MP204
Electricity and Magnetism

2018–2019, practice exam 1

Time allowed: 2 hours

Answer **ALL** questions

This is a **SAMPLE** exam, roughly reflecting the general structure of the MP204 exams for 2018 – 2019.

Remember that **ALL** questions are to be answered.

If any figures are relevant, please include them with your solution! Sketching appropriate pictures often helps you arrive at the correct solution.

1. (a) The electrostatic field in some region is given by

$$\mathbf{E} = (2x\ W/L^2)\hat{\imath} + (5\ W/L)\hat{k}$$

where W and L are positive constants.
Find the potential difference between the points $(L, 3L, 0)$ and $(4L, L, 0)$.
Find the charge density in the region.

[13 marks]

(b) An infinite charged plane carries surface charge density σ. Use Gauss' dielectric flux theorem to calculate the electric field at distance z from the plane. Explain very clearly the shape of the closed surface on which you apply the theorem. Plot the electric field magnitude as a function of z.

[12 marks]

2. (a) A long straight wire carries steady current I. The wire cross-section is circular and has radius R. The current density is uniform $\left(= \frac{I}{\pi R^2}\right)$ inside the wire. Using Ampere's law, calculate the magnetic field at distance r from the axis of the wire. Consider separately the cases $r < R$ (inside the wire) and $r > R$ (outside the wire). Plot the magnitude of the magnetic field as a function of r.

[20 marks]

(b) A magnetic field of $\mathbf{B} = a\sin(bz)e^{-by}\hat{\imath}$ is produced by a steady electric current. What is the density of that current?

[5 marks]

3. (a) The magnetic field in some region is uniform: $\mathbf{B} = B_0\hat{k}$. At time $t = 0$, a charged particle (mass m, charge q) is at the origin and has velocity $\mathbf{v} = \alpha\hat{\jmath}$.

Describe the motion (trajectory) of the particle. Sketch the trajectory, indicating clearly the coordinates of at least two points on the trajectory.

[8 marks]

(b) Ampere's theorem for the magnetic field due to a steady current is

$$\oint_C \mathbf{B} \cdot d\mathbf{l} \;=\; \mu_0 I$$

What does C denote? What does the symbol I refer to?

Starting from this integral relation, derive part of Maxwell's fourth equation (which concerns the curl of of the magnetic field). If you use a theorem about vector integrals on the way, name the theorem explicitly.

[10 marks]

(c) The magnetic field in a region changes with time, $\mathbf{B} = B_0 e^{-2t/t_0}\hat{\imath}$. Here B_0 and t_0 are positive constants.

A square loop lies in the y-z plane, and has sides of length L_1. Find the electromotive force (EMF) induced in this loop.

[7 marks]

4. (a) An electromagnetic system is described by the scalar and vector potentials

$$V = 3E_0 L e^{-x^2/L^2} e^{-2\omega t}\ , \qquad \mathbf{A} = \frac{E_0}{\omega} \sin\left(\frac{\omega}{c} y - \omega t\right) \hat{k}$$

where L, E_0 and ω are positive constants.

Calculate the electric and magnetic fields.

Calculate the charge density, and the current density.

[15 marks]

(b) Using Maxwell's equations in vacuum, derive a continuity equation for the energy carried by electromagnetic fields, in terms of the Poynting vector. The vector identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

might be helpful.

[10 marks]

Posssibly useful Equations

- Electrostatics: $\mathbf{E} = -\nabla V$; $V_{PQ} = -\int_Q^P \mathbf{E} \cdot d\mathbf{l}$

  Electric potential at $\mathbf{r}$ due to a point charge q_1 at $\mathbf{r_1}$: $V = \dfrac{q_1}{4\pi\epsilon_0} \dfrac{1}{|\mathbf{r} - \mathbf{r_1}|}$

  Gauss' law: $\oint_\Sigma \mathbf{E} \cdot d\mathbf{S} = \dfrac{Q_{\text{enclosed}}}{\epsilon_0}$

- Magnetostatics: Ampere's law: $\int_C \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{enclosed}}$

  Biot-Savart law: $d\mathbf{B} = \left(\dfrac{\mu_0 I}{4\pi}\right) \dfrac{d\mathbf{l}' \times \left(\widehat{\mathbf{r} - \mathbf{r}'}\right)}{|\mathbf{r} - \mathbf{r}'|^2}$

- Force on a charge: $\mathbf{F} = q\mathbf{E} + q\mathbf{v} \times \mathbf{B}$

  Magnetic force on a current element: $d\mathbf{F} = I d\mathbf{l} \times \mathbf{B}$

- Fields from potentials: $\mathbf{E} = -\nabla V - \dfrac{\partial \mathbf{A}}{\partial t}$, $\mathbf{B} = \nabla \times \mathbf{A}$

- The continuity equation: $\nabla \cdot \mathbf{J} + \dfrac{\partial \rho}{\partial t} = 0$

- Faraday's law: $\mathcal{E} = -\dfrac{d\Phi_B}{dt}$ or $\oint_C \mathbf{E} \cdot d\mathbf{l} = -\dfrac{d}{dt}\Phi_B = -\dfrac{d}{dt}\int_\Sigma \mathbf{B} \cdot d\mathbf{S}$

- Maxwell's Equations:

  1. $\nabla \cdot \mathbf{E} = \dfrac{\rho}{\epsilon_0}$
  2. $\nabla \cdot \mathbf{B} = 0$
  3. $\nabla \times \mathbf{E} = -\dfrac{\partial \mathbf{B}}{\partial t}$
  4. $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \dfrac{\partial \mathbf{E}}{\partial t} = \mu_0 (\mathbf{J} + \mathbf{J_D})$

- Poynting vector: $\mathbf{S} = \dfrac{1}{\mu_0} \mathbf{E} \times \mathbf{B}$ Speed of light: $c = 1/\sqrt{\mu_0 \epsilon_0}$

  Energy density of electromagnetic fields: $u = \dfrac{1}{2}\epsilon_0 E^2 + \dfrac{1}{2\mu_0} B^2$