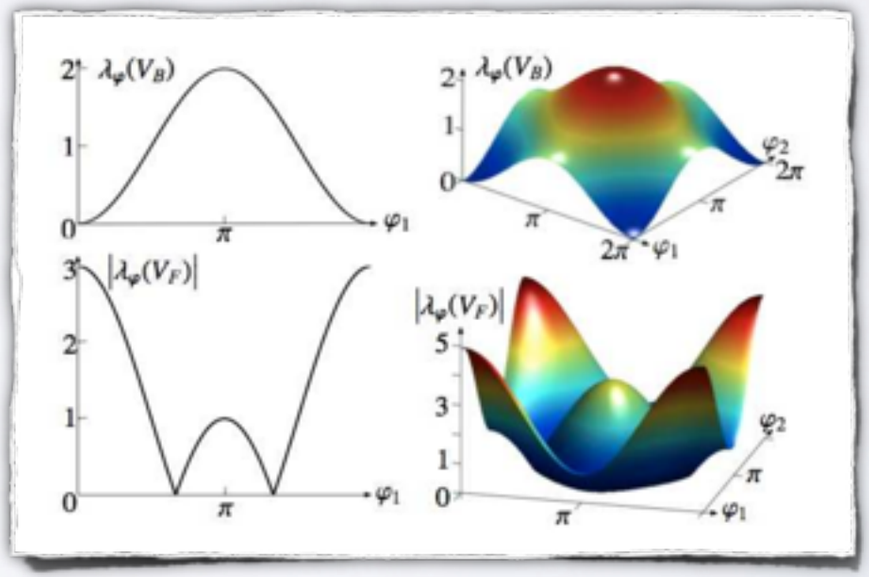
Higher-dimensions



- Still true that **PEPS** satisfy area law

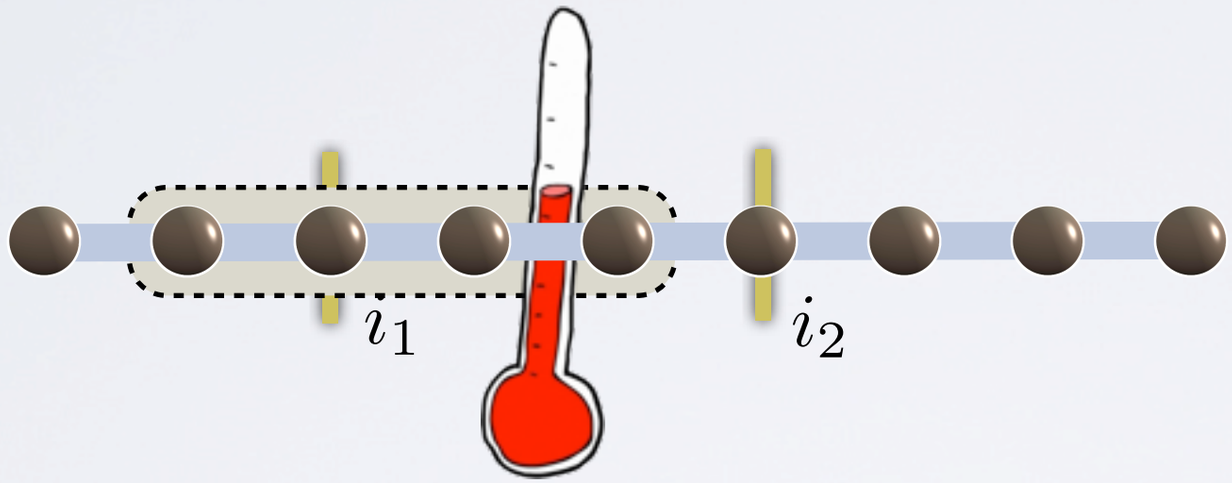
Verstraete, Wolf, Perez-Garcia, Cirac, *Phys Rev Lett* **96**, 220601 (2006)

- **Free bosons:** $S(\rho_A) \sim L^{D-1}$
- **Critical free fermions:** $S(\rho_A) \sim L^{D-1} \log_2(L)$



Plenio, Eisert, Dreissig, Cramer, *Phys Rev Lett* **94**, 060503 (2005)
Wolf, *Phys Rev Lett* **96**, 010404 (2006)
Cramer, Eisert, Plenio, *Phys Rev Lett* **98**, 220603 (2007)
Eisert, Cramer, Plenio, *Rev Mod Phys* **82**, 277 (2010)

Overview of the (rest of this) talk



- **Part 1:** Thermal area laws

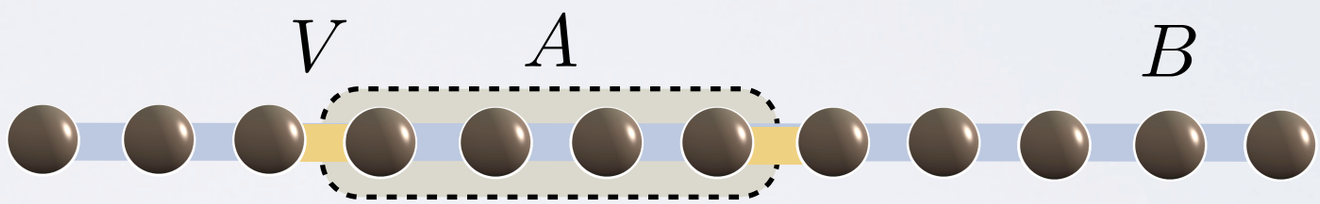
  Bernigau, Kastoryano, Eisert, in preparation (2012)

- **Part 2:** "Wick's theorem" for (continuous) matrix product states (short)

  Huebener, Mari, Eisert, arXiv:1207.6537, *Phys Rev Lett,* in press (2012)

Part 1: Thermal area laws and spectra

Area laws for the mutual information



- Entanglement entropy meaningless as correlation measure for **Gibbs states**

$$\rho = e^{-\beta H}/Z$$

- **Mutual information** (reduces to entanglement entropy for pure states)

$$I = S(\rho_A) + S(\rho_B) - S(\rho)$$

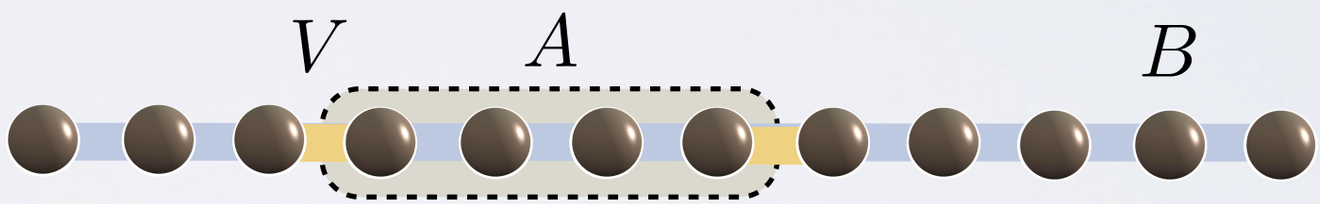
- Gibbs state minimizes **free energy** $F(\rho) = \mathrm{tr}(H\rho) - S(\rho)/\beta$,

so $H = H_{AB} + H_V$ gives $F(\rho) \leq F(\rho_A \otimes \rho_B)$, hence

$$I \leq \beta \mathrm{tr}(H_V(\rho_A \otimes \rho_B - \rho)) \leq \beta C \mathrm{Area}$$

Wolf, Verstraete, Hastings, Cirac, *Phys Rev Lett* **100**, 070502 (2008)
Bratteli, Robinson, *Operator algebras and quantum statistical mechanics* (1976)

Area laws for the mutual information



- **Area law** for all temperatures, remarkably simple argument
- **But:** Linear divergence in β
- True scaling? Capture this in free fermionic integrable systems?

$$I \leq \beta \mathrm{tr}(H_V(\rho_A \otimes \rho_B - \rho)) \leq \beta C \mathrm{Area}$$

Wolf, Verstraete, Hastings, Cirac, *Phys Rev Lett* **100**, 070502 (2008)
Bratteli, Robinson, *Operator algebras and quantum statistical mechanics* (1976)

Free fermionic models

- **Fermionic models** (such as XX model in 1D)**:**

$$H = \frac{1}{2}\sum_{i,j}\left(f_i^\dagger M_{i,j} f_j - f_i M_{i,j} f_j^\dagger\right)\ ,\ M = M^T$$

- Majorana operators:

$$x_j = (f_j^\dagger + f_j)/\sqrt{2}$$
$$p_j = i(f_j^\dagger - f_j)/\sqrt{2}$$

- Covariance matrix

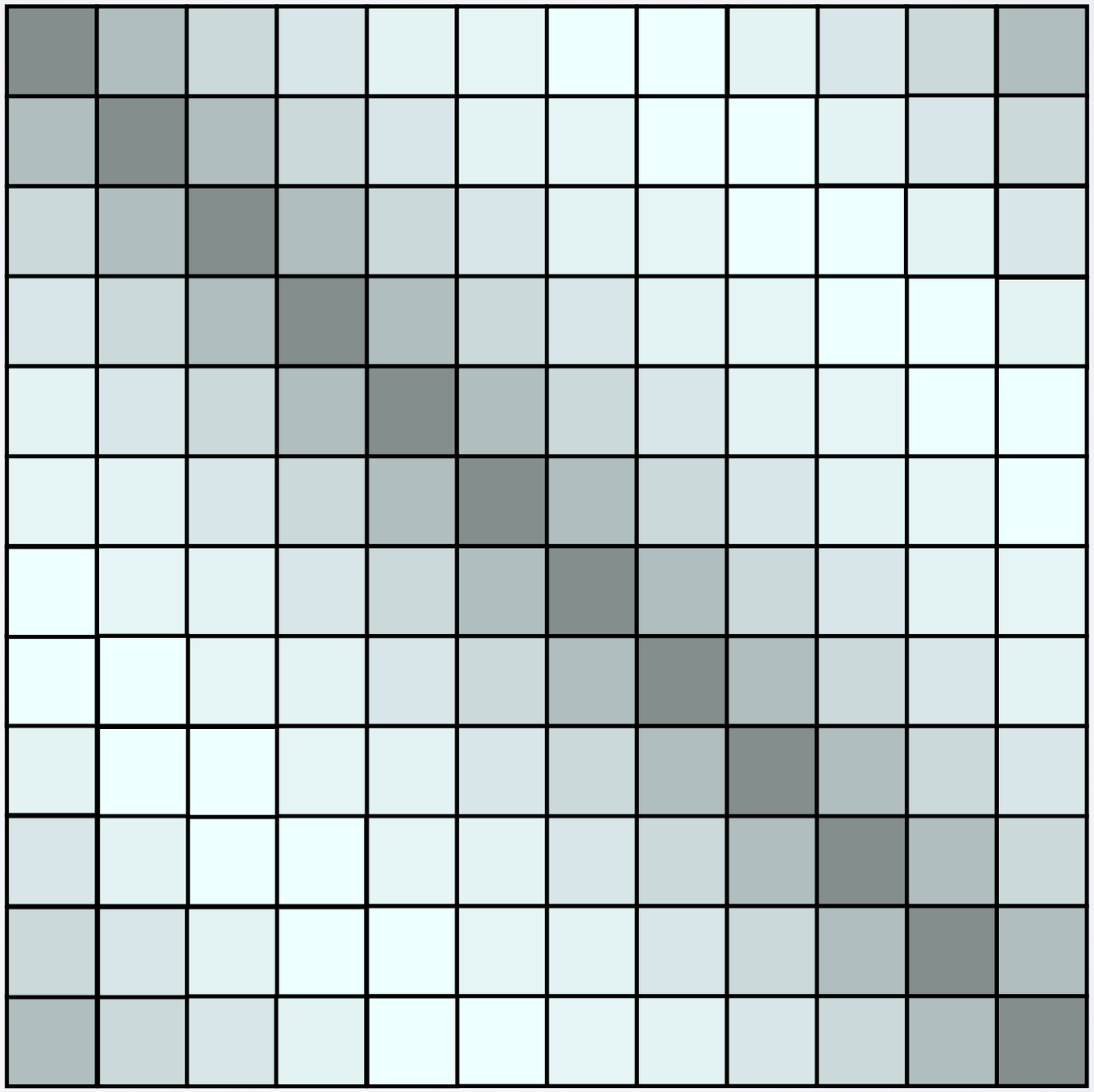
$$-i\Gamma_{i,j} = \langle [r_i, r_j] \rangle = \langle r_i r_j \rangle - \langle r_j r_i \rangle\ ,\ r = (x_1, \ldots, x_n, p_1, \ldots, p_n)$$

- Entanglement spectrum computable from principal submatrix Γ_A

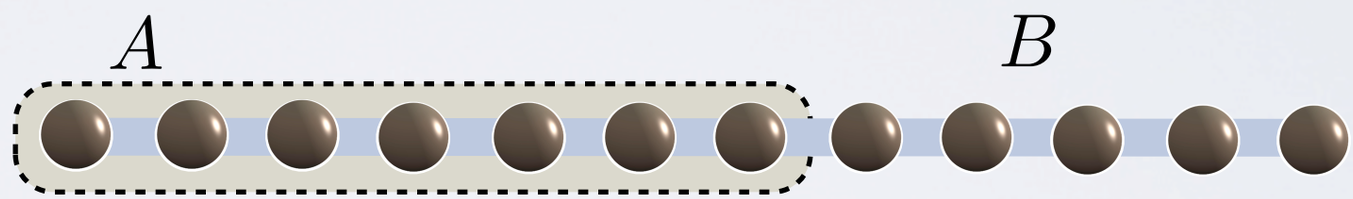
Circulant and Toeplitz matrices

M circulant, $M_{i,j} = M_{i-j}$ → Γ circulant

$\Gamma =$

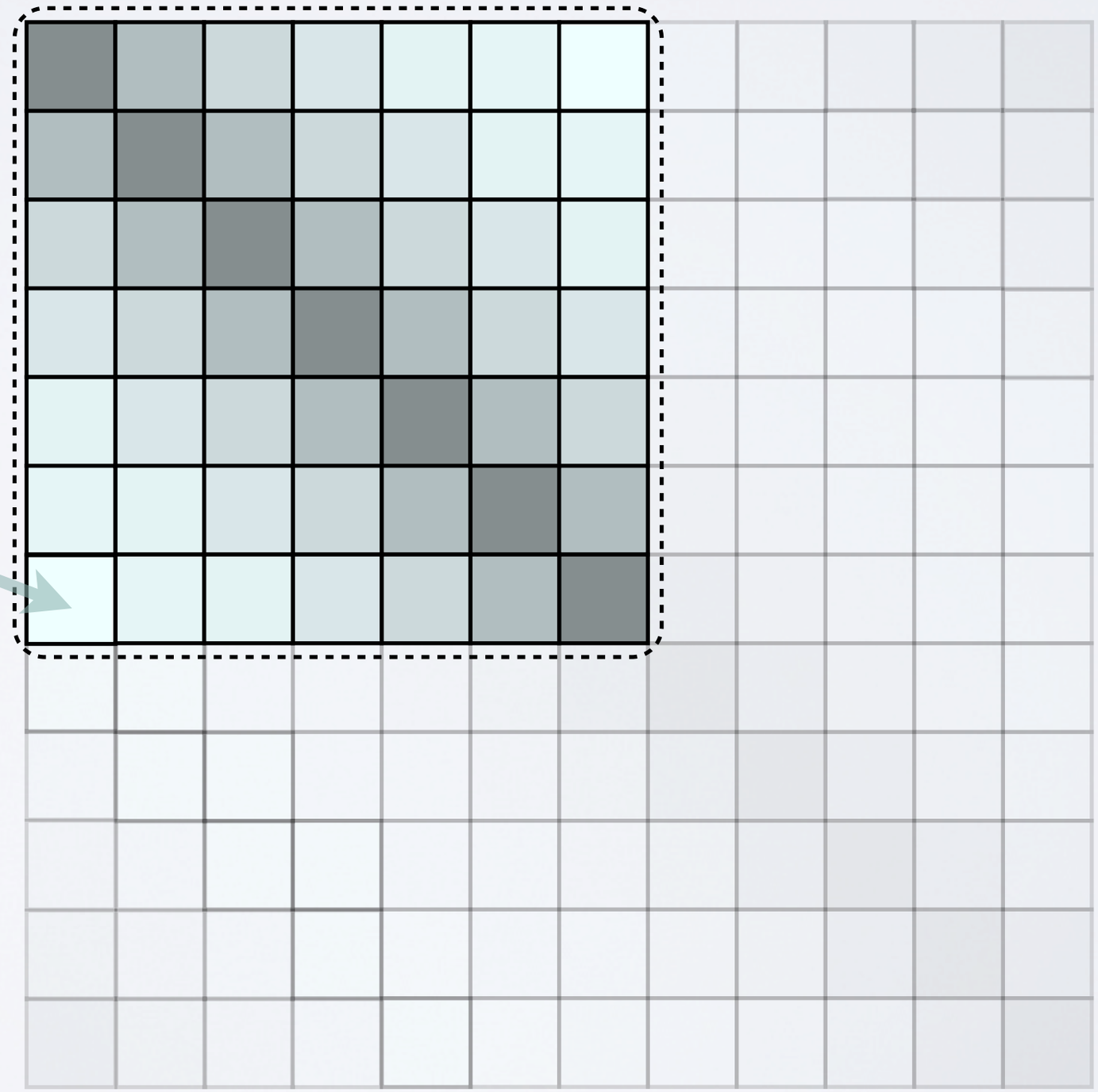


Circulant and Toeplitz matrices

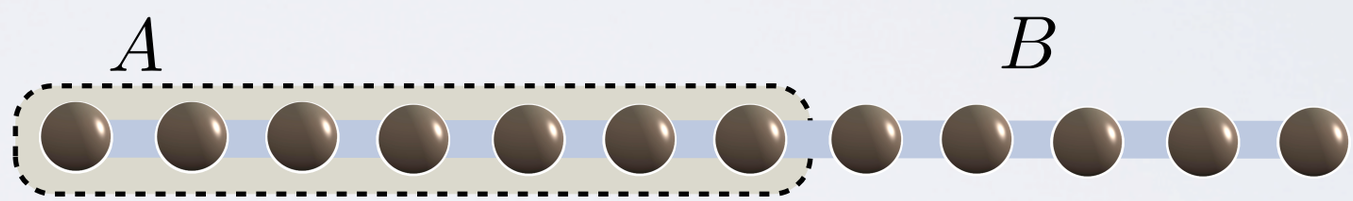


M circulant, $M_{i,j} = M_{i-j}$ → Γ circulant

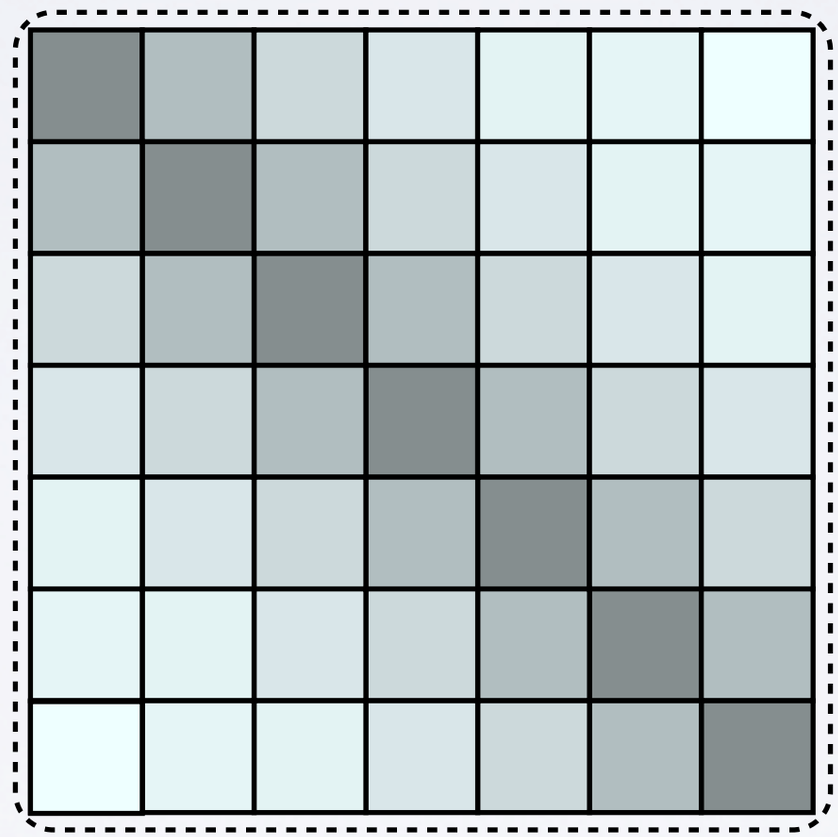
$\Gamma_A = T_L =$



Circulant and Toeplitz matrices



Toeplitz determinant $\det(T_L)$ $= \det$



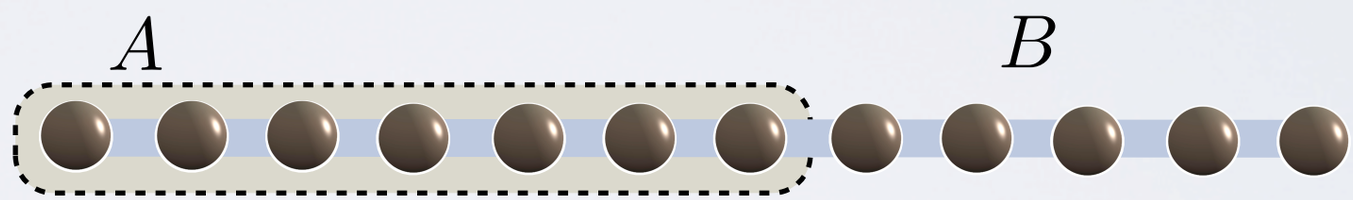
Symbol $g : [0, 2\pi) \to \mathbb{C}$

$$M_l = \frac{1}{2\pi} \int_0^{2\pi} d\phi g(\phi) e^{il\phi}$$

Its, Jin, Korepin, *J Phys A* **38**, 2975 (2005)
Eisert, Cramer, *Phys Rev A* **72**, 042112 (2005)
Orus, Latorre, Eisert, Cramer, *Phys Rev A* **73**, 060303 (2006)
Calabrese, Essler, J Stat Mech P08029 (2010)

Boettcher, Silbermann, *Analysis of Toeplitz operators* (2006)

Circulant and Toeplitz matrices



Toeplitz determinant $\det(T_L)$

Complex integration

Cauchy residue thm

Entanglement spectrum/entropies

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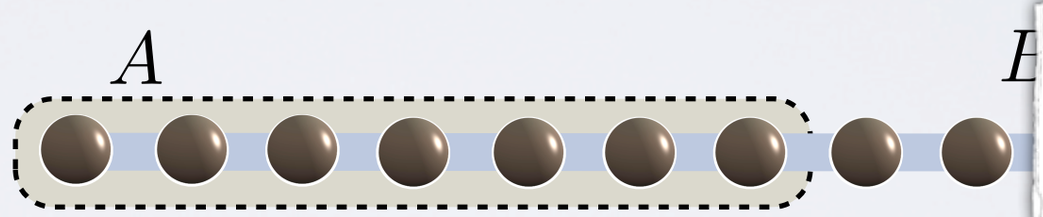
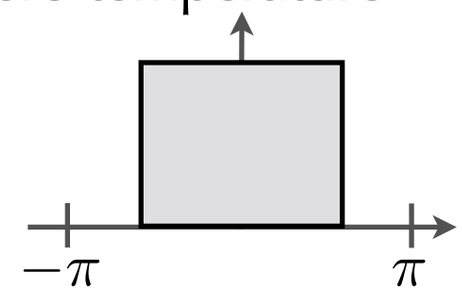


Zero temperature



Toeplitz determinant $\det(T_L)$

Complex integration

Cauchy residue thm

Entanglement spectrum/entropies

$$S(\rho_A) = \frac{1}{3}\log_2(L) + o(L)$$

Symbol $g : [0, 2\pi) \to \mathbb{C}$

$$M_l = \frac{1}{2\pi}\int_0^{2\pi} d\phi g(\phi) e^{il\phi}$$

Fisher-Hartwig: Gives asymptotic behavior of determinant from

$$g(\phi) = b(\phi)\prod_{r=1}^{R} t_{\beta_r}(\phi - \phi_r) u_{\alpha_r}(\phi - \phi_r)$$

"Regular part" "Jumps" "Zeros"

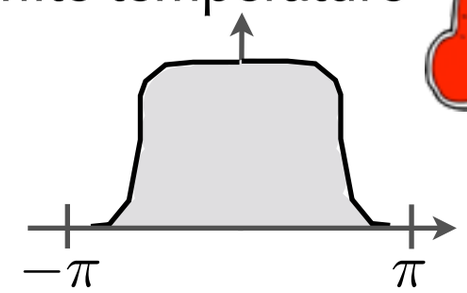
$$t_\beta(\phi) = e^{-i\beta(\pi-\phi)} \qquad u_\alpha = (2 - 2\cos(\phi))^\alpha$$

Boettcher, Silbermann, *Analysis of Toeplitz operators* (2006)

Toeplitz operator techniques for thermal symbols

Approximation by submatrices of infinite systems
+ by **continuum limit**

Finite temperature



Toeplitz determinant $\det(T_L)$

Complex integration

Cauchy residue thm

Asymptotic mutual information (here for infinite L)

$$I = \frac{1}{2\pi^2} \sum_{k=1}^{\infty} \int_0^{2\pi} \int_0^{2\pi} \frac{s\left(g(e^{i\theta})\right) - s\left(g(e^{i\phi})\right)}{g(e^{i\theta}) - g(e^{i\phi})} \times \sin\left(k(\theta - \phi)\right) \left(g'(e^{i\phi}) - g'(e^{i\theta})\right) d\theta d\phi$$

(Renyi) entropy function

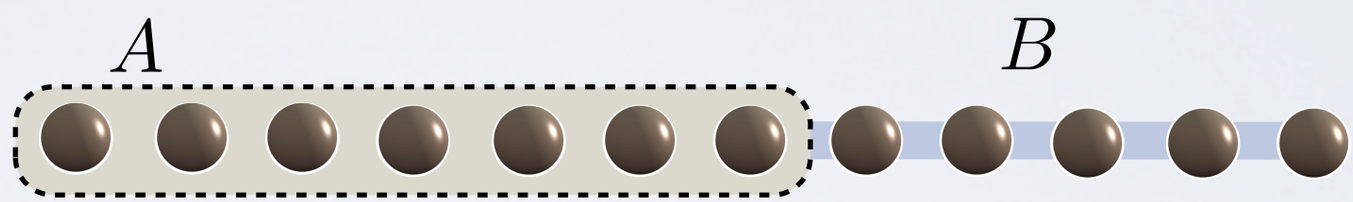
Szegö's strong limit theorem: Behavior for smooth, (but in temperature increasingly ill-behaved) symbol

Symbol $g : [0, 2\pi) \to \mathbb{C}$

$$M_l = \frac{1}{2\pi} \int_0^{2\pi} d\phi g(\phi) e^{il\phi}$$

Bernigau, Kastoryano, Eisert, in preparation (2012)

Thermal area laws



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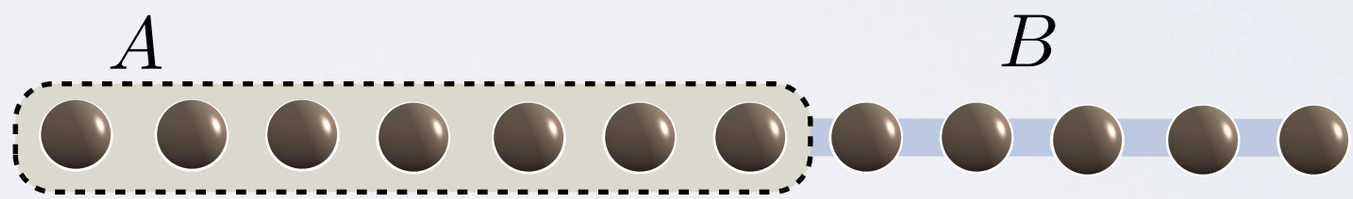
Methods of highly oscillatory function kernels

- **Theorem:** Asymptotic mutual information for thermal states

$$I = \frac{1}{8\pi^2} \int_0^{2\pi} \int_0^{2\pi} \frac{s\left(g(e^{i\theta})\right) - s\left(g(e^{i\phi})\right)}{g(e^{i\theta}) - g(e^{i\phi})} \times \tan^{-1}\left((\theta-\phi)/2\right)\left(g'(e^{i\phi}) - g'(e^{i\theta})\right) d\theta d\phi$$

Bernigau, Kastoryano, Eisert, in preparation (2012)

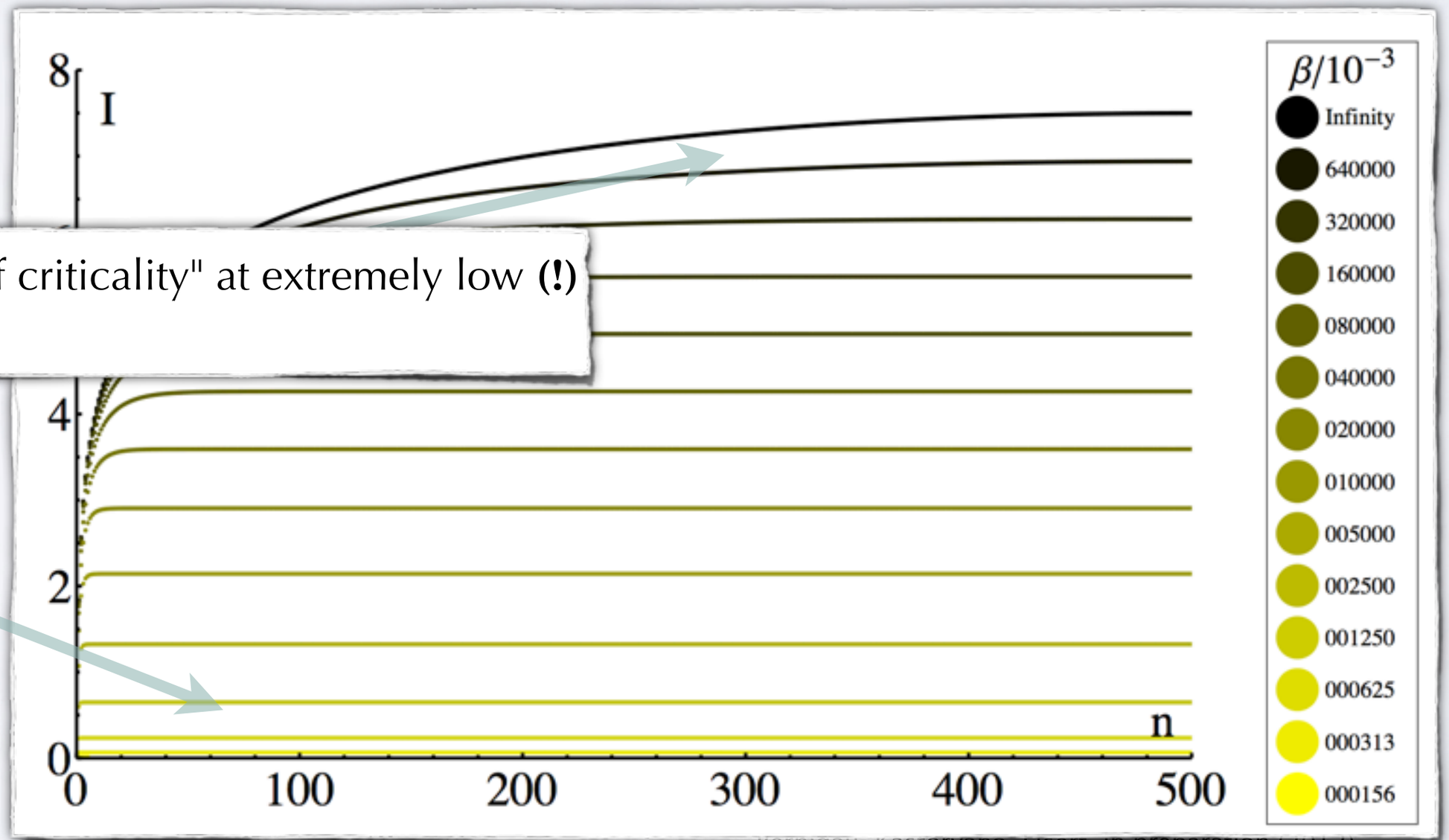
Thermal area laws



- **Example: XX model**

$$H = -\frac{1}{4}\sum_{\langle i,j\rangle}\left(\sigma_i^x\sigma_j^x + \sigma_i^y\sigma_j^y\right) - \lambda\sum_i \sigma_i^z,$$

Log scale

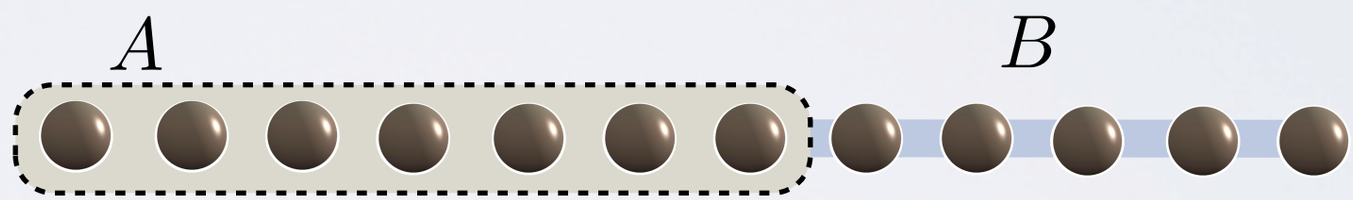


"Signatures of criticality" at extremely low (!) temperatures

"Area law"

Bernigau, Kastoryano, Eisert, in preparation (2012)

Thermal area laws



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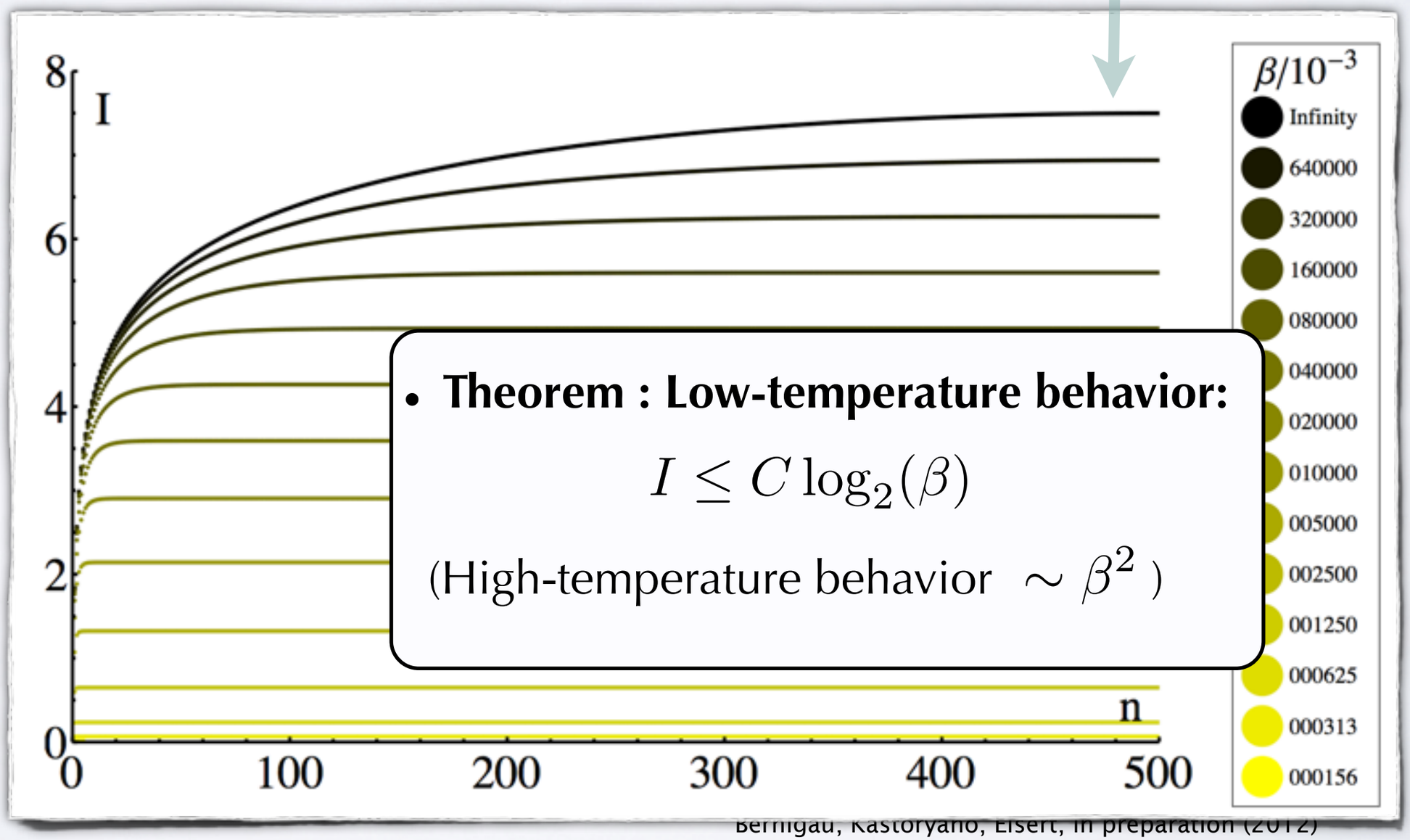
$$H = -\frac{1}{4}\sum_{\langle i,j\rangle}\left(\sigma_i^x\sigma_j^x + \sigma_i^y\sigma_j^y\right) - \lambda\sum_i \sigma_i^z,$$

Exponential improvement over $\sim \beta$ bound

Log scale



- **Theorem : Low-temperature behavior:**

$$I \leq C\log_2(\beta)$$

(High-temperature behavior $\sim \beta^2$)

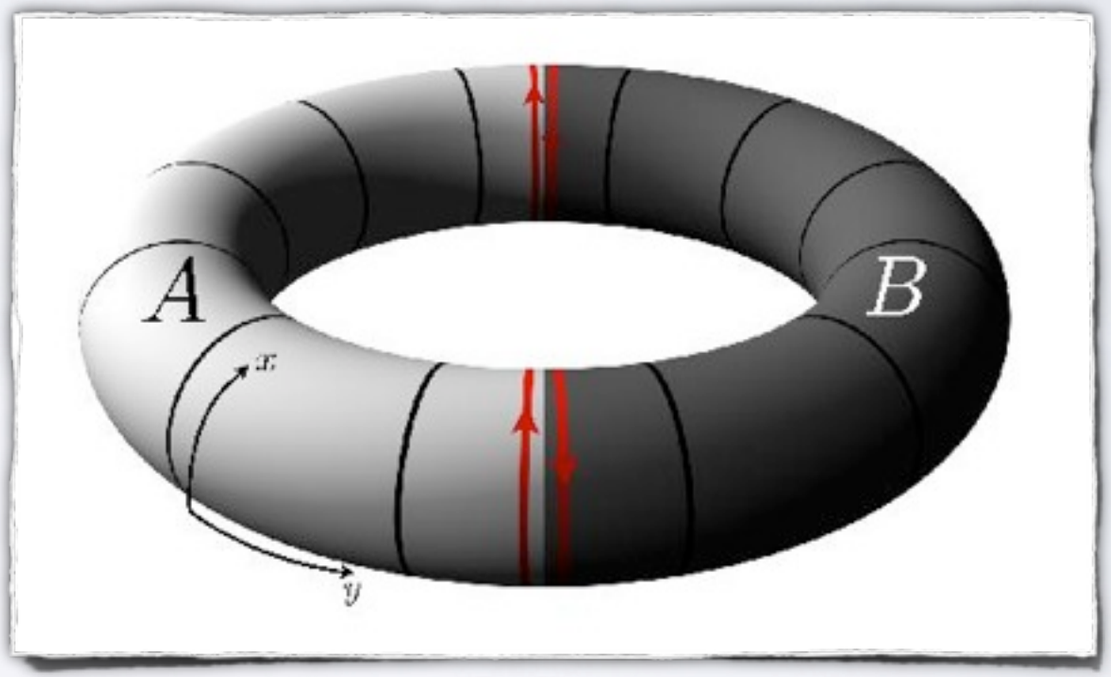
Bernigau, Kastoryano, Eisert, in preparation (2012)

Higher-dimensional thermal area laws and open systems

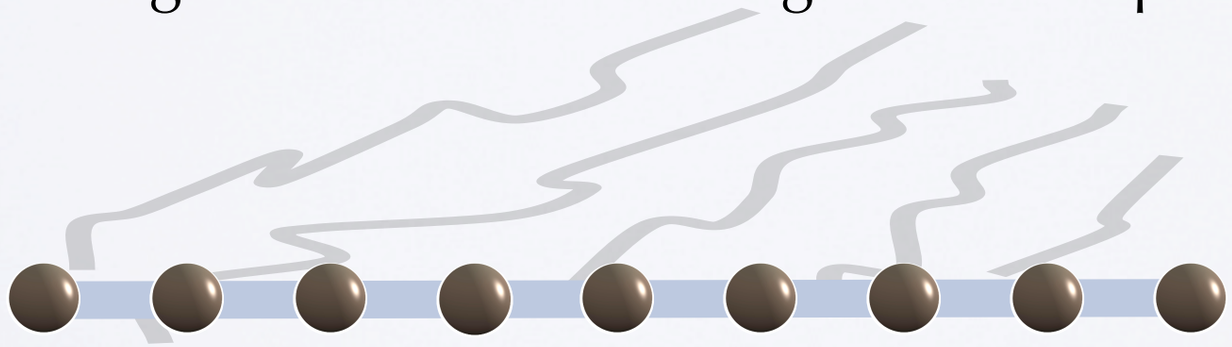
- **On torus:** *Analytical* thermal area laws, spectra, for higher-dimensional systems

$$I \leq \log \beta \, \mathrm{Area}$$

- Interplay of temperature and critical properties



- **Open, noise driven fermionic systems** (variants of Majorana wires):
  Classification, entanglement versus winding number capturing topological order



Wilming, Kastoryano, Eisert, in preparation (2012)
Eisert, Prosen, arXiv:1012.5013
Diehl, Micheli, Kantian, Kraus, Buechler, Zoller, *Nature Physics* **4**, 878 (2008)

Lesson of part 1

- **Lesson:**
- **First tight thermal area laws** ("entanglement" spectra) for specific models, exponentially tighter than via free energy
- New technical results on **Toeplitz operators** with smooth symbols
- Needs **very low** (!) temperature to **feel the criticality** of the ground state

- Fun playground to analytically study thermal spectra
- Good results on approximatability with MPO?

Part 2: "Wick's theorem" for (c)-matrix product states (short)

Matrix-product states

- **n-site translationally invariant matrix product state with bond dimension d:**
- $A[s_i] \in \mathbb{C}^{d\times d}$ matrices per site
- Gives $|\psi\rangle = \sum_{s_1,\ldots,s_n} \mathrm{tr}(A[s_n]\ldots A[s_1])|s_n,\ldots,s_1\rangle$

Correlation functions of operators, supported at sites $i_1 < \cdots < i_N$

$$\langle O^{(i_N)} O^{(i_{N-1})} \ldots O^{(i_1)}\rangle = \mathrm{tr}(M E^{i_N - i_{N-1} - 1} M \ldots M E^{\infty}) =: C^{(N)}(\mathbf{n})$$

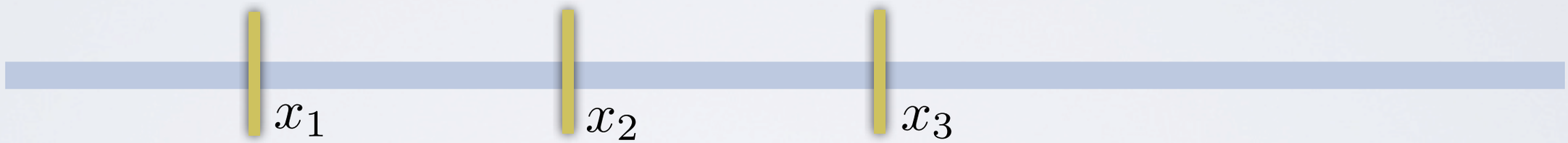
- Transfer matrix $E = \sum_s A^*[s] \otimes A[s]$

$$M = \sum_{n,m} A^*[m] \otimes A[n] \langle m|O|n\rangle$$

- Thermodynamic limit $E^{\infty} = \lim_{n\to\infty} E^n$

Fannes, Nachtergaele, Werner, *J Phys A* **24**, L185 (1991)
Perez-Garcia, Verstraete, Wolf, Cirac, *Quant Inf Comp* **7**, 401 (2007)

Continuous matrix-product states



- **One-dimensional non-relativistic bosonic quantum field**
- Field operators $\Psi(x)$ and $\Psi^\dagger(x)$ with $[\Psi(x), \Psi^\dagger(x)] = \delta(x - x')$
- $Q(x)$ and $R(x)$ are complex $d \times d$-matrices

$$|\psi\rangle = \mathrm{tr}_{\mathrm{aux}}(\mathcal{P} e^{\int_0^L dx Q(x) \otimes 1 + R(x) \otimes \Psi^\dagger(x))} |\Omega\rangle$$

Correlation functions of operators:

$$\langle \Psi^\dagger(x_2) \Psi^\dagger(x_3) \Psi(x_2) \Psi(x_1) \rangle = \mathrm{tr}(M^{[1]} e^{T\tau_2} M^{[3]} e^{T\tau_1} M^{[2]} e^{T\infty}) =: \mathcal{C}_{\mathbf{j}}^{(3)}(\boldsymbol{\tau})$$

- Thermodynamic limit $L \to \infty$
- Translation-invariant case $R(x) = R$ and $Q(x) = Q$
- Liouvillian matrix $T = Q^* \otimes 1 + 1 \otimes Q + R^* \otimes R$
- Vector of distances $\tau = (\tau_1, \tau_2, \ldots, \tau_{N-1}) \in \mathbb{R}^{N-1}$
- Matrices $M^{[j]}$ are $R^* \otimes R$, $1 \otimes R$, $R^* \otimes 1$ and so on

Verstraete, Cirac, *Phys Rev Lett* **104**, 190405 (2010)
Osborne, Eisert, Verstraete, *Phys Rev Lett* **105**, 260401 (2010)

Question

- Full quantum state: needs to specify all correlation functions

- **Question:** It is possible to characterize a (continuous) matrix product state from low-order correlation functions only?

Huebener, Mari, Eisert, arXiv:1207.6537, *Phys Rev Lett,* in press (2012)

Basic idea for continuous-matrix product states

Correlation functions

$$\langle O^{i_N} O^{N-1} \dots O^{(i_1)} \rangle = C^{(N)}(\boldsymbol{\tau})$$

Laplace transform

$$\mathcal{L}^{(N)}(\boldsymbol{s}) = \int_0^\infty d^{N-1}\boldsymbol{\tau} e^{-\boldsymbol{s}\cdot\boldsymbol{\tau}} \mathcal{C}^{(N)}(\boldsymbol{\tau}), \;\; s_1, \dots, s_N \in \mathbb{C}$$

When T non-degenerate

$$\mathcal{C}^{(N)}(\boldsymbol{\tau}) = \sum_{k_1,\dots,k_{N-1}=1}^{d^2} c^{(N)}(k_1,\dots,k_{N-1}) e^{\lambda_{k_1}\tau_1} \dots e^{\lambda_{k_{N-1}}\tau_{N-1}}$$

$$\mathcal{L}^{(N)}(\boldsymbol{s}) = \sum_{k_1,\dots,k_{N-1}}^{d^2} \frac{c^{(N)}(k_1,\dots,k_{N-1})}{(\lambda_{k_1} - s_1)\dots(\lambda_{k_{N-1}} - s_{N-1})}$$

Theorem: (If all two-point function transforms show all the poles (generically the case) all residues of all the poles of all N-point functions with $N \leq 3$ can be obtained)

can give explicit formulas expressing all N-point functions in terms of the **2- and 3-point functions only**

Huebener, Mari, Eisert, arXiv:1207.6537, *Phys Rev Lett*, in press (2012)

Basic idea for matrix product states

Correlation functions

$$\langle O^{i_N} O^{N-1} \dots O^{(i_1)} \rangle = C^{(N)}(\boldsymbol{\tau})$$

Z-transform

$$\mathcal{Z}^{(N)}(\boldsymbol{\tau}) = \sum_{n_1,\dots,n_N} s_1^{n_1} \dots s_N^{n_N} C^{(N)}(\boldsymbol{n}), \; s_1, \dots, s_N \in \mathbb{C}$$

...

Theorem: (If all two-point function transforms show all the poles (generically the case) all residues of all the poles of all N-point functions with $N \leq 3$ can be obtained)

can give explicit formulas expressing all N-point functions in terms of the **2- and 3-point functions only**

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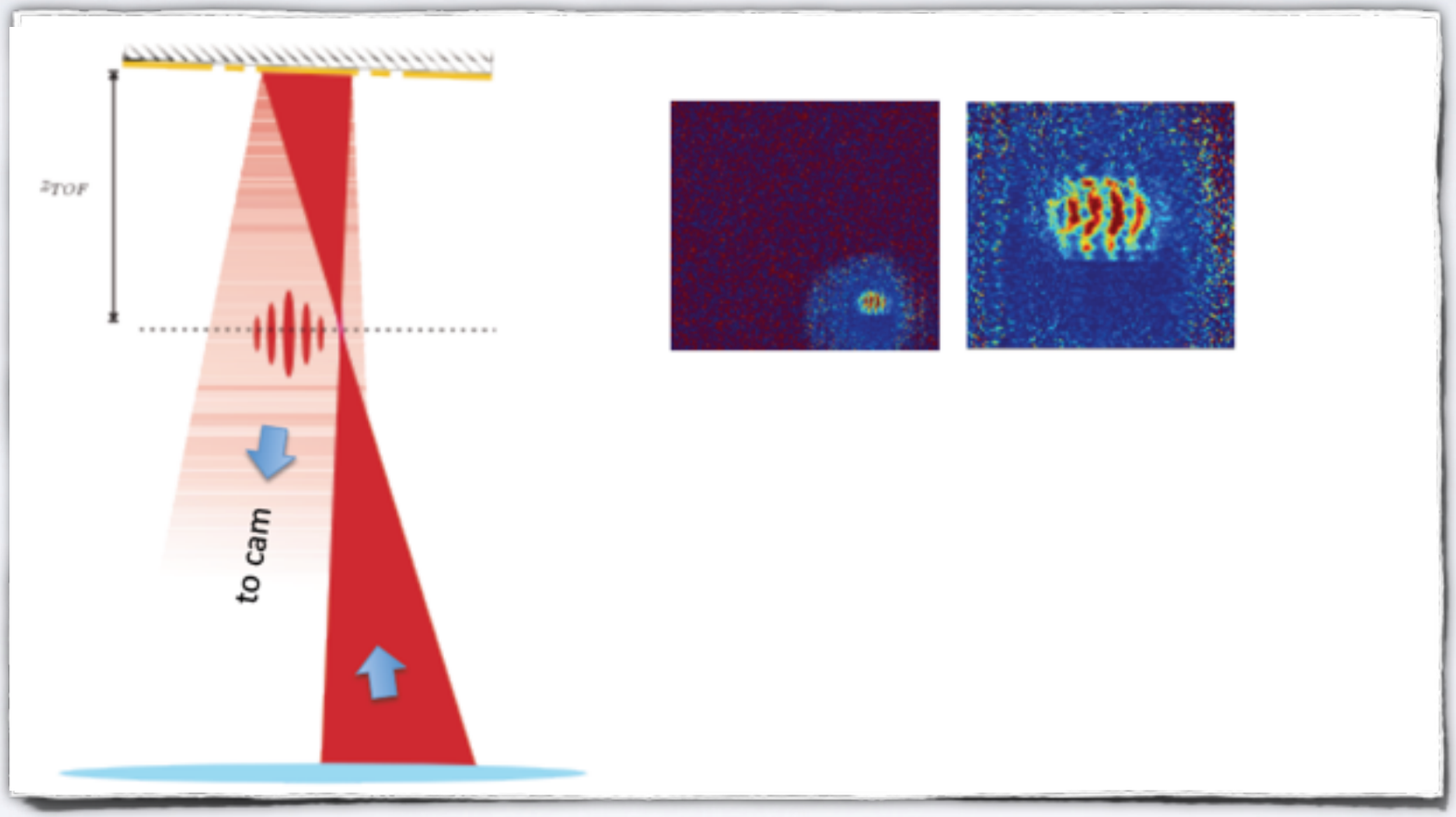
Lesson of part 2

- **Lesson:**
- **MPS and cMPS** can be generically reconstructed based on **2- and 3-point** functions only: "Wick's theorem" for (c)MPS

- Contribution to grasping many-body systems in **"MPS-world"**
- Structural insight into **low entanglement states**
- Relationship to **hidden Markov** models?
- Diagrammatic methods?

Huebener, Mari, Eisert, arXiv:1207.6537, *Phys Rev Lett,* in press (2012)

Lesson of part 2

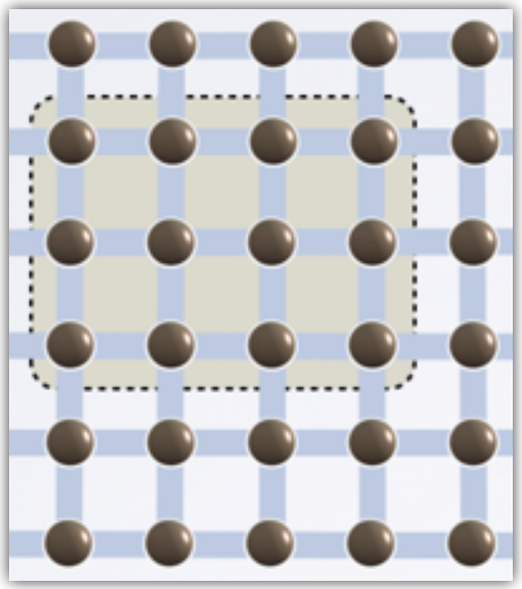


Langen, Schmiedmayer et al (2012)

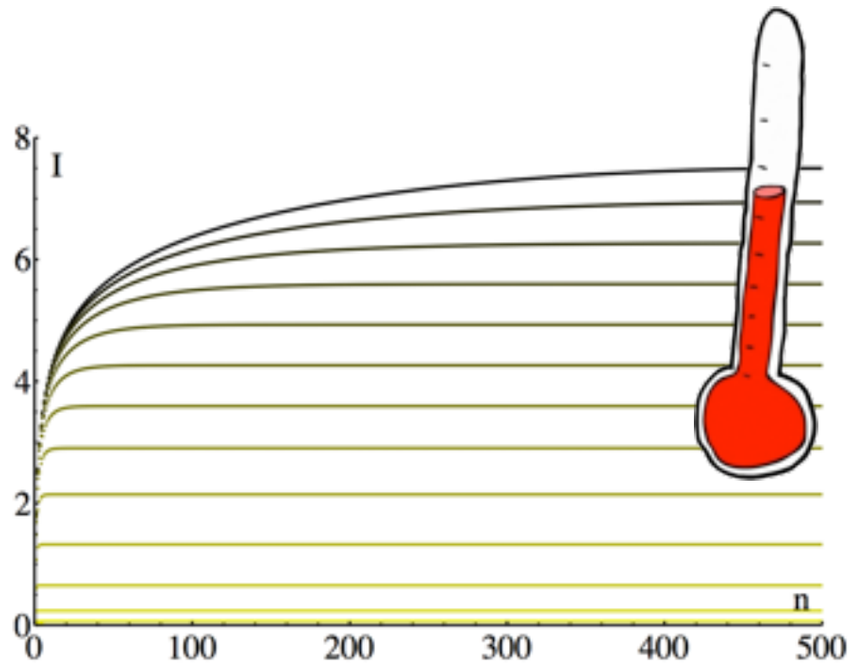
- Reconstructing unknown states from correlation functions?
- **Quantum state tomography** of quantum fields

Summary and outlook

- Entanglement spectra and area laws



- Toeplitz operator techniques capturing thermal spectra



- "Wick's theorem" for matrix-product states

Thanks for your attention!