On the nature of entanglement Hamiltonians

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- Work with Ming-Chiang Chung and Viktor Eisler -

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Consider system in quantum state $|\Psi\rangle$

- Divide in two parts $\alpha = 1, 2$
- Consider reduced density matrix $\rho_\alpha$
- Write as
  \[
  \rho_\alpha = \frac{1}{Z} e^{-\mathcal{H}_\alpha}
  \]

**Question:** What is the character of $\mathcal{H}_\alpha$?

This talk

- Two types of geometries
- Mainly free-particle models
- Features of $\mathcal{H}_\alpha$
- Relation of $\mathcal{H}_\alpha$ and $H_\alpha$

**Article:** EPL 96, 50006 (2011)
Fermionic hopping models

Hamiltonian, one dimension

\[ H = -\frac{1}{2} \sum_n t_n \left( c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n \right) \]

Ground state: Fermi sea, Slater determinant

Entanglement Hamiltonian: For \( L \) sites in subsystem

\[ \mathcal{H}_\alpha = \sum_{i,j=1}^{L} h_{i,j} c_i^\dagger c_j = \sum_{l=1}^{L} \varepsilon_l f_l^\dagger f_l \]

Determined by correlation matrix \( C_{i,j} = \langle c_i^\dagger c_j \rangle \) in subsystem

Matrix: \( \mathbf{h} = \ln \left[ \frac{(1 - \mathbf{C})}{\mathbf{C}} \right] \)

Eigenvalues: \( \varepsilon_l = \ln \left[ \frac{(1 - \zeta_l)}{\zeta_l} \right] \)

Elements: \( h_{i,j} = \sum_{l=1}^{L} \phi_l(i) \varepsilon_l \phi_l(i) \)
Homogeneous chain 1

Segment in half-filled infinite chain
Single-particle eigenvalues and eigenfunctions of $\mathcal{H}_\alpha$

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Homogeneous chain 2

Form of $\mathcal{H}_\alpha$ in real space

Features

- Inhomogeneous system
- Mainly nearest-neighbour hopping
- Large in centre, small near interfaces, $h_{n,n+1} \sim \pi \ n(1 - n/L)$

Note: Only nature of Fermi sea enters
Chain with two defects 1
Segment coupled to remainder by weak hopping, $t = 0.1$
Single-particle eigenvalues and eigenfunctions of $\mathcal{H}_\alpha$
Chain with two defects 2

Form of $\mathcal{H}_\alpha$ in real space

Features

- Inhomogeneous system
- Mainly nearest-neighbour hopping
- **New**: Amplitudes larger and alternating, also for longer-range hops
**Dimerized chain 1**
Segment in chain with alternating hopping $$(1 \pm \delta)$$, $$\delta = 0.1$$
Single-particle eigenvalues and eigenfunctions of $$\mathcal{H}_\alpha$$

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Dimerized chain 2

Form of $\mathcal{H}_\alpha$ in real space

Features

- Inhomogeneous system
- Mainly nearest-neighbour hopping
- Strong alternation and plateau in bulk

Note: Positions of boundaries matter
Dimerized chain 3

Move segment by one lattice constant
⇒ Large hopping across interfaces and in centre

Form of $\mathcal{H}_\alpha$ in real space

Features
- Alternation as before
- Largest hopping now in centre
- Pattern reflects physical Hamiltonian
Fermionic ladder 1

Hamiltonian

\[ H = \sum_q \gamma_q a_q^\dagger a_q - \sum_q \gamma_q b_q^\dagger b_q + \sum_q \delta (a_q^\dagger b_q + b_q^\dagger a_q) \]

- Diagonalize with \( a_q = u_q \alpha_q + v_q \beta_q \)

\[ H = \sum_q \omega_q (\alpha_q^\dagger \alpha_q - \beta_q^\dagger \beta_q), \quad \omega_q = \sqrt{\gamma_q^2 + \delta^2} \]

- Correlation matrix diagonal in momentum space gives for leg 1

\[ \varepsilon_q = \ln \left( \frac{\omega_q + \gamma_q}{\omega_q - \gamma_q} \right) \]

- Entanglement Hamiltonian

\[ \mathcal{H}_1 = \sum_q \varepsilon_q a_q^\dagger a_q \]

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Fermionic ladder 2

Single-particle eigenvalues and corresponding hopping amplitudes,
\[ \gamma_q = -\cos q \] and various \( \delta \)

**Features**
- Small \( \delta \): Dispersion steep, hopping long-ranged
- Large \( \delta \): Dispersion flat, hopping short-ranged
Fermionic ladder 3

• Strong-coupling limit, $\delta \gg \gamma_q \Rightarrow \varepsilon_q = 2\gamma_q/\delta$

$$\mathcal{H}_1 = \frac{2}{\delta} H_1,$$

Direct proportionality

• Two arbitrary legs with dispersions $\gamma_1q$ and $\gamma_2q$

$$\mathcal{H}_1 = \frac{1}{\delta} \sum_q (\gamma_1q - \gamma_2q) a_q^{\dagger} a_q$$

Both legs appear in entanglement Hamiltonian

$$\mathcal{H}_1 = \frac{1}{\delta} (H_1 - \tilde{H}_2)$$

• Large $\delta$ means $H_1, H_2$ are perturbations

**Question**: Can’t one use perturbation theory?  
**A**: Yes, we can!
Strongly coupled systems 1

System composed of two parts 1,2

\[ H = H_1 + H_2 + H_{12} \]

Assume \( H_{12} \) large, calculate ground state to first order

\[ |\psi_0^1 \rangle = |\psi_0 \rangle - \sum_{k \neq 0} |\psi_k \rangle \frac{< \psi_k | (H_1 + H_2) | \psi_0 \rangle}{E_k - E_0} \]

- Assumption 1: Only coupling to states with gap \( \Delta = E_k - E_0 \)
- Assumption 2: Matrix elements \( < \psi_k | H_1 | \psi_0 \rangle = < \psi_k | H_2 | \psi_0 \rangle \)

\[ |\psi_0^1 \rangle = |\psi_0 \rangle - \frac{2}{\Delta} \sum_{k \neq 0} |\psi_k \rangle < \psi_k | H_1 | \psi_0 \rangle \]

\[ = |\psi_0 \rangle - \frac{2}{\Delta} \hat{H}_1 |\psi_0 \rangle \]

\[ \hat{H}_1 = H_1 - < \psi_0 | H_1 | \psi_0 \rangle \]
Strongly coupled systems 2

Total density matrix

\[ \rho^1 = \langle \Psi_0 | \Psi_0 \rangle - \frac{2}{\Delta} \left[ \hat{H}_1 \langle \Psi_0 | \Psi_0 \rangle + \langle \Psi_0 | \Psi_0 \rangle \hat{H}_1 \right] \]

Reduced density matrix in part 1

\[ \rho^1_1 = \rho_1 - \frac{2}{\Delta} (\hat{H}_1 \rho_1 + \rho_1 \hat{H}_1) \]

• Assumption 3 : \( \rho_1 \) multiple of unit matrix

\[ \rho^1_1 = \frac{1}{Z} \exp(-\frac{4}{\Delta} H_1) \]

Result:

\[ \mathcal{H}_1 = \frac{4}{\Delta} H_1 \]

\( \mathcal{H}_1 \) proportional to \( H_1 \)
Discussion

Comments

- Condition 3 means maximal entanglement in $|\Psi_0>$
- Conditions 1 and 2 can be weakened

Generalizations

- Higher order perturbation theory (Läuchli and Schliemann 2012)
- Higher spin (Schliemann and Läuchli 2012)

Examples

- Spin-1/2 Heisenberg ladders (Poilblanc 2010)
- AKLT ladders (Cirac, Poilblanc, Schuch and Verstraete 2011)
- Left and right moving fermions (Qi, Katsura and Ludwig 2012)
- Coupled planes (Schliemann 2011)
Summary

Segments in Chains

- $\mathcal{H}_\alpha$ describes inhomogeneous system
- Only certain traces of $H_\alpha$ visible
- Spectra reflect connectivity

Legs of Ladders

- $\mathcal{H}_\alpha \sim H_\alpha$ for strong entanglement
- Can be seen in numerical single-particle spectra
- Thermal initial state when subsystems are decoupled
- **Note:** Considerations general, not limited to ladders