Maynooth University
National University of Ireland Maynooth

MATHEMATICAL PHYSICS

SEMESTER 2, REPEAT

2017–2018

MP204
Electricity and Magnetism

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Time allowed: 2 hours

Answer **ALL** questions

1. (a) Consider the static electric field $\mathbf{E} = 2E_0\hat{\imath} - 3E_0\hat{\jmath}$.

   Find the electric potential difference V_{QP} between the point Q at $(0, 3L, 0)$ and the point P at $(0, L, 0)$.

   Find the electric potential difference V_{RP} between the point R at $(4L, L, 0)$ and the point P at $(0, L, 0)$.

   The planar surface Σ_1 lies parallel to the x-y plane and has area A_1. Find the electric flux through this surface.

   [12 marks]

   (b) A circular ring of radius R lies in the x-y plane with its center at the origin. The ring carries a uniformly distributed positive charge Q.

   Find the electric potential due to the charged ring at the point $P(0, 0, z_0)$ on the z-axis.

   Approximate your result in the limit $z_0 \gg R$ and explain your result in terms of the potential due to a point charge.

   [12 marks]

   (c) A charged sphere has radius a_0. The charge density is

$$\rho = \begin{cases} \rho_0(r^2/a_0^2) & \text{for } r < a_0 \\ 0 & \text{for } r > a_0 \end{cases}$$

   where r is the distance from the center of the sphere.

   What is the total charge carried by the sphere?

   Using Gauss' law (Gauss' dielectric flux theorem), find the electric field inside the sphere as a function of r.

   [16 marks]

2. (a) The Biot-Savart law can be written as

$$d\mathbf{B} = \left(\frac{\mu_0 I}{4\pi}\right) \frac{d\mathbf{l}' \times \left(\widehat{\mathbf{r}-\mathbf{r}'}\right)}{|\mathbf{r}-\mathbf{r}'|^2}$$

Explain what this equation expresses.

Include an appropriate figure or figures, showing relevant vectors and/or distances. Indicate the direction of $d\mathbf{B}$ in your sketch.

[12 marks]

(b) Current I flows through a square-shaped wire loop. Each side of the square-shaped loop has length L.

Using the Biot-Savart law, calculate the magnetic field created by the current at the center of the loop.

Possibly useful integral:

$$\int \frac{du}{(u^2+a^2)^{3/2}} = \frac{u}{a^2(u^2+a^2)^{1/2}}$$

[18 marks]

3. (a) The magnetic field in some region is uniform: $\mathbf{B} = B_0\hat{k}$. At time $t = 0$, a charged particle (mass m, charge q) is at the origin and has velocity

$$\mathbf{v} = \alpha\hat{i} + w\hat{k}$$

At which subsequent times does the particle cross the z axis?

What is the z coordinate of the particle position at these instants?

Hint: The trajectory of the particle is helical.

[12 marks]

(b) Consider an electromagnetic wave travelling through empty space described by the electric and magnetic fields

$$\mathbf{E} = 3V_0 L \cos\left(\frac{1}{L}(y + ct)\right)\hat{i}\,, \qquad \mathbf{B} = \mathbf{G}\cos\left(\frac{1}{L}(y + ct)\right)\,,$$

where V_0 and L are positive constants and $\mathbf{G}$ is a constant vector.

Using Maxwell's equations in free space, find the constant vector $\mathbf{G}$.

In which direction is the electromagnetic wave propagating?

[10 marks]

(c) An electromagnetic field is described by the scalar potential V and vector potential $\mathbf{A}$, given by

$$V = \frac{2B_0}{\mu_0\epsilon_0}t\,, \quad \mathbf{A} = B_0\left[x\hat{i} + (y\sin\omega t - 3z)\hat{k}\right]\,,$$

where B_0 and ω are positive constants.

Find the electric and magnetic fields in this system.

[8 pts.]

Posssibly useful Equations

- Electrostatics: $\mathbf{E} = -\nabla V$; $V_{PQ} = -\int_Q^P \mathbf{E}\cdot d\mathbf{l}$

  Electric potential at $\mathbf{r}$ due to a point charge q_1 at $\mathbf{r_1}$: $V = \dfrac{q_1}{4\pi\epsilon_0}\dfrac{1}{|\mathbf{r}-\mathbf{r_1}|}$

  Gauss' law: $\displaystyle\oint_\Sigma \mathbf{E}\cdot d\mathbf{S} = \frac{Q_{\text{enclosed}}}{\epsilon_0}$

- Magnetostatics: Ampere's law: $\displaystyle\int_C \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enclosed}}$

  Biot-Savart law: $d\mathbf{B} = \left(\dfrac{\mu_0 I}{4\pi}\right)\dfrac{d\mathbf{l}'\times\left(\widehat{\mathbf{r}-\mathbf{r}'}\right)}{|\mathbf{r}-\mathbf{r}'|^2}$

- Force on a charge: $\mathbf{F} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}$

  Magnitic force on a current element: $d\mathbf{F} = Id\mathbf{l}\times\mathbf{B}$

- Fields from potentials: $\mathbf{E} = -\nabla V - \dfrac{\partial\mathbf{A}}{\partial t}$, $\mathbf{B} = \nabla\times\mathbf{A}$

- The continuity equation: $\nabla\cdot\mathbf{J} + \dfrac{\partial\rho}{\partial t} = 0$

- Maxwell's Equations:

$$① \quad \nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0} \qquad ② \quad \nabla\cdot\mathbf{B} = 0$$

$$③ \quad \nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$$

$$④ \quad \nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} = \mu_0\left(\mathbf{J} + \mathbf{J_D}\right)$$

- Poynting vector: $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$ Speed of light: $c = 1/\sqrt{\mu_0\epsilon_0}$

  Energy density of electromagnetic fields: $u = \dfrac{1}{2}\epsilon_0 E^2 + \dfrac{1}{2\mu_0}B^2$

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HINTS AND PARTIAL SOLUTIONS

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1. Question 1.

   (a) Question 1(a)

   Consider the static electric field $\mathbf{E} = 2E_0\hat{\imath} - 3E_0\hat{\jmath}$.

   Find the electric potential difference V_{QP} between the point Q at $(0, 3L, 0)$ and the point P at $(0, L, 0)$.

   Find the electric potential difference V_{RP} between the point R at $(4L, L, 0)$ and the point P at $(0, L, 0)$.

   The planar surface Σ_1 lies parallel to the x-y plane and has area A_1. Find the electric flux through this surface.

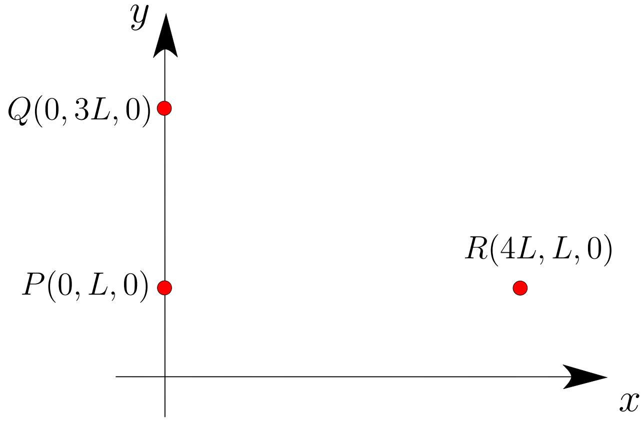
   [12 marks]

   [**Sample Answser:**]

   Calculating V_{QP}

   The line from P to Q is in the y direction. We can use the line integral along the straight line from P to Q.

$$V_{QP} = -\int_P^Q \mathbf{E}\cdot d\mathbf{l} = -\int_{y=L}^{y=3L} \left(2E_0\hat{\imath} - 3E_0\hat{\jmath}\right)\cdot\left(\hat{\jmath}\,dy\right)$$
$$= -\int_{y=L}^{y=3L} (-3E_0)\,dy = 3E_0\int_L^{3L} dy = 3E_0(2L) = 6E_0L$$



Calculating V_{RP}

To calculate V_{RP}, we can use the straight line from P to R, which is in the x direction.

$$V_{RP} = -\int_R^Q \mathbf{E}\cdot d\mathbf{l} = -\int_{x=0}^{x=4L} \left(2E_0\hat{i} - 3E_0\hat{j}\right)\cdot \left(\hat{i}\ dx\right)$$
$$= -\int_{x=0}^{x=4L} (2E_0)\, dx = -2E_0\int_0^{4L} dx = -2E_0(4L) = -8E_0L$$

Flux

A unit vector perpendicular to Σ_1 is $\hat{k}$, so that any surface element vector of Σ_1 is $d\mathbf{S} = \hat{k}dS$. Thus the flux is

$$\int_{\Sigma_1} \mathbf{E}\cdot d\mathbf{S} = \int_{\Sigma_1} \left(2E_0\hat{i} - 3E_0\hat{j}\right)\cdot \hat{k}dS = \int_{\Sigma_1} 0dS = 0$$

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(b) Question 1(b)

A circular ring of radius R lies in the x-y plane with its center at the origin. The ring carries a uniformly distributed positive charge Q.

Find the electric potential due to the charged ring at the point $P(0, 0, z_0)$ on the z-axis.

Approximate your result in the limit $z_0 \gg R$ and explain your result in terms of the potential due to a point charge.

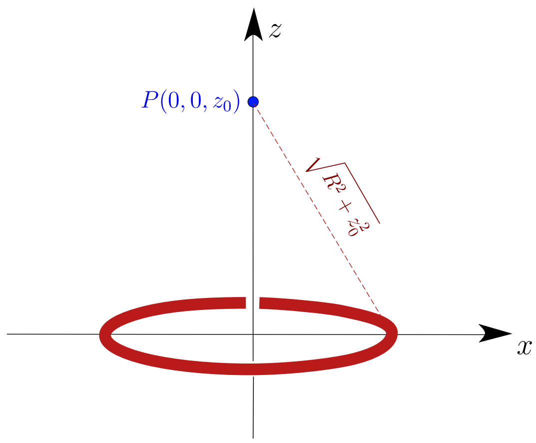
[12 marks]

[Sample Answser:]

Any part of the ring is at distance

$$\sqrt{R^2 + z_0^2}$$

from the point $P(0, 0, z_0)$.



If we consider an infintesimal segment of the ring carrying charge dq, this segment may be regarded as a point charge.

The potential at the point P due to this segment is

$$dV = \frac{1}{4\pi\epsilon_0} \frac{dq}{\sqrt{R^2 + z_0^2}}$$

The potential is independent of the location of the element along the ring!! Thus, each element dq produces the same potential at P. Integrating gives

$$V = \frac{1}{4\pi\epsilon_0} \frac{1}{\sqrt{R^2 + z_0^2}} \int dq = \frac{1}{4\pi\epsilon_0} \frac{Q}{\sqrt{R^2 + z_0^2}}$$

At large distances, $z_0 \gg R$, the potential becomes

$$V = \frac{1}{4\pi\epsilon_0} \frac{Q}{z_0}$$

which is the potential due to a single point charge. The interpretation is that, at large distances, the structure of the ring is not visible and the charge is perceived as clumped together as a point charge.

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(c) Question 1(c)
A charged sphere has radius a_0. The charge density is

$$\rho = \begin{cases} \rho_0(r^2/a_0^2) & \text{for } r < a_0 \\ 0 & \text{for } r > a_0 \end{cases}$$

where r is the distance from the center of the sphere.
What is the total charge carried by the sphere?
Using Gauss' law (Gauss' dielectric flux theorem), find the electric field inside the sphere as a function of r.

[16 marks]

[**Sample Answser:**]
Total charge:

$$Q \;=\; \int_0^{a_0} 4\pi r^2 \rho(r) dr \;=\; 4\pi \frac{\rho_0}{a_0^2} \int_0^{a_0} r^4 dr \;=\; \frac{4\pi\rho_0}{a_0^2}\frac{a_0^4}{5} \;=\; \frac{4}{5}\pi\rho_0 a_0^3$$

To find the electric field inside the sphere, we construct a spherical gaussian surface Σ of radius r concentric to the charged sphere. (A sketch is expected from the examinees.) The charge enclosed within this surface is

$$q_{\text{encl.}} \;=\; 4\pi \frac{\rho_0}{a_0^2} \int_0^{r} r^4 dr \;=\; \frac{4\pi\rho_0 r^5}{5a_0^2}$$

To find the electric flux through this surface, we need to use symmetry. By symmetry, the electric field has the same magnitude at all points of the surface and points radially outwards, hence is perpendicular to the surface (parallel to the surface-vector $d\mathbf{S}$) at all points of the surface. Therefore the flux is

$$\oint_\Sigma \mathbf{E} \cdot d\mathbf{S} \;=\; E \times \text{area} \;=\; E(4\pi r^2)$$

Thus Gauss's law (flux $= \frac{1}{\epsilon_0} \times$ enclosed charge) gives

$$E(4\pi r^2) \;=\; \frac{q_{\text{encl.}}}{\epsilon_0} \;=\; \frac{4\pi\rho_0 r^5}{5\epsilon_0 a_0^2} \qquad \Longrightarrow \qquad E \;=\; \left(\frac{\rho_0}{5\epsilon_0 a_0^2}\right) r^3$$

Although not asked for in this exam question, the calculation for the electric field outside the charged sphere would be very similar. Now the enclosed charge is the total charge on the sphere. Thus Gauss's law gives

$$E(4\pi r^2) \;=\; \frac{q_{\text{encl.}}}{\epsilon_0} \;=\; \frac{Q}{\epsilon_0} \qquad \Longrightarrow \qquad E \;=\; \frac{1}{4\pi\epsilon_0}\frac{Q}{r^2} \;=\; \left(\frac{\rho_0 a_0^3}{5\epsilon_0}\right)\frac{1}{r^2}$$

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2. Question 2.

   (a) Question 2(a)

   The Biot-Savart law can be written as

$$d\mathbf{B} = \left(\frac{\mu_0 I}{4\pi}\right) \frac{d\mathbf{l}' \times \left(\widehat{\mathbf{r}-\mathbf{r}'}\right)}{|\mathbf{r}-\mathbf{r}'|^2}$$

   Explain what this equation expresses.

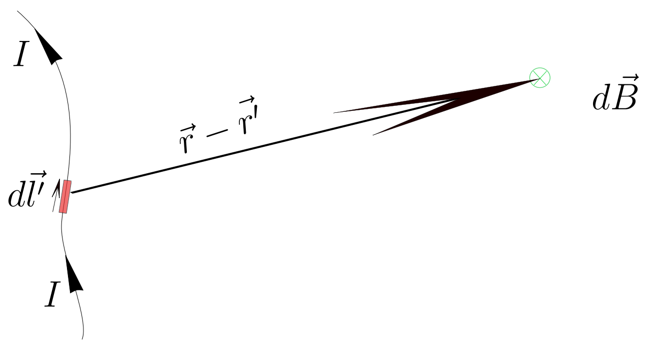
   Include an appropriate figure or figures, showing relevant vectors and/or distances. Indicate the direction of $d\mathbf{B}$ in your sketch.

   [12 marks]

   [**Sample Answser:**]

   The Biot-Savart law gives the magnetic field created by an infinitesimal element of a current-carrying wire.

   In the equation, $d\mathbf{l}'$ is an infinitesimal element of the wire, directed along the flow of current I. The magnetic field created at point $\mathbf{r}$ (call this point P) is $d\mathbf{B}$. The location of the wire element is at point $\mathbf{r}'$.



   The displacement of the point P from the source current element $d\mathbf{l}'$ is $\mathbf{r}-\mathbf{r}'$. In the equation, $\widehat{\mathbf{r}-\mathbf{r}'}$ is a unit vector in the direction of the displacement. The magnitude of the created magnetic field is inversely proportional to the square of the distance of the point P from the current element. The field magnitude is also proportional to the current strength I.

   The direction of the magnetic field is given by the cross-product of $d\mathbf{l}'$ and $\mathbf{r}-\mathbf{r}'$. In the figure (assuming there is one), the field is likely perpendicular to the plane of the figure. In the electronic figure shown here, the field points into the plane of the paper.

In the equation, the μ_0 is a constant known as the permeability of free space.

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(b) Question 2(b)

Current I flows through a square-shaped wire loop. Each side of the square-shaped loop has length L.

Using the Biot-Savart law, calculate the magnetic field created by the current at the center of the loop.

Possibly useful integral:

$$\int \frac{du}{(u^2 + a^2)^{3/2}} = \frac{u}{a^2(u^2 + a^2)^{1/2}}$$

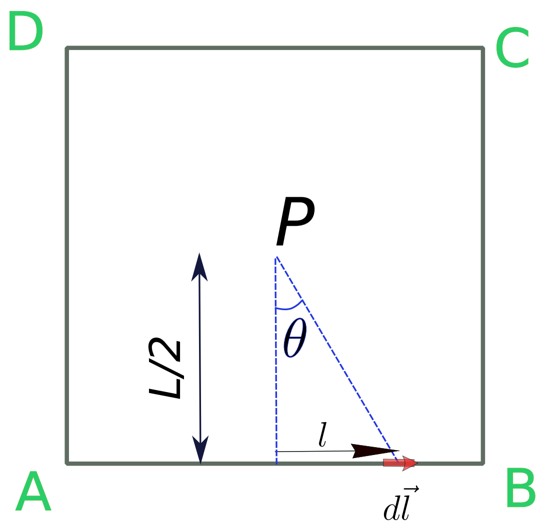
[18 marks]

[**Sample Answser:**]

A sketch would be very useful.

Let's first concentrate on one of the sides, say AB. Choose an infinitesimal element of this side, of length $|d\mathbf{l}|$. Define $\mathbf{l}$ as its displacement from the center of AB. The line joining this element to the center of the loop (call it point P) makes an angle θ with the line joining the point P to the center of the side AB.

It would be useful to sketch a top view of the square wire and an element $d\mathbf{l}$, showing clearly the angle θ, made at the center of the loop, the distance l, and the direction of the current.



Point P is at distance $L/2$ from the wire segment AB. The distance to the element $d\mathbf{l}$ is

$$\sqrt{(L/2)^2 + l^2} \;=\; \frac{L/2}{\cos\theta}$$

The Biot-Savart law says that the magnetic field element is due to the current in an element of the wire is

$$d\mathbf{B} \;=\; \frac{\mu_o I}{4\pi}\frac{d\mathbf{l}\times\hat{r}}{r^2}$$

where $\mathbf{r}$ is the displacement from the wire element in $d\mathbf{l}$ to the point where we are calculating the field, and $\hat{r}$ is a unit vector in the direction of $\mathbf{r}$. In class we also used the more cumbersome notation

$$d\mathbf{B} \;=\; \frac{\mu_o I}{4\pi}\frac{d\mathbf{l}\times\widehat{(\mathbf{r}-\mathbf{r}')}}{|\mathbf{r}-\mathbf{r}'|^2}$$

to mean the same thing. Let's use the first notation to avoid clutter. In the present case, $d\mathbf{l}$ makes angle $\frac{\pi}{2}-\theta$ with $\hat{r}$, hence the magnitude of the cross product is

$$|d\mathbf{l}\times\hat{r}| \;=\; |d\mathbf{l}|\times 1\times\sin(\frac{\pi}{2}-\theta) \;=\; dl\frac{L/2}{\sqrt{(L/2)^2+l^2}}$$

Also, $r^2 = (L/2)^2 + l^2$. Thus the magnitude of the field element is

$$dB \;=\; \frac{\mu_o I}{4\pi}\frac{(L/2)dl}{[(L/2)^2+l^2]^{3/2}}$$

The limits of integration from A to B (over one side of the square) are $l = -L/2$ to $l = +L/2$. Thus the total field due to the entire segment AB is

$$\begin{aligned} B &= \frac{\mu_o I L}{8\pi} \int_{-L/2}^{+L/2} \frac{dl}{\left[l^2 + (L/2)^2\right]^{3/2}} = \frac{\mu_o I L}{8\pi} \left[\frac{l}{(L/2)^2 \left[l^2 + (L/2)^2\right]^{1/2}} \right] \\ &= \frac{\mu_o I}{2\pi L} \left(\frac{L/2}{\left[(L/2)^2 + (L/2)^2\right]^{1/2}} - \frac{(-L/2)}{\left[(L/2)^2 + (L/2)^2\right]^{1/2}} \right) \\ &= \frac{\mu_o I}{2\pi L} \sqrt{2} \end{aligned}$$

By symmetry, each side of the square produces the same magnitude of magnetic field at P. Using the right hand rule also shows that each segment creates a field in the same direction, hence the fields due to the four sides add up. The total field is

$$4 \times \frac{\mu_o I}{2\pi L} \sqrt{2} = \frac{\mu_o I}{\pi L} 2\sqrt{2}$$

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3. Question 3.

   (a) Question 3(a)

   The magnetic field in some region is uniform: $\mathbf{B} = B_0\hat{k}$. At time $t = 0$, a charged particle (mass m, charge q) is at the origin and has velocity

   $$\mathbf{v} = \alpha\hat{i} + w\hat{k}$$

   At which subsequent times does the particle cross the z axis?

   What is the z coordinate of the particle position at these instants?

   Hint: The trajectory of the particle is helical.

   [12 marks]

   [**Sample Answser:**]

   The magnetic field exerts a force $q\mathbf{v} \times \mathbf{B}$ perpendicular to itself, hence in directions perpendicular to z. In the z direction, the velocity thus remains constant, $v_z = w$.

   Perpendicular to the z direction, the magnetic field causes cyclotron motion, with speed α. If the cyclotron radius is R, then

   $$q\alpha B = \frac{m\alpha^2}{R} \qquad \Longrightarrow \qquad R = \frac{m\alpha}{qB}$$

   The time it takes to complete one cyclotron revolution is

   $$T = \frac{\text{circumference}}{\text{speed}} = \frac{2\pi R}{\alpha} = \frac{2\pi m}{qB}$$

   Therefore, the particle will cross the z axis at times

   $$T = \frac{2\pi m}{qB},\ 2T = \frac{4\pi m}{qB},\ 3T = \frac{6\pi m}{qB},\ 4T = \frac{8\pi m}{qB},\ \ldots$$

   At these instants the z coordinate will be

   $$wT = \frac{2\pi mw}{qB},\ 2wT = \frac{4\pi mw}{qB},\ 3wT = \frac{6\pi mw}{qB},\ 4wT = \frac{8\pi mw}{qB},\ \ldots$$

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(b) Question 3(b)

Consider an electromagnetic wave travelling through empty space described by the electric and magnetic fields

$$\mathbf{E} \;=\; 3V_0 L \cos\left(\frac{1}{L}(y+ct)\right)\hat{\imath}\,, \qquad \mathbf{B} \;=\; \mathbf{G}\cos\left(\frac{1}{L}(y+ct)\right)\,,$$

where V_0 and L are positive constants and $\mathbf{G}$ is a constant vector.

Using Maxwell's equations in free space, find the constant vector $\mathbf{G}$.

In which direction is the electromagnetic wave propagating?

[10 marks]

[**Sample Answser:**]

Using Maxwell's third equation:

$$\begin{aligned}
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla\times\mathbf{E} \;=\; -\left[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right]\hat{k} \\
&= -\left[0 - 3V_0 L\left(-\frac{1}{L}\right)\sin\left(\frac{1}{L}(y+ct)\right)\right]\hat{k} \\
&= -3V_0 \sin\left(\frac{1}{L}(y+ct)\right)\hat{k}
\end{aligned}$$

Integrating gives

$$\begin{aligned}
\mathbf{B} &= -3V_0\left(\frac{1}{c/L}\right)(-1)\cos\left(\frac{1}{L}(y+ct)\right)\hat{k} + \begin{bmatrix}\text{time-independent}\\ \text{constant vector}\end{bmatrix} \\
&= -\frac{3V_0 L}{c}\cos\left(\frac{1}{L}(y+ct)\right)\hat{k} \;+\; \begin{bmatrix}\text{time-independent}\\ \text{constant vector}\end{bmatrix}
\end{aligned}$$

Comparing with the given form for the magnetic field, the constant vector (constant of integration) is seen to be zero. In addition, the constant $\mathbf{G}$ is seen to be

$$\mathbf{G} \;=\; -\frac{3V_0 L}{c}\hat{k}$$

The spatial dependence appears as $y+ct$. This means the wave travels in the negative y direction.

If the fields were functions of $y-ct$, it would be a wave traveling in the positive y direction.

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(c) Question 3(c)
An electromagnetic field is described by the scalar potential V and vector potential $\mathbf{A}$, given by

$$V = \frac{2B_0}{\mu_0\epsilon_0}t\,, \quad \mathbf{A} = B_0\left[x\hat{i} + (y\sin\omega t - 3z)\hat{k}\right]\ ,$$

where B_0 and ω are positive constants.
Find the electric and magnetic fields in this system.

[8 pts.]

[**Sample Answser:**]

$$\mathbf{E} \;=\; -\nabla V - \frac{\partial\mathbf{A}}{\partial t} \;=\; -0 - B_0\left(\omega y\cos\omega t\right)\hat{k} \;=\; -B_0\omega y\cos\omega t\,\hat{k}$$

$$\mathbf{B} \;=\; \nabla\times\mathbf{A} \;=\; \left(\partial_y A_z - \partial_z A_y\right)\hat{i} \;+\; 0\hat{j} \;+\; 0\hat{k} \;=\; B_0\sin\omega t\,\hat{i}$$

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