# Real Space Entanglement Spectra

## for real space quantum Hall wave functions

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***Entanglement Spectra in Complex Quantum Wavefunctions***

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Ollscoil na hÉireann Má Nuad

# Entanglement Spectra in the FQHE

In the FQHE, many systems with different topological orders exist.

How to distinguish them?
Especially if only Ground States are really accessible in numerics?

- Torus degeneracy (not this talk)

- Topological entanglement entropy! Kitaev-Preskill and Levin-Wen, PRL 96, 2006
  Needs scaling of subsystem size, extracting constant term – hard!
  Orbital, Particle Cuts introduced Haque, Zozulya, Schoutens, PRL 98, 2007

- **Entanglement Spectra!** Li, Haldane, PRL 101,2008
  Captures much more information, get info on excitations from ground state |Ψ>.

For cut into subsystems A and B, write $\rho_A = Tr_B \rho = Tr_B |\Psi><\Psi| \propto e^{-H_A}$

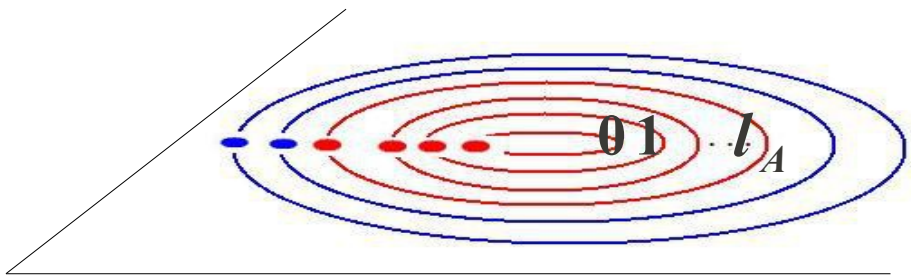
CONJECTURE (reading Li & Haldane + many later works) :
When $|\Psi>$ is the ground state of the full system,
$H_A$ is an effective Hamiltonian for subsystem A

Caution:it's not always true in this generality
However, for gapped systems, reasonable subsystems (partitioning),
degrees of freedom of excitations present in ground state,
we can at least hope for qualitative agreement.

# “Well-known” cuts for FQH-systems

Haque, Zozulya, Schoutens, PRL 98, 2007

## Orbital ES (OES)




Describes $N_A$ particles in the inner orbitals.

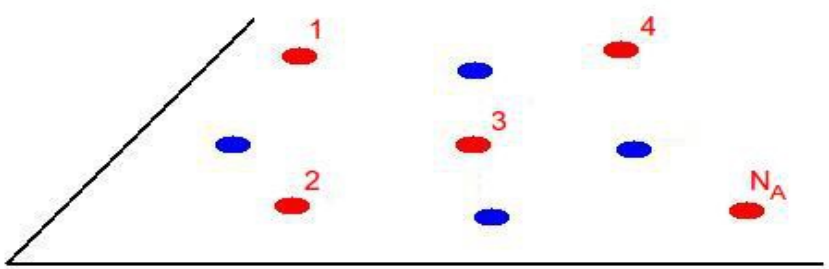
“fuzzy” spatial cut,

convenient for momentum space wave functions.

→ $H_A$ should describe FQH system on disk hence edge spectrum

Li, Haldane, PRL 101,2008

## Particle ES (PES)




Describes $N_A$ particles on the full disk/sphere

(at the same total flux as before)

Sterdyniak, Regnault, Bernevig, PRL 106, 2011

Density is lowered by (at least) half.

→ See many (bulk) quasiholes

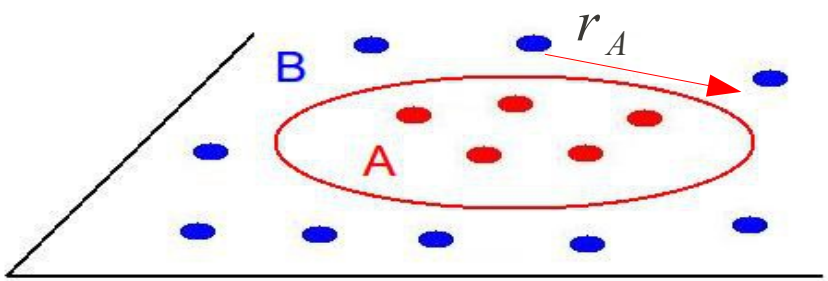
At lowest angular momenta, also edge

(all holes pushed to outside)

# Project: Real Space Cut

## Real Space ES (RSES)

Rodriguez, Slingerland, Simon arxiv:1111.3634, PRL 108, 2012
Sterdyniak et al. Arxiv:1111.2810, PRB 85, 2012
Dubail, Read, Rezayi arxiv:1111.2811, PRB 85, 2012




$H_A$ describes $N_A$ particles on a disc ("hard" cut).
→ edge excitations

Expect: convenient for real space wavefunctions

with $\nu_A = \frac{N_A}{N_\Phi^A} = \nu$ → CFT (Wen,Cappelli et al. Lopez-Fradkin... )

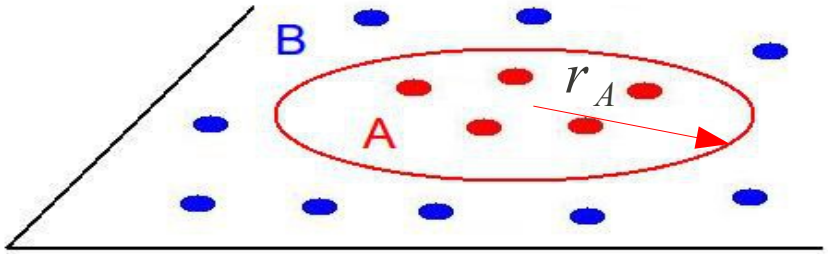
Goals:

- Calculate RSES efficiently using real space methods (MC, or even simpler for multiplicities)
  Can do "large" systems (bigger than diagonalization) with good trial wave functions

- Find relations between RSES and OES/PES

- Apply RSES to study of edges,
  - Where OES doesn't work well – e.g. integer fillings
  - Where angular momentum space wave function is difficult to obtain
  - Where edge CFT is not "obvious"
  - e.g. Jain states, BS hierarchy

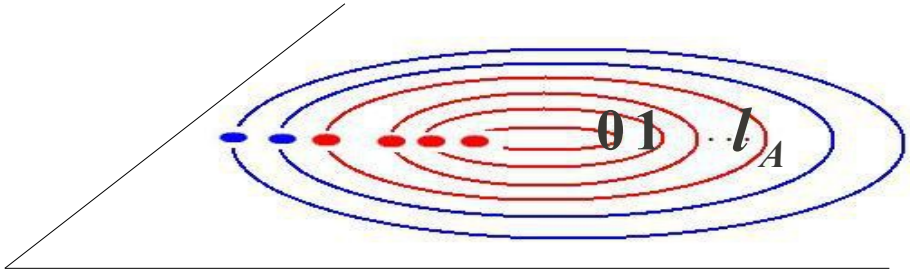
# Real Space vs. Orbital Entanglement Spectrum




A: $\vec{r}_1, \vec{r}_2, .., \vec{r}_{N_A} \setminus |\vec{r}_i| \leq r_A$ , i=1,.., $N_A$

B: $\vec{r}_{N_A+1}, .., \vec{r}_N \setminus |\vec{r}_i| > r_A$ , i= $N_A+1$,.., $N$

Real Space Partition




A: $N_A$ part. in {0,1,.., $l_A$} ; $\sqrt{l_A} \sim r_A$

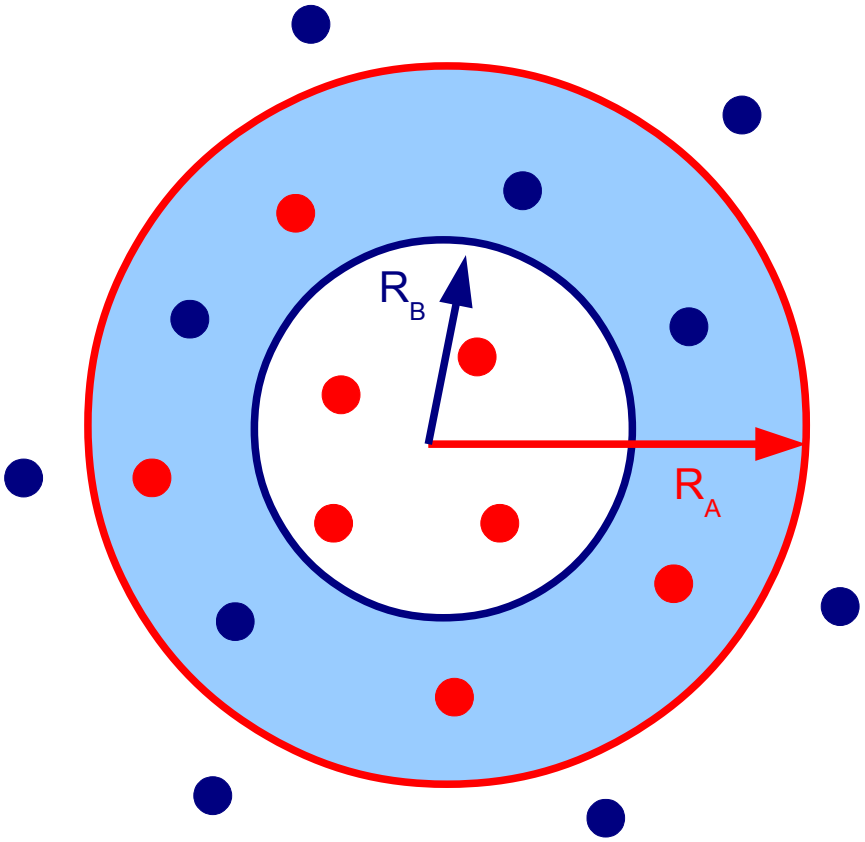
B: $N - N_A$ part. In {$l_A+1$ , $l_A+2$ , .. }

Orbital Partition in FQHE
(Haque, Zozulya, Schoutens PRL 98)
Orbitals centered on rings

Both partitions should describe edge excitations for FQH systems
Expect Real space partition to do better at high filling (esp. filling 1)
Because a larger Hilbert space is available for the A subsystem

NOTE: actual spectra are qualitative – e.g. no edge potential introduced.
Symmetry features (numbers of multiplets) should be preserved + expect CFT at large N

# RSES vs. PES




A continuous family of cuts.

$N_A$ Red particles inside the red circle
$N_B$ Blue particles outside the blue circle

If $R_A = R_B$,
we have no overlap region
and we get the RSES

If $R_B = 0$ and $R_A$ contains the full system,
all particles can go anywhere
and we get the PES

Conclusion: RSES and PES are adiabatically connected!

→ Must have the same numbers of angular momentum states.

(Of course the entanglement energies are different)

It gets better:

RSES and PES both count the number of *electron hole states* (next slides)

It is nontrivial to show that this is the same as the number of quasihole states (cf. Ardonne's talk)

# PES/RSES density matrix blocks as overlap matrices

For PES/RSES, the reduced density matrix preserves sectors with fixed angular momentum $L_z^A$ (and for RSES, fixed $N_A$). Below: $\rho_A$ in these sectors.

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PES: plane RSES: disc

$$\rho_A^{N_A, L_z^A}\left(\boxed{z_1, .., z_{N_A}}; z'_1, .., z'_{N_A}\right) = \int_B \boxed{\prod_{j=N_A+1}^{N} d z_j}\, \xi^{L_z^A}\left(z_1, .., z_{N_A}, z_{N_A+1}, .., z_N\right) \times \bar{\xi}^{L_z^A}\left(z'_1, .., z'_{N_A}, z_{N_A+1}, .., z_N\right)$$

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Here, $\xi^{L_z^A}(z_1, .., z_N) = \int d\phi\, e^{2\pi i L_z^A \phi} \psi\left(z_1 e^{-2\pi i \phi}, .., z_{N_A} e^{-2\pi i \phi}, z_{N_A+1}, .., z_N\right)$

In other words, $\xi$ is simply the projection of $\psi$ onto the desired $L_z^A$ sector

**We can now reinterpret these blocks.** First, write $\xi^{L_z^A}(Z_A, Z_B) = \xi_{Z_A}^{L_z^A}(Z_B)$

Note: $\xi_{Z_A}^{L_z^A}(Z_B)$ is the wave function for: a system of $N_B$ electrons (positions $Z_B$) with $N_A$ fixed electron holes at positions $Z_A$

$$\boxed{(\rho_A)_{Z_A, Z'_A}^{L_z^A, N_A} = \int d Z_B\, \xi_{Z_A}^{L_z^A}(Z_B)\, \bar{\xi}_{Z'_A}^{L_z^A}(Z_B)}$$

→ $\rho_A$ is the overlap matrix of *all* such Electron hole wave functions!

NB: The rank of the matrix of overlaps is the same as the number of independent wave functions

## Conclusions from overlap matrix interpretation

The rank of $\rho_A$ at given $L_z^A$ and $N_A$ is the number of independent wave functions for $N_B$ electrons with $N_A$ electron holes at this $L_z^A$

- In particular, this is finite if the wave functions live in a finite dimensional Hilbert space e.g. they are built from orbitals in a finite number of Landau levels on the sphere
- Any vanishing properties of the original wave function are present automatically for these wave functions in the A-system → much stronger bounds on ranks.

Again, we see that ranks(RSES)=rank(PES).

Looking at the same functions of $Z_A$, $Z_B$, just smaller domains for RSES

## Actual calculation of the RSES

Choose finite (d x d) submatrices of the $\rho_A^{N_A, L_z^A}$ I.e. a set of d "indices" $Z_{A,i}$ (i=1...d)
Also discretize the $Z_B$ integral

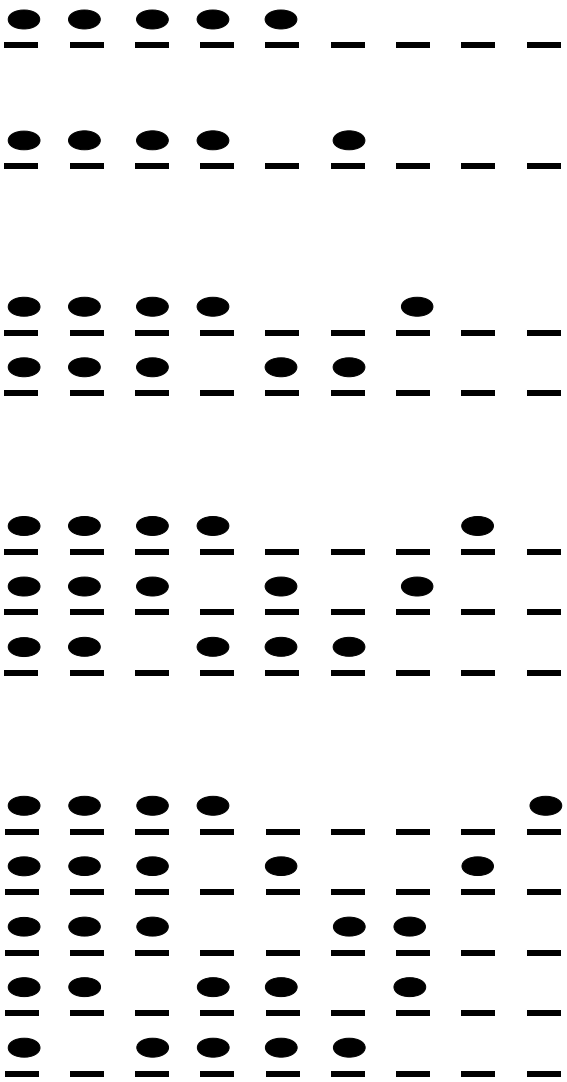
Calculate the ranks and spectra of these submatrices.
If the configurations $Z_A$ are chosen cleverly (I. Rodriguez' talk) then:

- We get convergence to the true spectrum of $\rho_A^{N_A, L_z^A}$ as d increases

- We can get the ranks already for much smaller d (if the wave function cooperates)
  Rank = number of nonzero ("large") eigenvalues.
  (Have calculated ranks for systems of ~100 particles on a laptop.)

# First test case: filling 1.

State counting



ΔL=0, 1 state

ΔL=1, 1 state

ΔL=2, 2 states, 2 = 1+1

ΔL=3, 3 states, 3 = 2+1 = 1+1+1

ΔL=4, 5 states, 4 = 3+1 = 2+2 = 2+1+1 = 1+1+1+1

Etc. ΔL=n, p(n) states, partition numbers

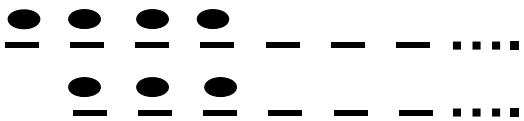
Generating function for $N_A$ to infinity: $$\prod_{k>0} \frac{1}{1-q^k}$$

OES doesn't work for edge counting
(Still have pure state after trace)

RSES gives “correct” result (see left)
Know this analytically
Rodriguez, Sierra, PRB 80, 2009

Reproduced perfectly by numerics

# Second test case: filling 2 (and higher integers).



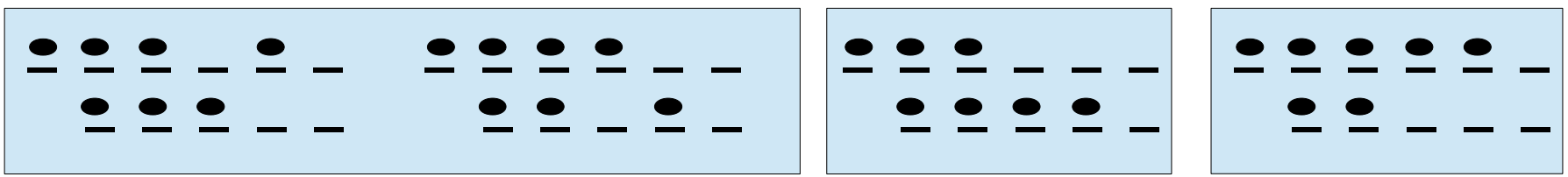
$N_{A,1}$ (here 4)
$N_{A,2}$ (here 3)

Naive approach: fix particle numbers in the LLs

Counting: $\Delta L=\Delta L_1+\Delta L_2=n_1+(n-n_1)$ gives number of states $\Sigma_{n1}\, p(n_1)p(n-n_1)$

Gives 1,2,5,10,20,...

However in reality, we can get excitations with various values of $N_{A,1}$ and $N_{A,2}$ for given $N_A$.



Here $\Delta L=1$, 4 states,

2 states with $N_{A,1}=N_{A,2}$
1 state with $N_{A,1}-N_{A,2}=2$
1 state with $N_{A,1}-N_{A,2}=-2$

This Is reflected by branches in the PES

Each branch: 1,2,5,10,20,...

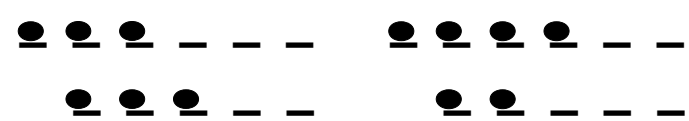
Total state counting: for $N_A$ odd, as above (on sphere), find

1 branch starting from $\Delta L=0$, 2 branches starting from $\Delta L=1$, 2 from $\Delta L=4$, 2 from $\Delta L=9$ etc.

Hence overall generating function

$$Z_{\nu=2,\, N_A \text{ odd}} = \left(\sum_{m\in\mathbb{Z}} q^{m^2}\right)\left(\prod_{k>0}\frac{1}{1-q^k}\right)^2$$

But note: there is an even-odd effect!



2 branches at $\Delta L=0$

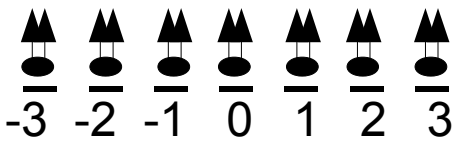
# On to Jain states, a reminder

Jain trial ground state (Laughlin for n=1)

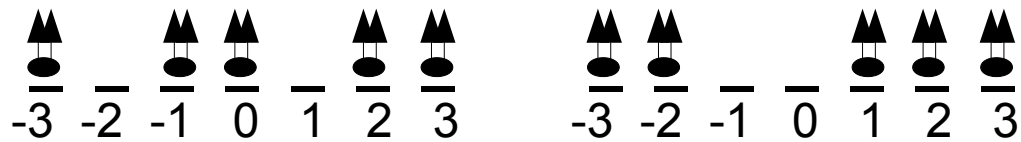
$$\Psi_{\frac{n}{nk\pm 1}}\left(z_1,\ldots,z_N\right)=\boldsymbol{P_{LLL}}\left(\prod_{i<j}^{N}\left(z_i-z_j\right)^k \phi_{\nu=\pm n}\left(z_1,..,z_N,\bar{z}_1,..,\bar{z}_N\right)\right)$$

Fermions at filling n with k fluxes attached (become bosons for odd k)

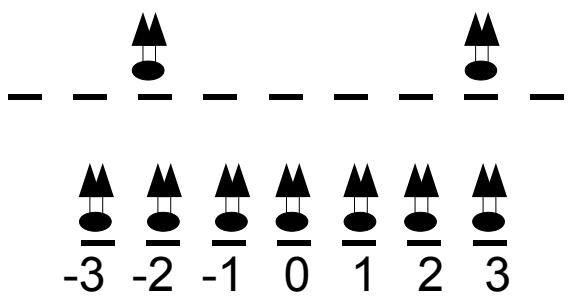
To get excited states, replace the fermionic state (examples below for n=1):




Ground state

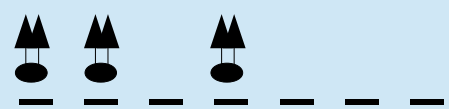



Two (bulk) quasihole states, $L_z$=1




Two quasielectron state, $L_z$=1

Trial ground state for disk is just lowest $L_z$ state as for integer filling



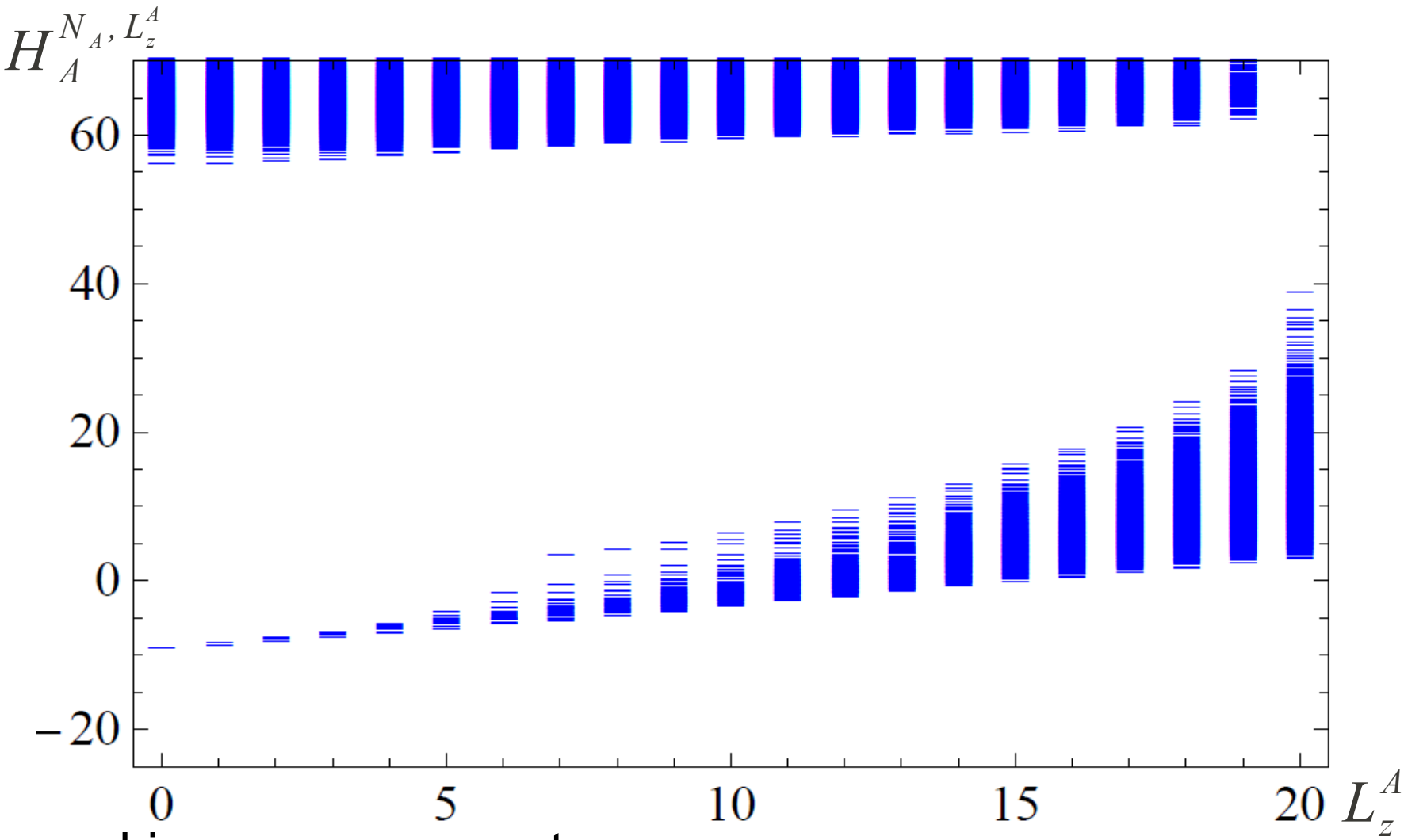
Trial states for edge quasiholes "same" as for integer filling. (Also if multiple CF Landau levels occupied)

Counting of states is usually reduced by LLL Projection, for bulk excitations.

For quasiholes in Laughlin states, no reduction (Projection does not come into play).

Laughlin $\nu = 1/2$ : $\Psi_{1/2}(z_1, .., z_N) = \prod_{i<j} (z_i - z_j)^2 e^{-\sum_i |z_i^2|/4}$

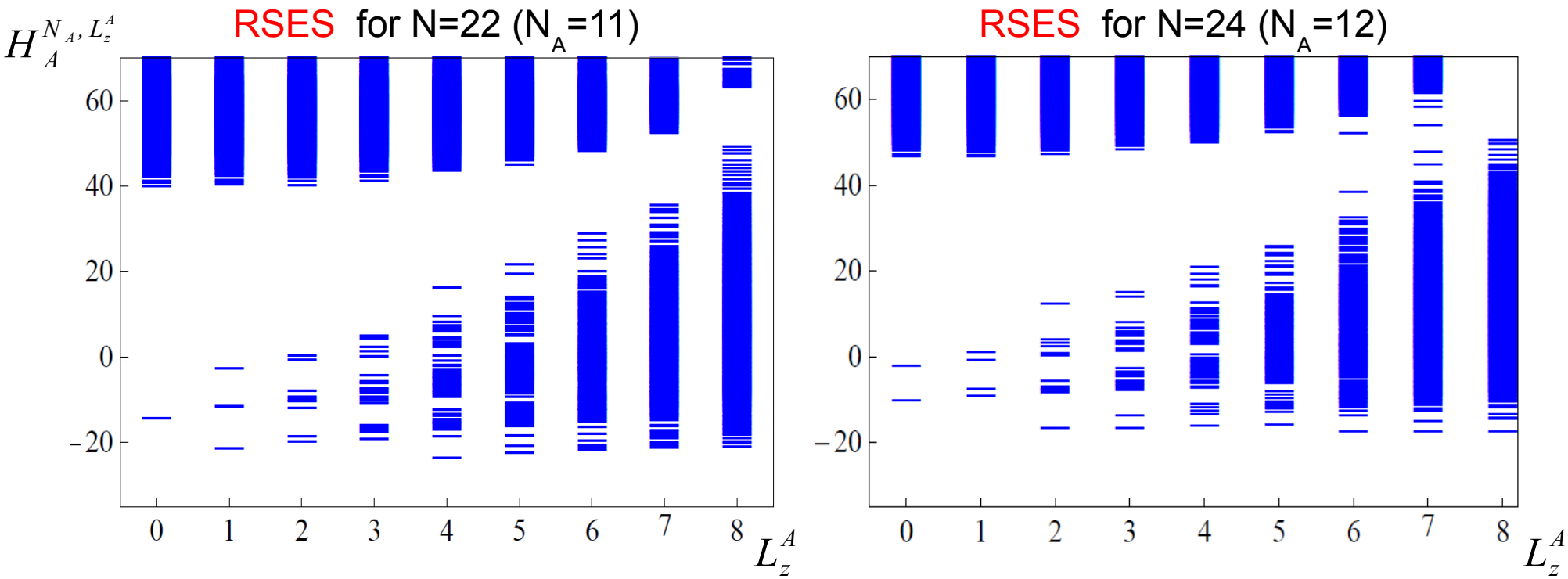
RSES for $N = 18$ $(N_A = 9)$




Linear energy spectrum,
counting as predicted by CFT (or previous slides)

Can get counting for much larger N, but nicely converged spectrum is more difficult

Jain $\nu$ = 2/5 : $\Psi_{2/5}(z_1,..,z_N) = \boldsymbol{P}\left(\phi_2(z_1,..,z_N,\bar{z}_1,..,\bar{z}_N)\prod_{i<j}(z_i - z_j)^2\right) e^{-\sum_i |z_i^2|/4}$




Clearly observe the predicted branches (actually, postdicted)

Each branch : 1,2,5,10,20,… ...NO reduction from LLL projection at low $L_z^A$

Looking at several system sizes, we may conjecture:

As $N_A$ goes to infinity, the multiplicy at fixed $L_z^A$ approaches the result for free fermions

**Q**: - Will this turn into a conformal spectrum as $N_A$ grows?

- Will the higher $L_z$ branches start at higher enanglement energies? (would make sense)
- Comparison to actual energy spectra?

Energy spectrum (approximate) for filling 12/5 with Coulomb interaction, taken from Sreejith, Jolad, Sen & Jain, Phys. Rev. B 84, 245104 (2011)

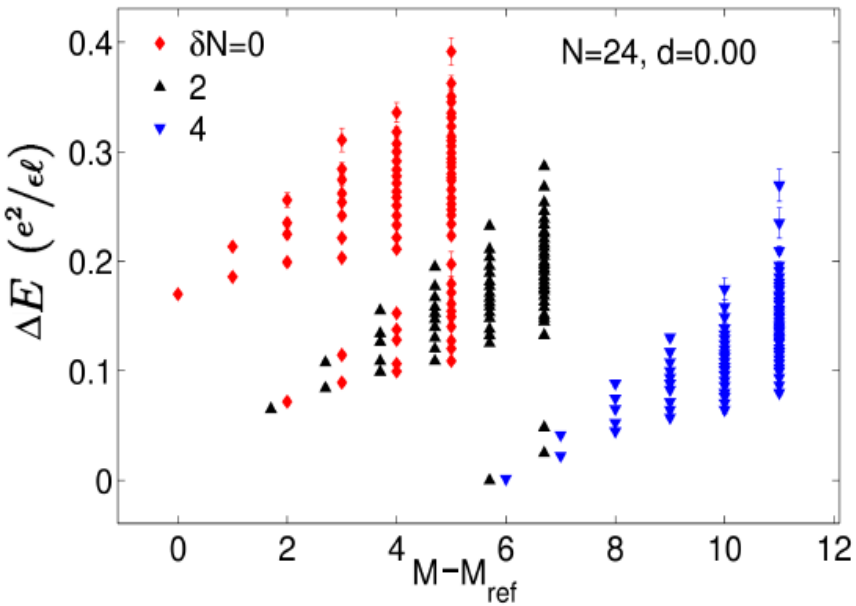
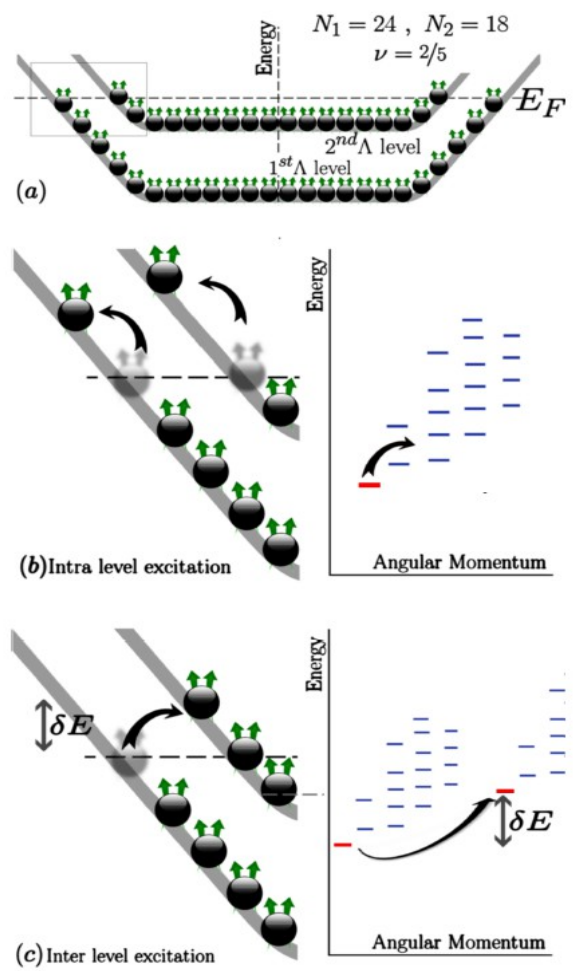



FIG. 3. (Color online) Energy spectrum of different sectors, $[N_1, N_2] = [(N+\delta N)/2, (N-\delta N)/2]$, of $^2/_5$th state obtained through CF diagonalization. The spectrum is evaluated for the Coulomb interaction, in the presence of a neutralizing positively charged disk that also provides a confinement potential. The energy is quoted in units of $e^2/(\epsilon\ell)$, where $\ell$ is the magnetic length and $\epsilon$ is the background dielectric constant; the zero of energy is set at the ground state of the $\delta N = 0$ sector. The following parameters are chosen: $N = 24$, $\delta N = 0, 2, 4$, setback distance $d = 0.0$. The different symbols are slightly offset horizontally to avoid clutter. Angular momentum measured relative to the sector-ground-state of $[N/2, N/2]$

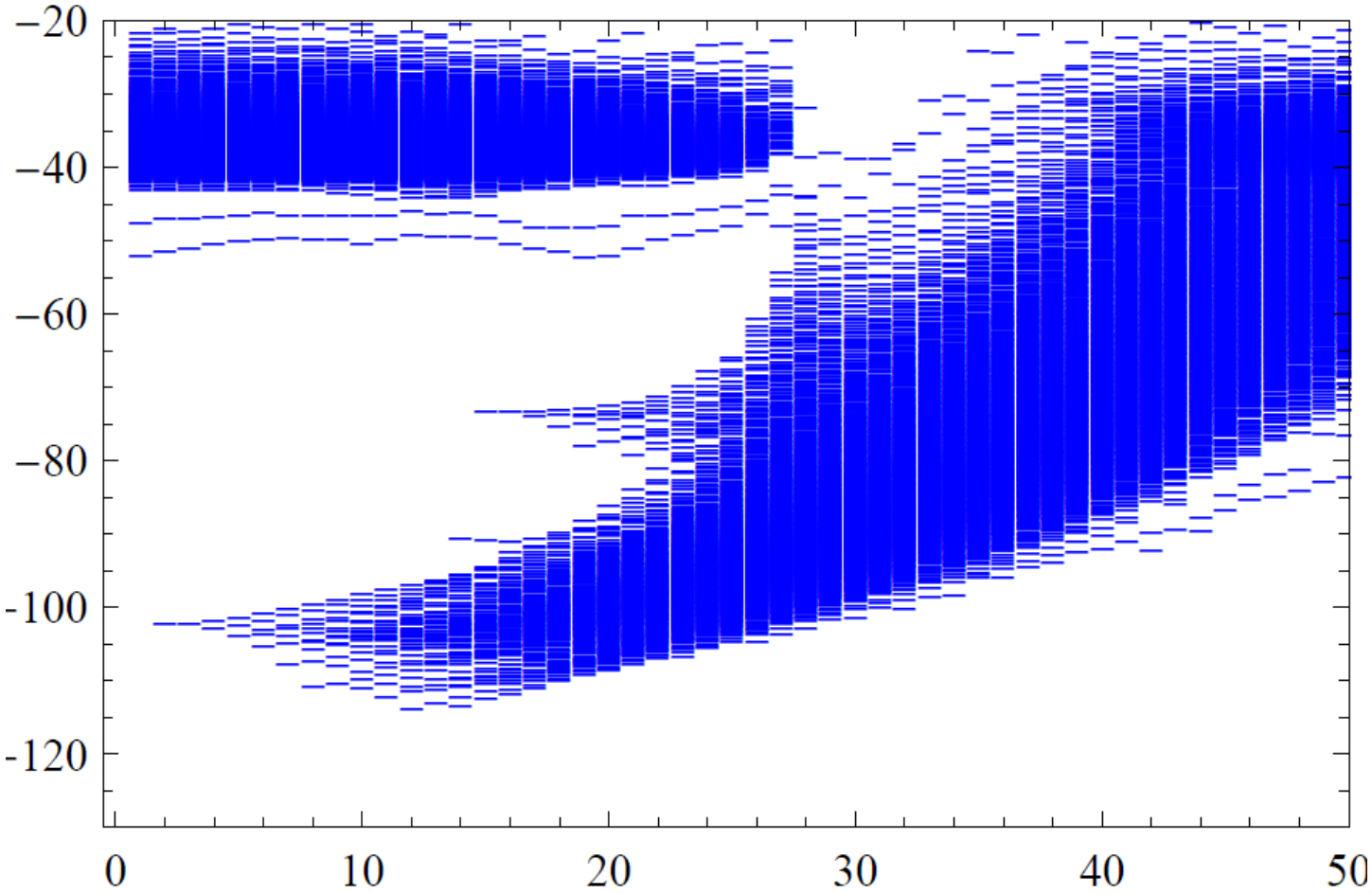
These authors find branches
- described by the same mechanism
- nevertheless some differences (?), e.g. branches overlap



# Reverse Flux Attachment *(very much in progress)*

RSES for Jain, filling 2/3, N=12 ($N_A$=6).

Note: *not* obtained as PH-dual of Laughlin 1/3 (though should be same universality)

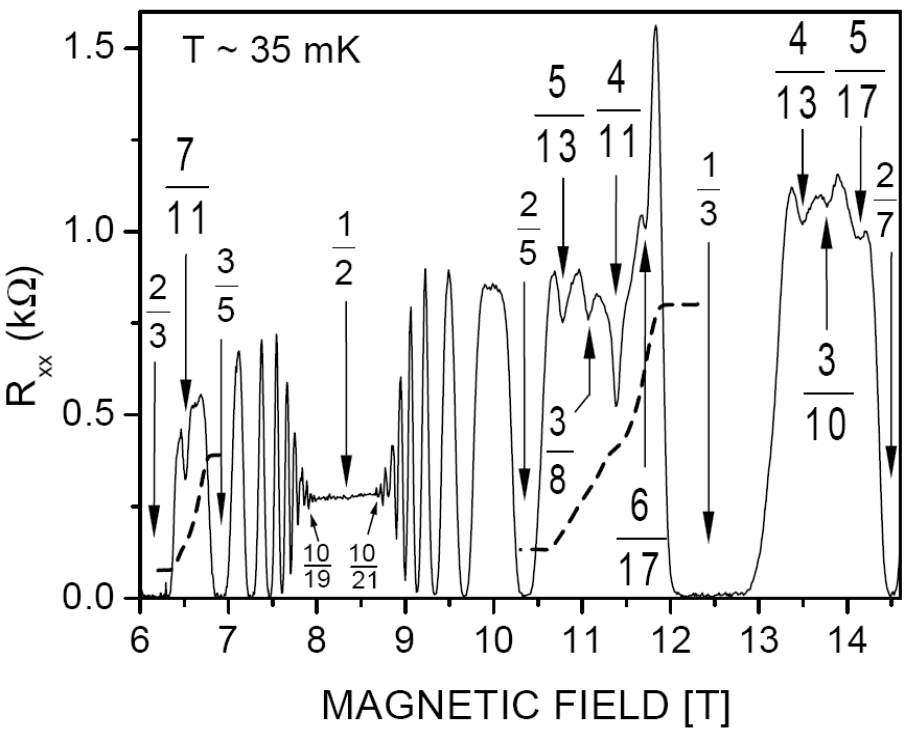



Despite two CF Landau Levels, see no series of branches.
Also counting is as for Laughlin: 1,1,2,3,5,7,11...
Obviously, here the LLL-Projection is essential.

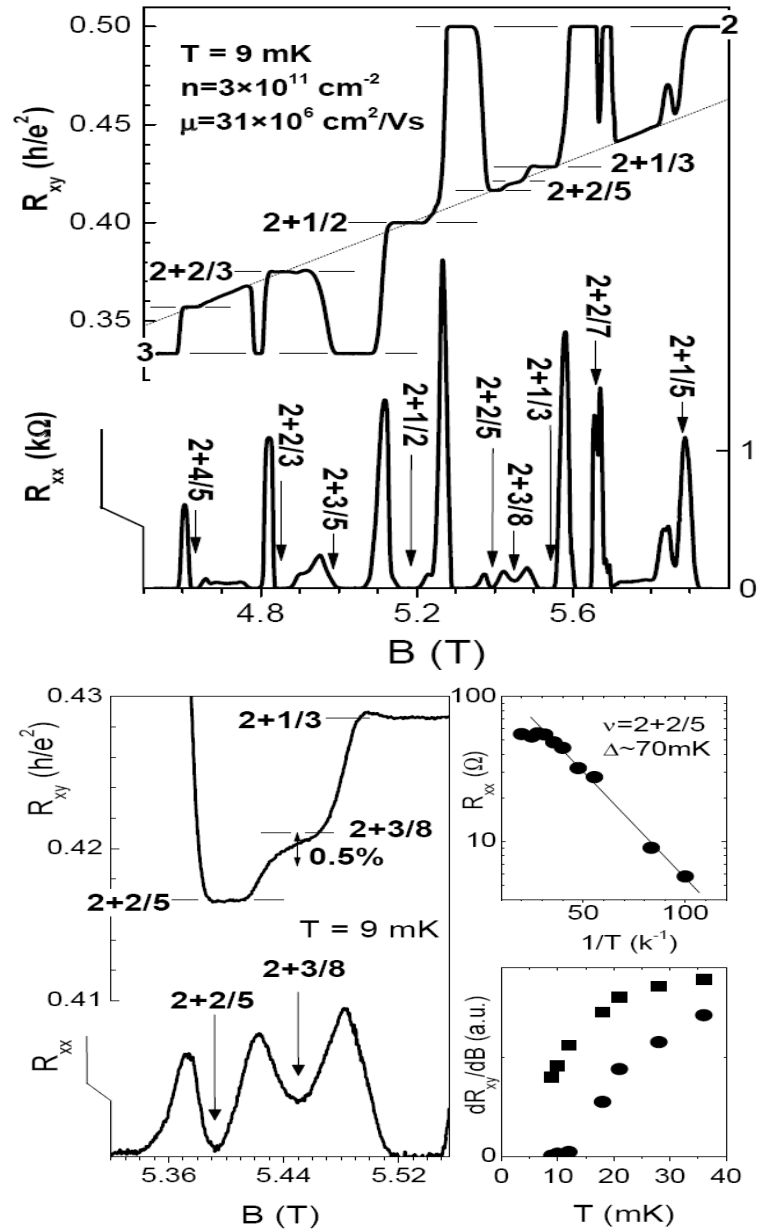
# Plateaus observed in the first and second LL (plots below)

Very different at first sight
- first LL Jain/ HH hierarchy fractions
- second LL dominated by filling ½ plateau

Idea: (Bonderson-Slingerland, PRB 78, 2007)
Build a hierarchy on top of the filling 2+½ plateau
(presumed MR-Pfaffian like)
Condense *Abelian* (Laughlin-like) quasiparticles
Obtain states with MR-like Topological order
At e.g. 2+2/5, 2+3/8, 2+p/(3p-1).




Pan et al. PRL 90, 016801 (2003)




Xia et al. PRL 93, 2004,

# BS Motivation and expectations

BS Wave function for filling 2+2/5

$$\Psi_{12/5}(\mathbf{z}) = \int \prod_i d^2u_i \mathrm{Pf}\left(\frac{1}{z_i - z_j}\right) \prod_{i<j}(z_i - z_j)^2 \prod_{i,j}(z_i - u_j) \prod_{i<j}(u_i - u_j)^{1/2} \prod_{i<j}(u_i^* - u_j^*)^{(1/2+2)}$$
$$\approx \Psi_{\mathrm{MR},\,\nu=1\,\mathrm{bosons}} \Psi_{2/3}^{(CF)}$$

This is a decent trial wave function at 2+2/5 (competes with the k=3 Read Rezayi state)

Bonderson, Feiguin, Möller, JKS, PRL 108, 2012 (arXiv:0901.4965)

Excitations can be inserted in Pfaffian layer or in Laughlin-like layer.
Find topological spectrum generated by Abelian and non-Abelian excitations (charge e/5)

Edge state trial wave functions not so obvious (especially for numerical testing),
Would expect some combination of Pfaffian ($SU(2)_2$) and Abelian counting

It is hard to study any proposed trial wave functions for edge excitations
by comparison with the “real” spectrum

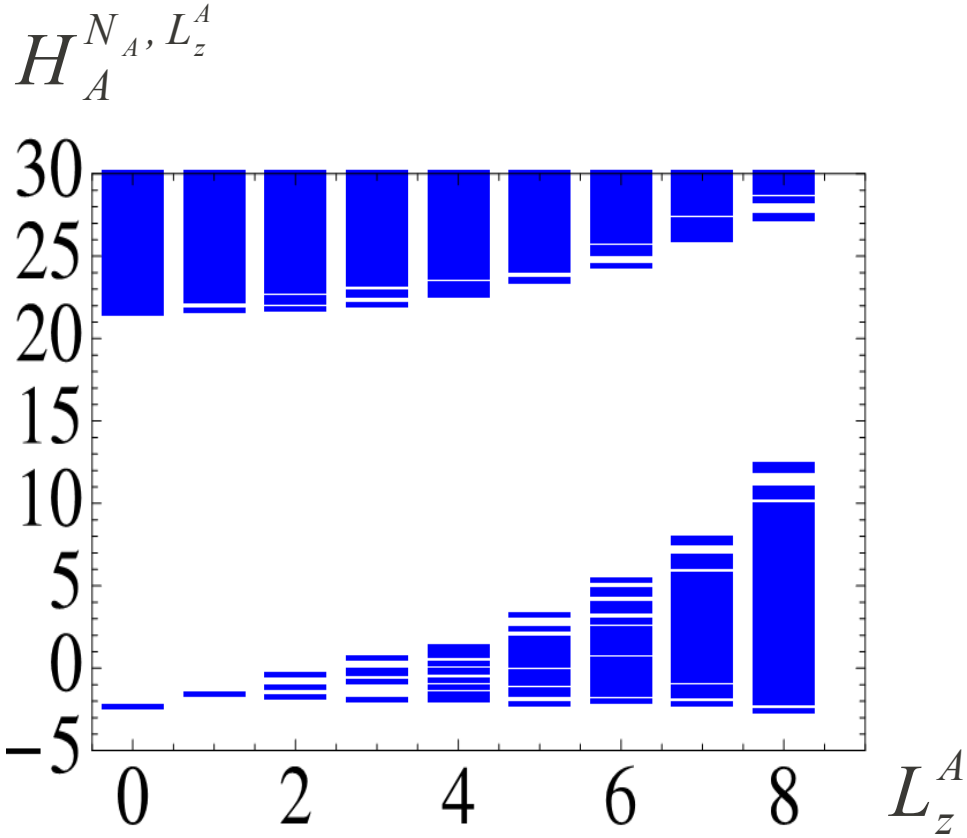
- competition with RR
- canno do very large sizes, so competition with excitations from other nearby plateaus
- no model Hamiltonian and probably trial wave functions won't perform very well with Coulomb

But it can be evaluated reasonably easily (...) in real space

- good RSES project

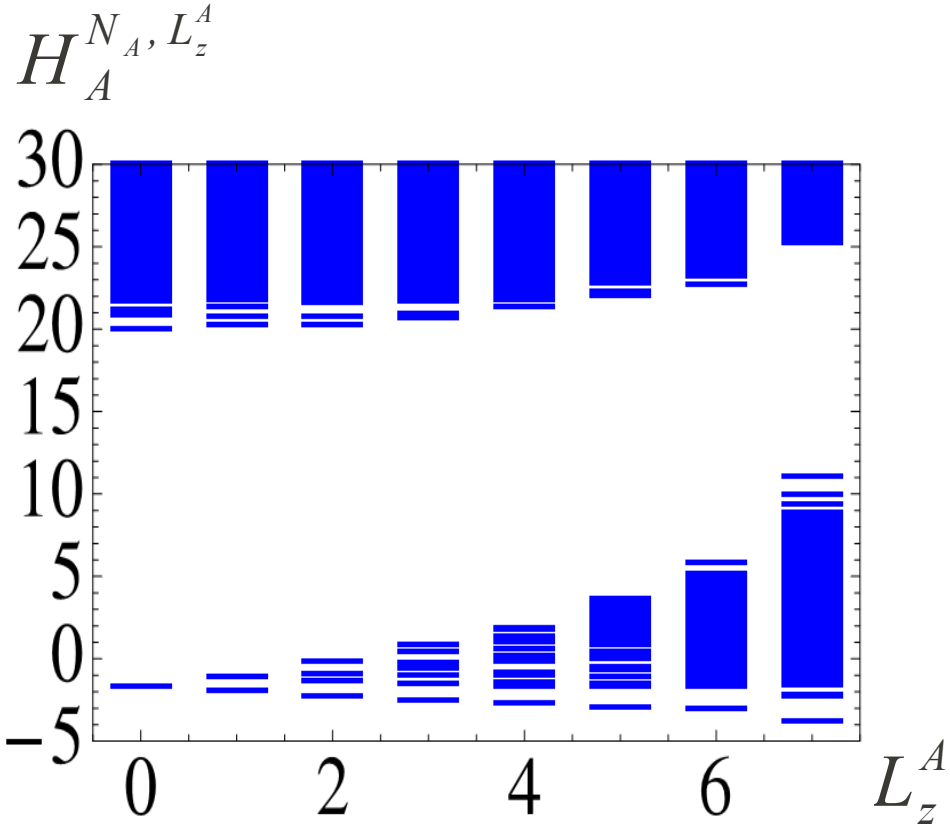
MR Pfaffian $\nu = 1$ : $$\psi_{\text{MR (bosons)}}(z_1, .., z_N) = Pf\left(\frac{1}{z_i - z_j}\right)\prod_{i<j}(z_i - z_j)e^{-\sum_i |z_i^2|/4}$$

Ranks of the RSES for $N = 40$, $N_A = 20$




1,1,3,5,10,16,28,...

Ranks of the RSES for $N = 42$, $N_A = 21$

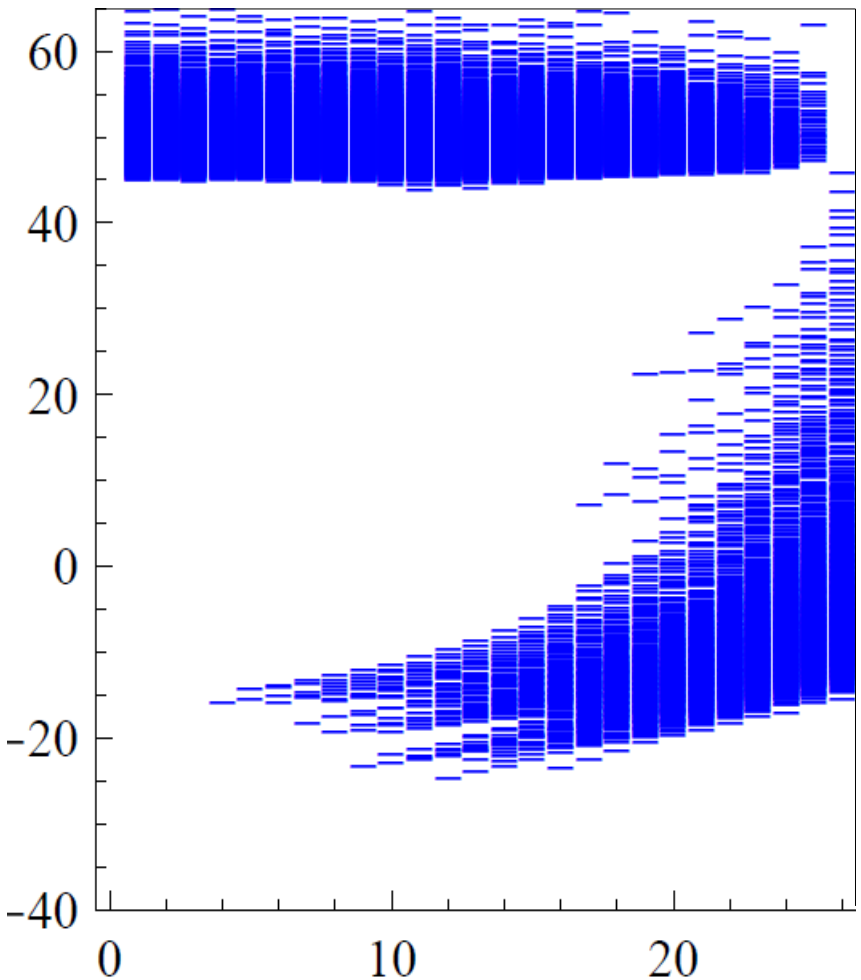



1,2,4,7,13,21,35,...

All ranks as known from CFT/polynomial counting (e.g. Milovanovic & Read, PRB 53, 1996)

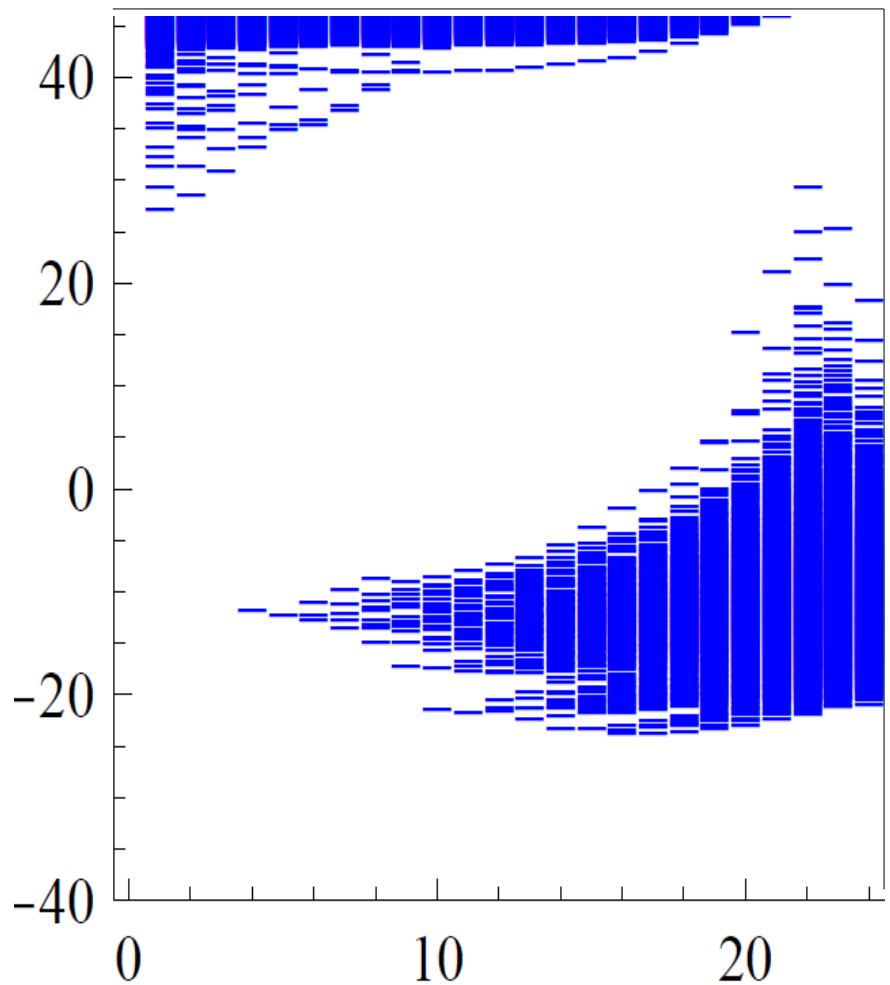
RSES for BS at filling 12/5 (*very* preliminary)

RSES ranks for N=10 ($N_A$=5)




1,2,4,7,13,21,...

RSES ranks for N=12 ($N_A$=6)




1,1,3,5,10,16...

Same as Pfaffian (good). So far see nothing extra (bad?)

# Conclusions

RSES can be calculated using real space methods

- Allows for calculation of ranks ($L_z$ counting) for large systems,
  if wave function is easily evaluated in real space and has good entanglement gap
- Eigenvalues can also be calculated (both PES and RSES), for smaller systems.
- Tested this on Laughlin, MR Pfaffian, Integer fillings (works well)
- Calculated spectra for Jain – works well for filling 2/5
- Working on reverse flux attachment states and BS states (ongoing)

Also found:
- RSES and PES adiabatically connected – ranks equal.
- RSES/PES count:
  number of independent states for $N_B$ electrons with $N_A$ electron holes

Could try to prove from here that they count the number of quasihole states
(at the same excess flux) but this is still nontrivial (cf. E. Ardonne's talk)