On the nature of entanglement Hamiltonians

Ingo Peschel

Fachbereich Physik, Freie Universität Berlin

• Work with Ming-Chiang Chung and Viktor Eisler •

Dresden, November 2012

Introduction

Consider system in quantum state $|\Psi\rangle$

- Divide in two parts $\alpha = 1, 2$
- Consider reduced density matrix ρ_α
- Write as
$$\rho_\alpha = \frac{1}{Z}\, e^{-\mathcal{H}_\alpha}$$

Question: What is the character of $\mathcal{H}_\alpha$?

This talk

- Two types of geometries
- Mainly free-particle models
- Features of $\mathcal{H}_\alpha$
- Relation of $\mathcal{H}_\alpha$ and H_α

Article: EPL **96**, 50006 (2011)

Fermionic hopping models

Hamiltonian, one dimension

$$H = -\frac{1}{2}\sum_n t_n \left(c_n^\dagger c_{n+1} + c_{n+1}^\dagger c_n\right)$$

Ground state: Fermi sea, Slater determinant

Entanglement Hamiltonian: For L sites in subsystem

$$\mathcal{H}_\alpha = \sum_{i,j=1}^{L} h_{i,j} c_i^\dagger c_j = \sum_{l=1}^{L} \varepsilon_l \, f_l^\dagger f_l$$

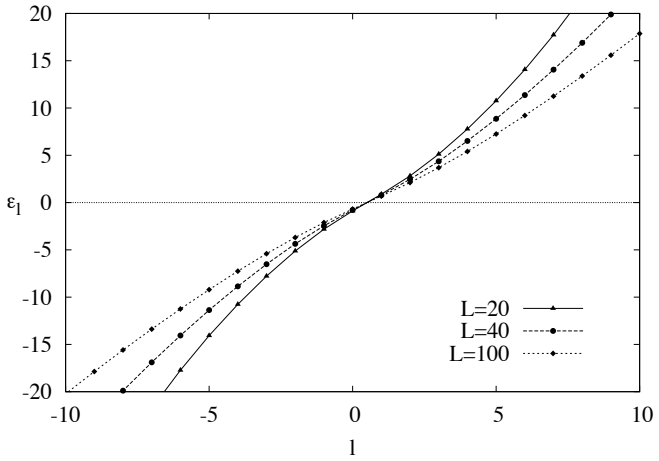
Determined by correlation matrix $C_{i,j} = \langle c_i^\dagger c_j \rangle$ in subsystem

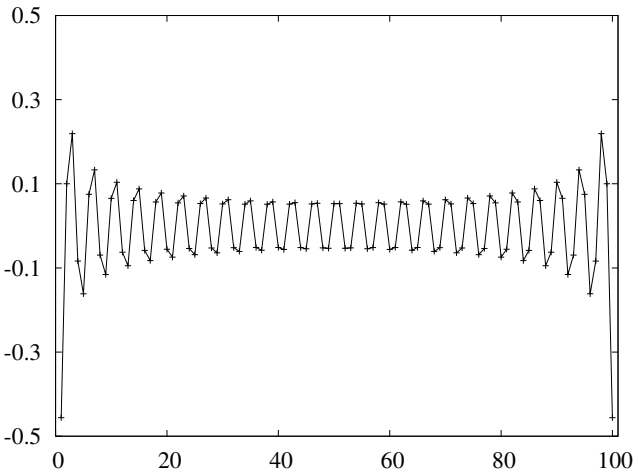
$$\begin{aligned} \text{Matrix}: \quad \mathbf{h} &= \ln\left[(\mathbf{1}-\mathbf{C})/\mathbf{C}\right] \\ \text{Eigenvalues}: \quad \varepsilon_l &= \ln\left[(1-\zeta_l)/\zeta_l\right] \\ \text{Elements}: \quad h_{i,j} &= \sum_{l=1}^{L} \phi_l(i)\, \varepsilon_l\, \phi_l(i) \end{aligned}$$

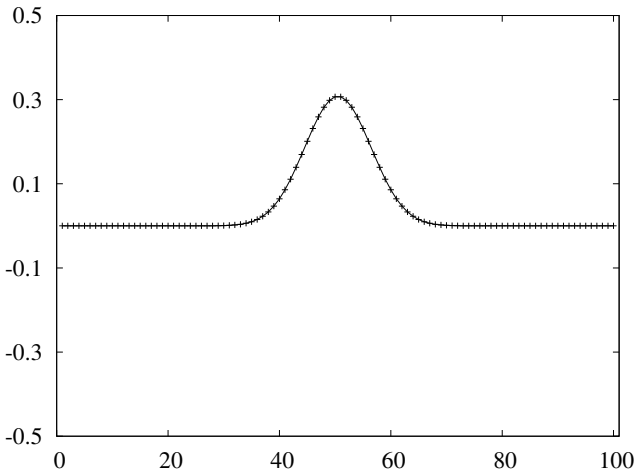
Homogeneous chain 1

Segment in half-filled infinte chain

Single-particle eigenvalues and eigenfunctions of $\mathcal{H}_\alpha$

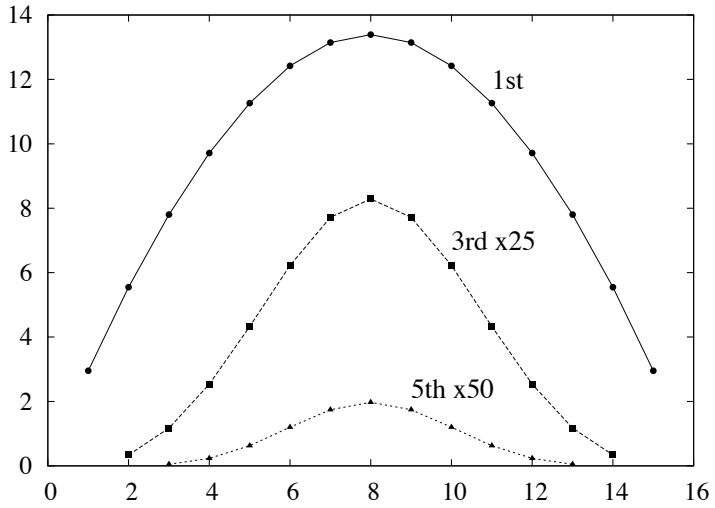






Homogeneous chain 2

Form of $\mathcal{H}_\alpha$ in real space



Features

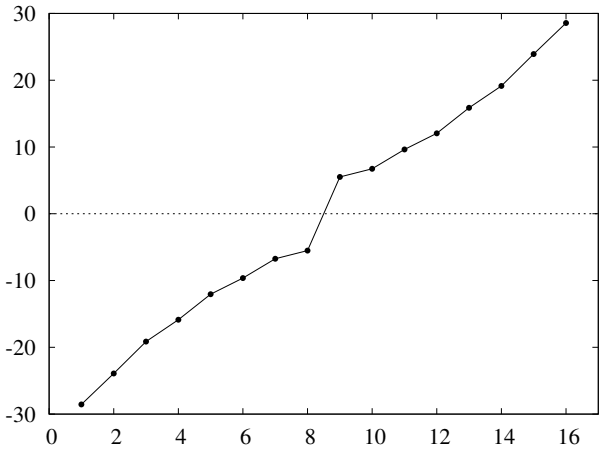
- Inhomogeneous system
- Mainly nearest-neighbour hopping
- Large in centre, small near interfaces, $h_{n,n+1} \simeq \pi\ n(1 - n/L)$

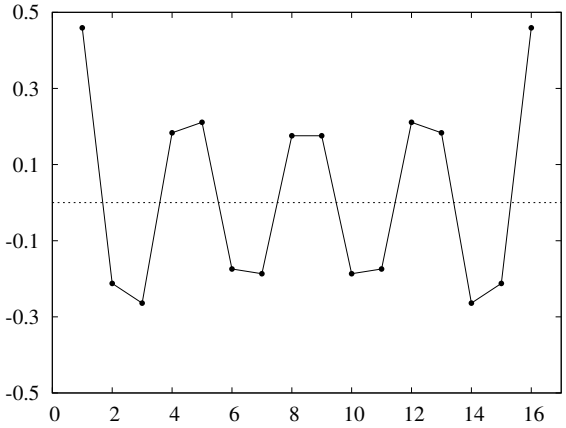
Note: Only nature of Fermi sea enters

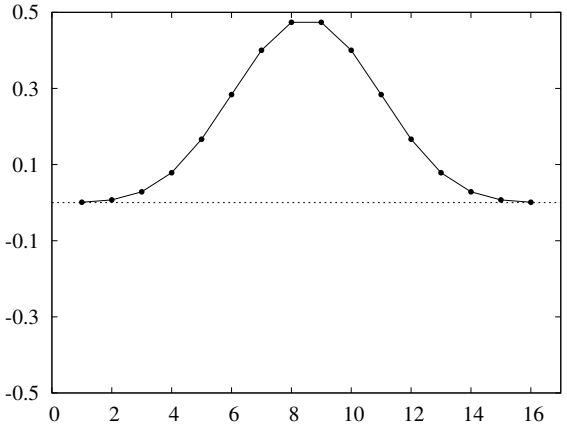
Chain with two defects 1

Segment coupled to remainder by weak hopping, $t = 0.1$

Single-particle eigenvalues and eigenfunctions of $\mathcal{H}_\alpha$

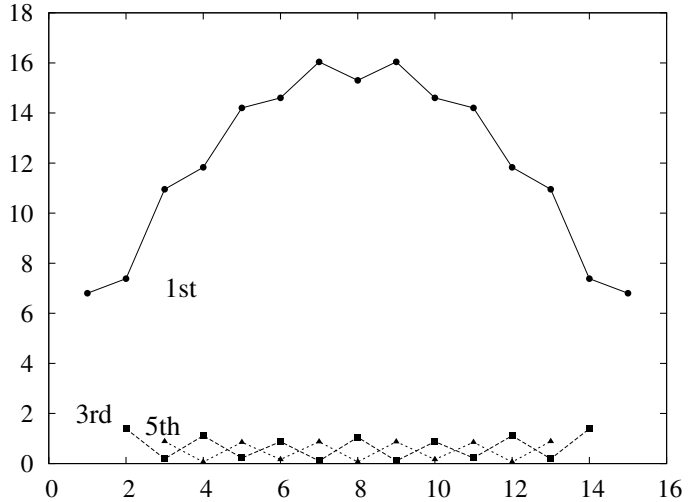






Chain with two defects 2

Form of $\mathcal{H}_\alpha$ in real space



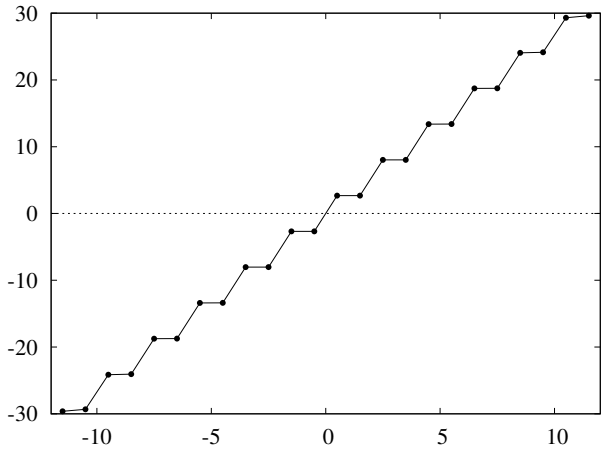
Features

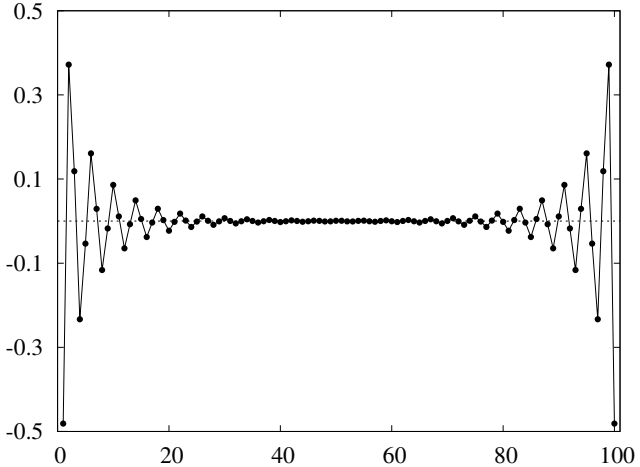
- Inhomogeneous system
- Mainly nearest-neighbour hopping
- **New**: Amplitudes larger and alternating, also for longer-range hops

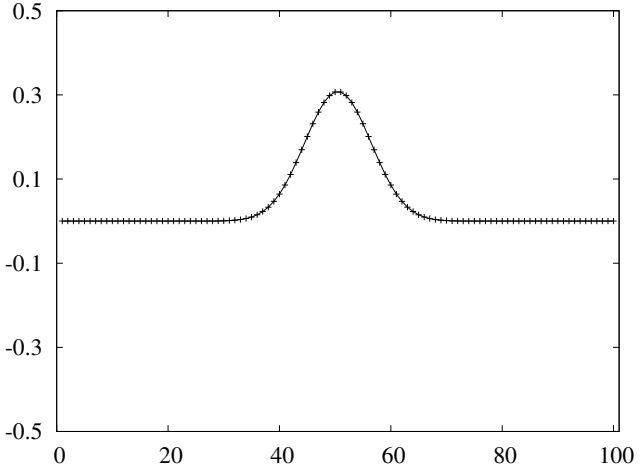
Dimerized chain 1

Segment in chain with alternating hopping $(1 \pm \delta)$, $\delta = 0.1$

Single-particle eigenvalues and eigenfunctions of $\mathcal{H}_\alpha$

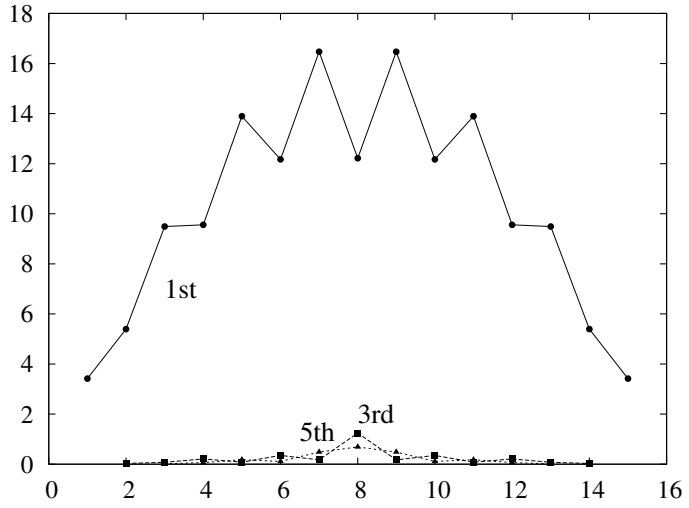






Dimerized chain 2

Form of $\mathcal{H}_\alpha$ in real space



Features

- Inhomogeneous system
- Mainly nearest-neighbour hopping
- Strong alternation and plateau in bulk

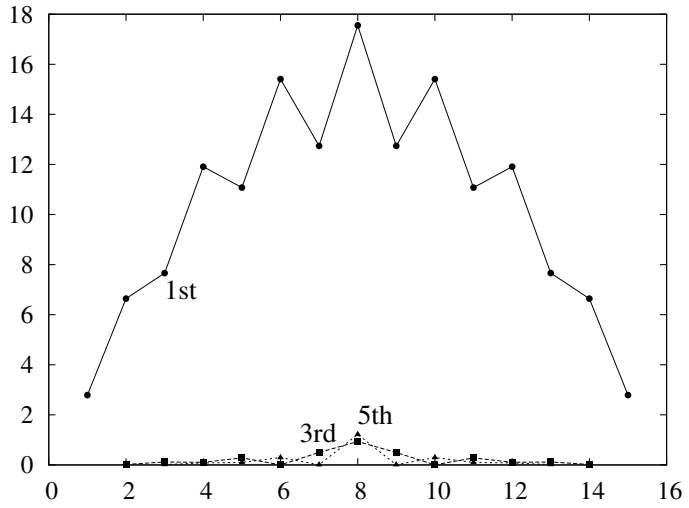
Note: Positions of boundaries matter

Dimerized chain 3

Move segment by one lattice constant

$\Rightarrow$ Large hopping across interfaces and in centre

Form of $\mathcal{H}_\alpha$ in real space



Features

- Alternation as before
- Largest hopping now in centre
- Pattern reflects physical Hamiltonian

Fermionic ladder 1

Hamiltonian

$$H = \sum_q \gamma_q \, a_q^\dagger a_q - \sum_q \gamma_q \, b_q^\dagger b_q + \sum_q \delta \left(a_q^\dagger b_q + b_q^\dagger a_q \right)$$

- Diagonalize with $a_q = u_q \alpha_q + v_q \beta_q$

$$H = \sum_q \omega_q (\alpha_q^\dagger \alpha_q - \beta_q^\dagger \beta_q) \ , \qquad \omega_q = \sqrt{\gamma_q^2 + \delta^2}$$

- Correlation matrix diagonal in momentum space gives for leg 1

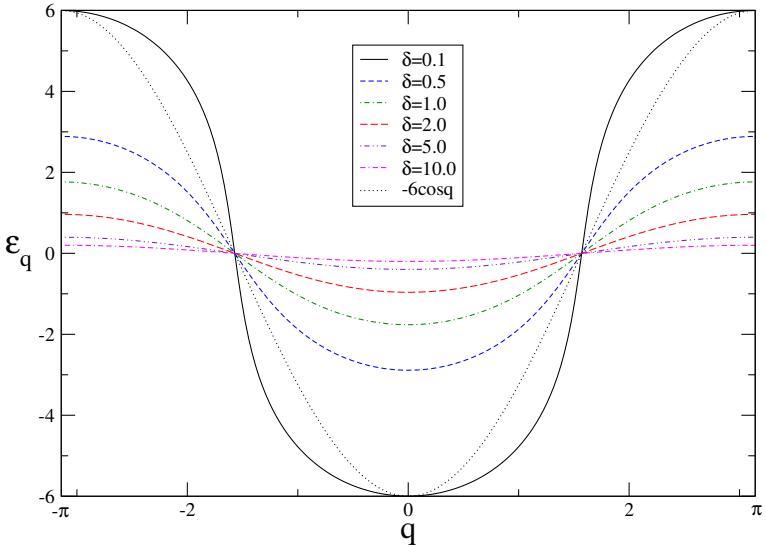
$$\varepsilon_q = \ln \left(\frac{\omega_q + \gamma_q}{\omega_q - \gamma_q} \right)$$

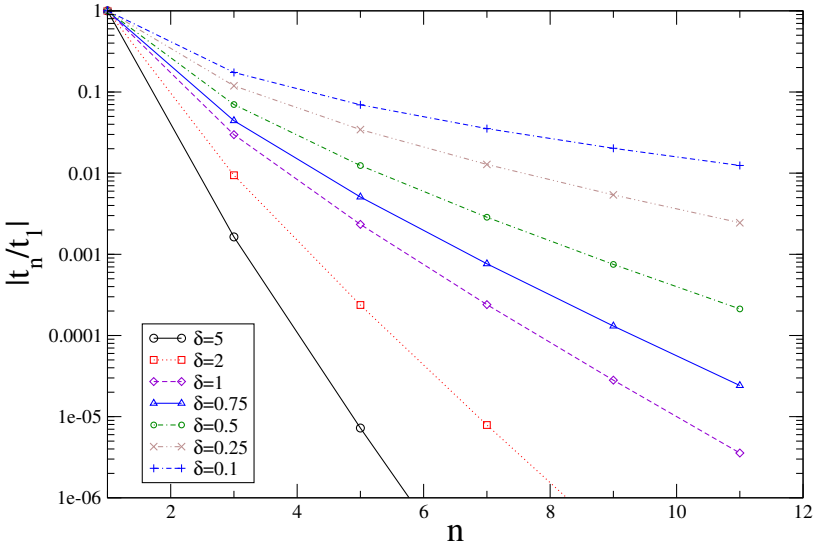
- Entanglement Hamiltonian

$$\mathcal{H}_1 = \sum_q \varepsilon_q a_q^\dagger a_q$$

Fermionic ladder 2

Single-particle eigenvalues and corresponding hopping amplitudes, $\gamma_q = -\cos q$ and various δ





Features

- Small δ : Dispersion steep, hopping long-ranged
- Large δ : Dispersion flat, hopping short-ranged

Fermionic ladder 3

- Strong-coupling limit, $\delta \gg \gamma_q \;\Rightarrow\; \varepsilon_q = 2\gamma_q/\delta$

$$\mathcal{H}_1 = \frac{2}{\delta} H_1,$$

Direct proportionality

- Two arbitrary legs with dispersions γ_{1q} and γ_{2q}

$$\mathcal{H}_1 = \frac{1}{\delta} \sum_q (\gamma_{1q} - \gamma_{2q})\, a_q^\dagger a_q$$

Both legs appear in entanglement Hamiltonian

$$\mathcal{H}_1 = \frac{1}{\delta} \left(H_1 - \tilde{H}_2\right)$$

- Large δ means H_1, H_2 are perturbations

Question: Can't one use perturbation theory ? **A**: Yes, we can !

Strongly coupled systems 1

System composed of two parts 1,2

$$H = H_1 + H_2 + H_{12}$$

Assume H_{12} **large**, calculate ground state to first order

$$|\Psi_0^1 >= |\Psi_0 > - \sum_{k \neq 0} |\Psi_k > \frac{< \Psi_k|(H_1 + H_2)|\Psi_0 >}{E_k - E_0}$$

- Assumption 1 : Only coupling to states with gap $\Delta = E_k - E_0$
- Assumption 2 : Matrix elements $< \Psi_k|H_1|\Psi_0 >=< \Psi_k|H_2|\Psi_0 >$

$$\begin{aligned} |\Psi_0^1 > &= |\Psi_0 > - \frac{2}{\Delta} \sum_{k \neq 0} |\Psi_k >< \Psi_k|H_1|\Psi_0 > \\ &= |\Psi_0 > - \frac{2}{\Delta} \hat{H}_1 |\Psi_0 > \end{aligned}$$

$$\hat{H}_1 = H_1 - < \Psi_0|H_1|\Psi_0 >$$

Strongly coupled systems 2

Total density matrix

$$\rho^1 = |\Psi_0 >< \Psi_0| - \frac{2}{\Delta}\left[\hat{H}_1 \, |\Psi_0 >< \Psi_0| + |\Psi_0 >< \Psi_0| \, \hat{H}_1\right]$$

Reduced density matrix in part 1

$$\rho_1^1 = \rho_1 - \frac{2}{\Delta}(\hat{H}_1\rho_1 + \rho_1\hat{H}_1)$$

- Assumption 3 : ρ_1 multiple of unit matrix

$$\rho_1^1 = \frac{1}{Z}\exp(-\frac{4}{\Delta}H_1)$$

Result:

$$\mathcal{H}_1 = \frac{4}{\Delta}H_1$$

$\mathcal{H}_1$ proportional to H_1

Discussion

Comments

- Condition 3 means maximal entanglement in $|\Psi_0>$
- Conditions 1 and 2 can be weakened

Generalizations

- Higher order perturbation theory (Läuchli and Schliemann 2012)
- Higher spin (Schliemann and Läuchli 2012)

Examples

- Spin-1/2 Heisenberg ladders (Poilblanc 2010)
- AKLT ladders (Cirac, Poilblanc, Schuch and Verstraete 2011)
- Left and right moving fermions (Qi, Katsura and Ludwig 2012)
- Coupled planes (Schliemann 2011)

Summary

Segments in Chains

- $\mathcal{H}_\alpha$ describes inhomogeneous system
- Only certain traces of H_α visible
- Spectra reflect connectivity

Legs of Ladders

- $\mathcal{H}_\alpha \sim H_\alpha$ for strong entanglement
- Can be seen in numerical single-particle spectra
- Thermal initial state when subsystems are decoupled
- **Note:** Considerations general, not limited to ladders