Maynooth University
National University of Ireland Maynooth

THEORETICAL PHYSICS

SEMESTER 2, MAY EXAM

2020–2021

MP204
Electricity and Magnetism

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Answer **ALL FOUR** questions

1. (a) An electromagnetic field is described by the scalar potential V and vector potential $\mathbf{A}$, given by

$$V \;=\; \alpha_1 \left(c^2 yt - cz^2\right) \;, \qquad \mathbf{A} \;=\; \alpha_1 \left(c^2 t^2 + x^2 + y^2\right) \hat{\jmath}$$

where α_1 is a positive constant, and $c = 1/\sqrt{\mu_0 \epsilon_0}$ is the speed of light. Find the electric and magnetic fields.

[8 marks]

(b) A semi-infinite line of charge lies along the positive x-axis, extending from the origin ($x = 0$) to $x = \infty$. The linear charge density is uniform, equal to λ everywhere on the semi-infinite line.

Calculate the electric field generated by this line of charge at the point $(0, L, 0)$ on the y-axis.

Specify clearly the magnitude, the components, and the direction of the field. How does the direction depend on L?

Possibly useful integrals:

$$\int \frac{du}{(u^2 + a^2)^{3/2}} = \frac{u}{a^2 \sqrt{u^2 + a^2}} \;; \qquad \int \frac{u du}{(u^2 + a^2)^{3/2}} = -\frac{1}{\sqrt{u^2 + a^2}} \;.$$

[17 marks]

2. Consider the static electric field:

$$\mathbf{E} = \beta(2x + 3y)\hat{\imath} + \beta(3x + z)\hat{\jmath} + \beta(y + 2z)\hat{k}\,.$$

Here β is a positive constant.

(a) Find the electric potential difference V_{QP} between the point Q at $(L, 5L, 0)$ and the point P at $(L, L, 0)$.
Find the electric potential difference V_{RP} between the point R at $(3L, L, 0)$ and the point P at $(L, L, 0)$.
Here L is a positive constant.

[14 marks]

(b) Consider the rectangular surface Σ_1 lying in the x-y plane, with three corners at points P, Q, R, and the fourth corner at point S at $(3L, 5L, 0)$. Find the magnitude of the electric flux through this surface.

[11 marks]

3. Steady current I flows through an infinitely long straight wire placed parallel to the z axis, from $z = +\infty$ to $z = -\infty$. (The current direction is 'anti-parallel' to the z axis.) The wire runs through the point $(-b, 0, 0)$. Here b is a positive constant.

   Reminder: An infinitely long straight wire produces a magnetic field of strength $\mu_0 I/(2\pi d)$ at a point at perpendicular distance d from the wire.

   (a) Find the magnitude of the magnetic field created at the **origin** due to the current-carrying wire. Find also the x-, y-, and z- components of the magnetic field created at the origin.

   [5 marks]

   (b) At the point $M(3b, 4b, 0)$ in the x-y plane, find the magnitude of the magnetic field created by the current. Find also the x-, y-, and z- components of the magnetic field at this point.

   [11 marks]

   (c) Now consider the point $N(3b, 4b, 5b)$, not in the x-y plane. Find the x-, y-, and z- components of the magnetic field at this point.

   [3 marks]

   (d) Consider the square-shaped surface Σ_2 lying in the y-z plane, having sides of length $4b$. The center of the square is at the origin. Two of its sides are parallel to the y-axis, and the other two are parallel to the z azis.

   What is the magnetic flux through this surface? Explain, or show your calculations.

   [6 marks]

4. (a) The magnetic field in some region is uniform: $\mathbf{B} = B_0\hat{k}$. At time $t = 0$, a charged particle (mass m, charge q) is at the origin and has velocity $\mathbf{v} = w\hat{\imath} + \alpha_2\hat{k}$. Here w and α_2 are positive constants.

What is the magnitude and direction of the force exerted at $t = 0$ on the particle ?

Explain how the x, y and z components of the velocity change with time.

[6 marks]

(b) The magnetic field in some region is time-dependent and points in the positive z direction: $\mathbf{B}(t) = B_0\left[2 + \sin(t/T)\right]\hat{k}$. A circular loop of radius r_0 lies in the xy plane. Here B_0, r_0 and T are positive constants.

Find the EMF induced in the loop.

[5 marks]

(c) A metallic slab has width $2a$, and lies between $y = -a$ and $y = +a$, so that its surfaces are parallel to the z-x plane. The slab has infinite extent in the x and z directions.

Steady current flows through the slab in the z direction. The current density has magnitude J_0 everywhere within the slab:

$$\mathbf{J} = \begin{cases} J_0\hat{k} & \text{for } |y| < a \\ 0 & \text{for } |y| > a \end{cases}$$

Using Ampere's law, find the magnetic field created by the current at locations outside the slab, both for $y > a$ and for $y < -a$.

Explain clearly any loop that you introduce, and any symmetry arguments that you use.

[14 marks]

Posssibly useful Equations

- Electrostatics: $\mathbf{E} = -\nabla V$; $V_{PQ} = -\int_Q^P \mathbf{E}\cdot d\mathbf{l}$

  Electric potential at $\mathbf{r}$ due to a point charge q_1 at $\mathbf{r_1}$: $V = \frac{q_1}{4\pi\epsilon_0}\frac{1}{|\mathbf{r}-\mathbf{r_1}|}$

  Gauss' law: $\oint_\Sigma \mathbf{E}\cdot d\mathbf{S} = \frac{Q_{\text{enclosed}}}{\epsilon_0}$

- Magnetostatics: Ampere's law: $\int_C \mathbf{B}\cdot d\mathbf{l} = \mu_0 I_{\text{enclosed}}$

  Biot-Savart law: $d\mathbf{B} = \left(\frac{\mu_0 I}{4\pi}\right)\frac{d\mathbf{l}'\times\left(\widehat{\mathbf{r}-\mathbf{r}'}\right)}{|\mathbf{r}-\mathbf{r}'|^2}$

- Force on a charge: $\mathbf{F} = q\mathbf{E} + q\mathbf{v}\times\mathbf{B}$

  Magnetic force on a current element: $d\mathbf{F} = I d\mathbf{l}\times\mathbf{B}$

- Fields from potentials: $\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$, $\mathbf{B} = \nabla\times\mathbf{A}$

- The continuity equation: $\nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t} = 0$

- Faraday's law: $\mathcal{E} = -\frac{d\Phi_B}{dt}$ or $\oint_C \mathbf{E}\cdot d\mathbf{l} = -\frac{d}{dt}\Phi_B = -\frac{d}{dt}\int_\Sigma \mathbf{B}\cdot d\mathbf{S}$

- Maxwell's Equations:

  ① $\nabla\cdot\mathbf{E} = \frac{\rho}{\epsilon_0}$ ② $\nabla\cdot\mathbf{B} = 0$

  ③ $\nabla\times\mathbf{E} = -\frac{\partial\mathbf{B}}{\partial t}$

  ④ $\nabla\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t} = \mu_0(\mathbf{J} + \mathbf{J_D})$

- Poynting vector: $\mathbf{S} = \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B}$ Speed of light: $c = 1/\sqrt{\mu_0\epsilon_0}$

  Energy density of electromagnetic fields: $u = \frac{1}{2}\epsilon_0 E^2 + \frac{1}{2\mu_0}B^2$

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PARTIAL SOLUTIONS/ HINTS/ COMMENTS

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GENERAL COMMENTS:

(1) The exam was an "online assessment", with 2 hours for the exam and an extra hour for the upload.

(2) There might be typographical errors or incomplete explanations in these pages. Please use with caution.

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1. Question 1

   (a) Question 1(a)

   An electromagnetic field is described by the scalar potential V and vector potential $\mathbf{A}$, given by

$$V = \alpha_1 \left(c^2 yt - cz^2\right) , \qquad \mathbf{A} = \alpha_1 \left(c^2t^2 + x^2 + y^2\right) \hat{j}$$

   where α_1 is a positive constant, and $c = 1/\sqrt{\mu_0\epsilon_0}$ is the speed of light. Find the electric and magnetic fields.

   [8 marks]

   [**(Partial) solution / Hints :**]

   This is a simple "plug-in" problem; no understanding or analysis is really needed, only the ability to calculate vector derivatives and the ability to distinguish vectors and scalars.

   Electric field:

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} = -\left(\alpha_1 c^2 t\hat{j} - 2\alpha_1 cz\hat{k}\right) - \left(2\alpha_1 c^2 t\hat{j}\right)$$
$$= -3\alpha_1 c^2 t\hat{j} + 2\alpha_1 cz\hat{k}$$

   The second term is really needed! The more familiar equation $\mathbf{E} = -\nabla V$ is only valid for electrostatics.

Magnetic field:

$\mathbf{B} = \nabla \times \mathbf{A}$ $= (\partial_x A_y)\,\hat{k}$ $= 2\alpha_1 x\hat{k}$	$\mathbf{A}$ has only a y-component, which depends on x but has no z-dependence. Hence $\nabla \times \mathbf{A} = (\partial_x A_y)\,\hat{k}$

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(b) Question 1(b)

A semi-infinite line of charge lies along the positive x-axis, extending from the origin ($x = 0$) to $x = \infty$. The linear charge density is uniform, equal to λ everywhere on the semi-infinite line.

Calculate the electric field generated by this line of charge at the point $(0, L, 0)$ on the y-axis.

Specify clearly the magnitude, the components, and the direction of the field. How does the direction depend on L?

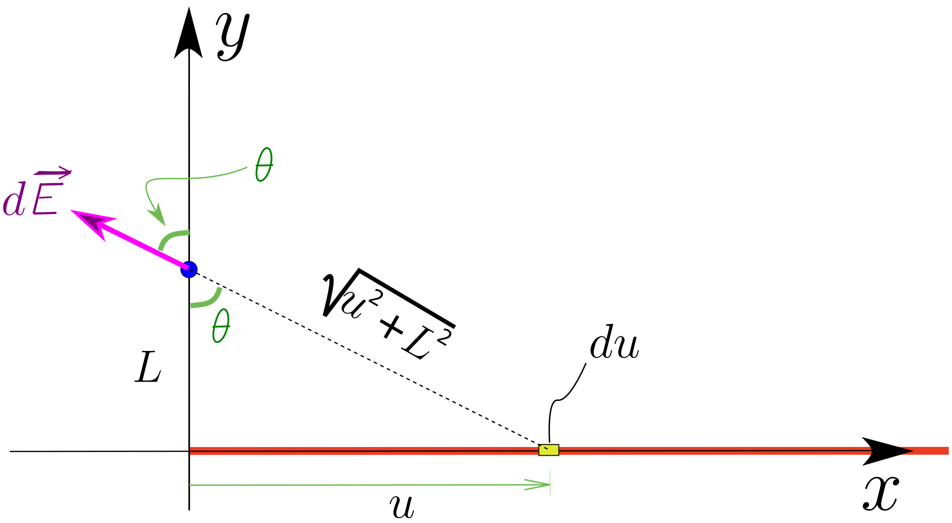
Possibly useful integrals:

$$\int \frac{du}{(u^2 + a^2)^{3/2}} = \frac{u}{a^2\sqrt{u^2 + a^2}}\,; \qquad \int \frac{udu}{(u^2 + a^2)^{3/2}} = -\frac{1}{\sqrt{u^2 + a^2}}\,.$$

[17 marks]

[(Partial) solution / Hints :]

A clear sketch of the situation is essential and is expected from examinees.



Consider an infinitesimal slice of the rod at distance u from the origin, of width du. (We could have called this variable x as well in this case.) The charge carried by this inifinitesimal element is λdu.

From the figure: the element is at distance $\sqrt{u^2+L^2}$ from the point $(0, L, 0)$ where we need to calculate the field. The field has both x and y components.

The electric field created by the small element has the magnitude

$$dE = \frac{\lambda du}{4\pi\epsilon_0}\frac{1}{u^2+L^2}$$

The x- and y- components of this field element are

$$dE_x = -dE\sin\theta = -\frac{\lambda du}{4\pi\epsilon_0}\frac{1}{u^2+L^2}\frac{u}{\sqrt{u^2+L^2}} = -\frac{\lambda}{4\pi\epsilon_0}\frac{udu}{(u^2+L^2)^{3/2}}$$

$$dE_y = dE\cos\theta = \frac{\lambda du}{4\pi\epsilon_0}\frac{1}{u^2+L^2}\frac{L}{\sqrt{u^2+L^2}} = \frac{\lambda L}{4\pi\epsilon_0}\frac{du}{(u^2+L^2)^{3/2}}$$

The expressions for $\sin\theta$ and $\cos\theta$ used above are obtained by noticing that the same angle θ appears in the right-angled triangle with sides u, L and $\sqrt{u^2+L^2}$.

The components of the total electric field are found by integrating the two expressions above. The line being semi-infinite, the integration limits are $u = 0$ to $y = \infty$.

$$\begin{aligned} E_x &= \int dE_x = -\frac{\lambda}{4\pi\epsilon_0}\int_0^\infty \frac{udu}{(u^2+L^2)^{3/2}} \\ &= -\frac{\lambda}{4\pi\epsilon_0}\left[-\frac{1}{\sqrt{u^2+L^2}}\right]_{u=0}^{u=\infty} \\ &= -\frac{\lambda}{4\pi\epsilon_0}\left(-0+\frac{1}{L}\right) = -\frac{\lambda}{4\pi\epsilon_0 L} \end{aligned}$$

and

$$\begin{aligned} E_y &= \int dE_y = \frac{\lambda L}{4\pi\epsilon_0}\int_0^\infty \frac{du}{(u^2+L^2)^{3/2}} \\ &= \frac{\lambda L}{4\pi\epsilon_0}\left[\frac{u}{L^2\sqrt{u^2+L^2}}\right]_{u=0}^{u=\infty} \end{aligned}$$

The upper limit might be tricky:

$$\lim_{u\to\infty}\frac{u}{L^2\sqrt{u^2+L^2}} = \lim_{u\to\infty}\frac{1}{L^2\sqrt{1+L^2/u^2}} == \frac{1}{L^2}$$

whereas the lower limit is easy:

$$\left[\frac{u}{L^2\sqrt{u^2+L^2}}\right]_{u=0} = 0\,.$$

Thus

$$E_y = \frac{\lambda L}{4\pi\epsilon_0}\left(\frac{1}{L^2}-0\right) = \frac{\lambda}{4\pi\epsilon_0 L}$$

The **magnitude** of the electric field at $(0, L, 0)$ is

$$E = \sqrt{E_x^2+E_y^2} = \sqrt{\left(-\frac{\lambda}{4\pi\epsilon_0 L}\right)^2+\left(\frac{\lambda}{4\pi\epsilon_0 L}\right)^2} = \frac{\sqrt{2}\lambda}{4\pi\epsilon_0 L}\,.$$

The components are

$$\boxed{E_x = -\frac{\lambda}{4\pi\epsilon_0 L}, \qquad E_y = \frac{\lambda}{4\pi\epsilon_0 L}}$$

The direction: because the two components are equal, the electric field points at angle $\frac{\pi}{4} = 45°$ to the y axis or to the negative x axis. The direction does not depend on L, as $E_x = -E_y$ for any value of L.

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2. Question 2

Consider the static electric field:

$$\mathbf{E} = \beta(2x+3y)\hat{\imath} + \beta(3x+z)\hat{\jmath} + \beta(y+2z)\hat{k}\,.$$

Here β is a positive constant.

(a) Question 2(a)

Find the electric potential difference V_{QP} between the point Q at $(L, 5L, 0)$ and the point P at $(L, L, 0)$.

Find the electric potential difference V_{RP} between the point R at $(3L, L, 0)$ and the point P at $(L, L, 0)$.

Here L is a positive constant.

[14 marks]

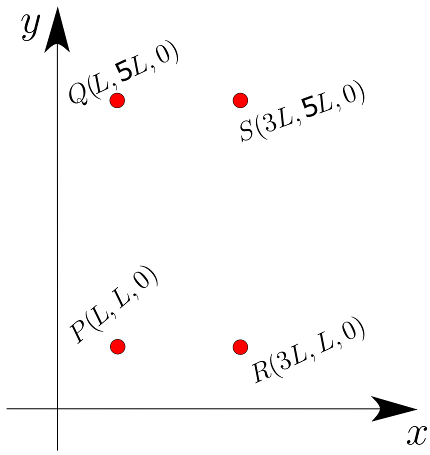
[(Partial) solution / Hints :]

It would be useful to first sketch the positions of P, Q, R on the x-y plane, so that one knows the directions of the lines joining them. (Point S is also shown, for the next question.)

<u>Calculating V_{QP}</u>

The line from P to Q is in the y direction. We can use the line integral along the straight line from P to Q. On this line, $x = L$ and $z = 0$.

$$\begin{aligned}
V_{QP} &= -\int_P^Q \mathbf{E}\cdot d\mathbf{l} \\
&= -\int_{y=L}^{y=5L} \Big(\beta(2x+3y)\hat{\imath} + \beta(3x+z)\hat{\jmath} + \beta(y+2z)\hat{k}\Big)\cdot\,\Big(\hat{\jmath}\;dy\Big) \\
&= -\beta\int_{y=L}^{y=5L}(3x+z)\,dy = -\beta\int_L^{5L}(3L+0)dy \\
&= -3\beta L\int_L^{5L} dy = -3\beta L(4L) = -12\beta L^2
\end{aligned}$$



Calculating V_{RP}

To calculate V_{RP}, we can use the straight line from P to R, which is in the x direction. On this line, $y = L$ and $z = 0$.

$$
\begin{aligned}
V_{RP} &= -\int_R^Q \mathbf{E} \cdot d\mathbf{l} \\
&= -\int_{x=L}^{x=3L} \left(\beta(2x+3y)\hat{\imath} + \beta(3x+z)\hat{\jmath} + \beta(y+2z)\hat{k}\right) \cdot \left(\hat{\imath}\ dx\right) \\
&= -\beta \int_{x=L}^{x=3L} (2x+3y)\,dx = -\beta \int_{x=L}^{x=3L} (2x+3L)\,dx \\
&= -\beta\left(2\left(\frac{(3L)^2}{2} - \frac{L^2}{2}\right) + 3L(3L-L)\right) \\
&= -\beta(8L^2 + 6L^2) = -14\beta L^2
\end{aligned}
$$

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(b) Question 2(b)

Consider the rectangular surface Σ_1 lying in the x-y plane, with three corners at points P, Q, R, and the fourth corner at point S at $(3L, 5L, 0)$. Find the magnitude of the electric flux through this surface.

[11 marks]

[**(Partial) solution / Hints :**]

Refer to the sketch made for part (a), for the geometry.
A unit vector perpendicular to Σ_1 is $\hat{k}$, so that any surface element vector of Σ_1 is $d\mathbf{S} = \hat{k}dS$. (Up to sign, which is not well-defined for this question.)
Thus the flux is

$$\int_{\Sigma_1} \mathbf{E} \cdot d\mathbf{S} = \int_{\Sigma_1} \Big(\beta(2x+3y)\hat{\imath} + \beta(3x+z)\hat{\jmath} + \beta(y+2z)\hat{k}\Big) \cdot \hat{k}dS$$
$$= \beta \int_{\Sigma_1} (y+2z)dS = \beta \int_{\Sigma_1} (y+0)dS$$

The last step uses the fact that the surface lies in the x-y plane, so that $z = 0$ everywhere on the surface.
The integration limits are determined by the geometry of the rectangle:

$$\int_{\Sigma_1} dS = \int_{x=L}^{3L} dx \int_{y=L}^{5L} dy$$

so that the flux is

$$\beta \int_{x=L}^{3L} dx \int_{y=L}^{5L} dy\, y = \beta(2L) \int_{L}^{5L} dy\, y = 2\beta L \left(\frac{25L^2}{2} - \frac{L^2}{2}\right) = 24\beta L^3$$

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3. Question 3

Steady current I flows through an infinitely long straight wire placed parallel to the z axis, from $z = +\infty$ to $z = -\infty$. (The current direction is 'anti-parallel' to the z axis.) The wire runs through the point $(-b, 0, 0)$. Here b is a positive constant.

Reminder: An infinitely long straight wire produces a magnetic field of strength $\mu_0 I/(2\pi d)$ at a point at perpendicular distance d from the wire.

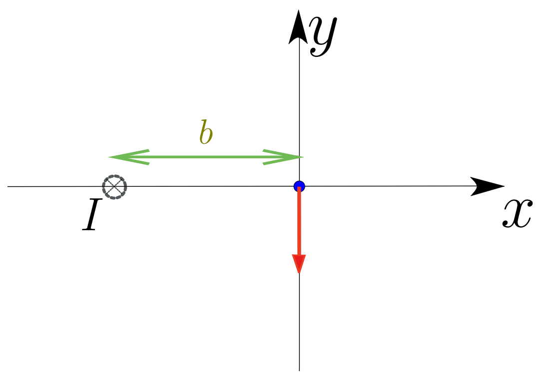
(a) Question 3(a)

Find the magnitude of the magnetic field created at the **origin** due to the current-carrying wire. Find also the x-, y-, and z- components of the magnetic field created at the origin.

[5 marks]

[(Partial) solution / Hints :]

A top view of the x-y plane is very useful. The current flows perpendicular to the plane, direction pointing opposite to the z axis, hence into the paper.



The perpendicular distance to the origin is b. Hence the magnitude of the magnetic field, created at the origin, is $\mu_0 I/(2\pi b)$.

The direction is found using the right hand rule (either the rule for the cross-product in the Biot-Savart law, or the rule for the direction of magnetic field due to current-carrying wires). At the origin, the direction is found to be in the negative y direction. Thus

$$B_x = 0; \quad B_y = -\frac{\mu_0 I}{2\pi b}; \quad B_z = 0.$$

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(b) Question 3(b)

At the point $M(3b, 4b, 0)$ in the x-y plane, find the magnitude of the magnetic field created by the current. Find also the x-, y-, and z-components of the magnetic field at this point.

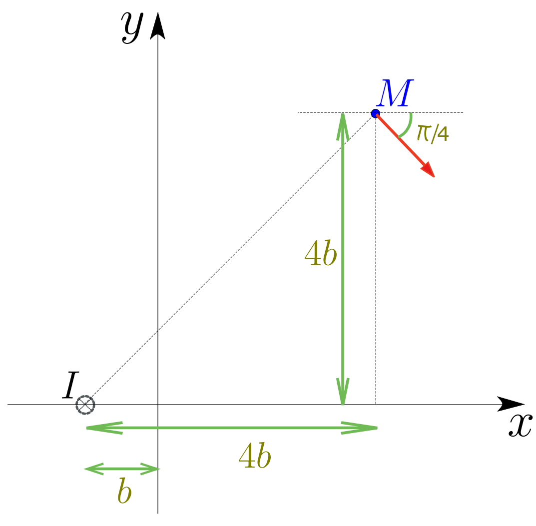
[11 marks]

[(Partial) solution / Hints :]

There is little hope of doing this correctly without a sketch. A top view of the x-y plane is essential.

The perpendicular distance of the point $M(3b, 4b, 0)$ to the wire is is the distance between points $(-b, 0, 0)$ and $M(3b, 4b, 0)$. Using Pythagoras, this is

$$\sqrt{(4b)^2 + (4b)^2} = 4\sqrt{2}b.$$



Hence the magnitude of the magnetic field is

$$B = \frac{\mu_0 I}{2\pi(\text{distance})} = \frac{\mu_0 I}{8\sqrt{2}\pi b}$$

The magnetic field at $M(3b, 4b, 0)$ points perpendicular to the line joining M to the nearest point on the wire, as shown by the arrow in the figure. The field direction is in the x-y plane, hence $B_z = 0$.

To find the x and y coordinates, we need the angle. The field points 'southeast', making angle $-\pi/4$ with the positive x direction. Hence the components are

$$B_x = B\cos(-\pi/4) = B\cos(\pi/4) = \frac{\mu_0 I}{8\sqrt{2}\pi b}\frac{1}{\sqrt{2}} = \frac{\mu_0 I}{16\pi b}$$

and

$$B_y = B\sin(-\pi/4) = -B\sin(\pi/4) = -\frac{\mu_0 I}{8\sqrt{2}\pi b}\frac{1}{\sqrt{2}} = -\frac{\mu_0 I}{16\pi b}$$

Thus the magnetic field created at point M has the components

$$B_x = +\frac{\mu_0 I}{16\pi b}, \quad B_y = -\frac{\mu_0 I}{16\pi b}, \quad B_z = 0.$$

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(c) Question 3(c)

Now consider the point $N(3b, 4b, 5b)$, not in the x-y plane. Find the x-, y-, and z- components of the magnetic field at this point.

[3 marks]

[(Partial) solution / Hints :]

The point $N(3b, 4b, 5b)$ lies directly above the point M, at the same distance and direction from the infinite wire. Therefore by symmetry the magnetic field created at N is exactly the same as that created at M, and hence has components

$$B_x = +\frac{\mu_0 I}{16\pi b}, \quad B_y = -\frac{\mu_0 I}{16\pi b}, \quad B_z = 0.$$

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(d) Question 3(d)

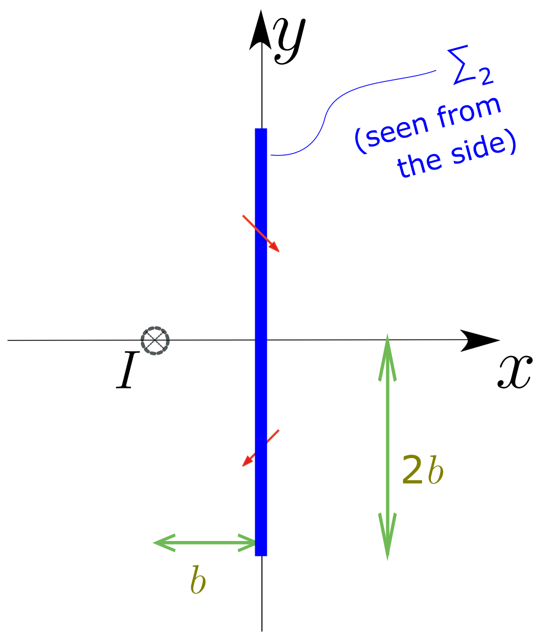
Consider the square-shaped surface Σ_2 lying in the y-z plane, having sides of length $4b$. The center of the square is at the origin. Two of its sides are parallel to the y-axis, and the other two are parallel to the z azis.

What is the magnetic flux through this surface? Explain, or show your calculations.

[6 marks]

[**(Partial) solution / Hints :**]

A reasonable drawing will show that the magnetic field component perpendicular to the surface, i.e., B_x, is positive for $y < 0$ and negative for $y > 0$.



The magnetic field also has a y component which is negative everywhere on Σ_2, but this does not contribute to the flux. (Why?)

The symmetry of the situation means that the contributions to the flux are equally positive and negative in the two halves of the surface. This is seen from the magnetic field represented by the red arrows in the figure. In the $y > 0$ half of the surface, the x-component of the field is positive (arrow rightward), while in the $y < 0$ half, the x-component of the field is negative (arrow leftward)

$\Longrightarrow$ the magnetic flux through Σ_2 is zero.

Of course, to get credits for this problem, examinees are expected to explain the geometry and symmetry argument, not just state that the flux is zero.

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4. Question 4

(a) Question 4(a)

The magnetic field in some region is uniform: $\mathbf{B} = B_0\hat{k}$. At time $t = 0$, a charged particle (mass m, charge q) is at the origin and has velocity $\mathbf{v} = w\hat{\imath} + \alpha_2\hat{k}$. Here w and α_2 are positive constants.

What is the magnitude and direction of the force exerted at $t = 0$ on the particle ?

Explain how the x, y and z components of the velocity change with time.

[6 marks]

[**(Partial) solution / Hints :**]

The force is

$$\mathbf{F} = q\mathbf{v} \times \mathbf{B} = q\left(w\hat{\imath} + \alpha_2\hat{k}\right) \times B_0\hat{k} = -qwB_0\hat{\jmath}$$

Magnitude qwB_0, direction in the negative y direction.

The magnetic field exerts a force $q\mathbf{v} \times \mathbf{B}$ perpendicular to itself, hence in directions perpendicular to z. In the z direction, the velocity thus remains constant, $v_z = \alpha_2$.

Perpendicular to the z direction, the magnetic field causes cyclotron motion. Thus the x and y components of the velocity will each oscillate.

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(b) Question 4(b)

The magnetic field in some region is time-dependent and points in the positive z direction: $\mathbf{B}(t) = B_0\left[2 + \sin(t/T)\right]\hat{k}$. A circular loop of radius r_0 lies in the xy plane. Here B_0, r_0 and T are positive constants.

Find the EMF induced in the loop.

[5 marks]

[(Partial) solution / Hints :]
Since the magnetic field is perpendicular to the loop, the flux through the loop is

$$\Phi_B \;=\; B(\text{area}) \;=\; B_0\,[2+\sin(t/T)]\,\pi r_0^2$$

The EMF according to Faraday's law is

$$\mathcal{E} \;=\; -\frac{d\Phi_B}{dt} \;=\; -\,B_0\pi r_0^2\frac{1}{T}\cos(t/T)$$

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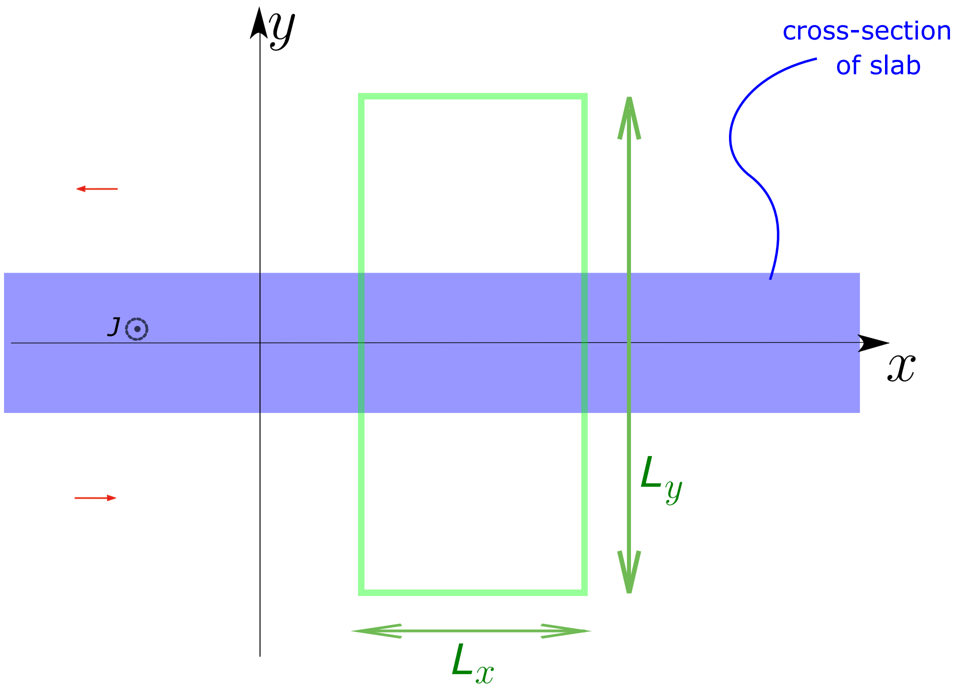
(c) Question 4(c)
A metallic slab has width $2a$, and lies between $y=-a$ and $y=+a$, so that its surfaces are parallel to the z-x plane. The slab has infinite extent in the x and z directions.
Steady current flows through the slab in the z direction. The current density has magnitude J_0 everywhere within the slab:

$$\mathbf{J} \;=\; \begin{cases} J_0\hat{k} & \text{for } |y|<a \\ 0 & \text{for } |y|>a \end{cases}$$

Using Ampere's law, find the magnetic field created by the current at locations outside the slab, both for $y>a$ and for $y<-a$.
Explain clearly any loop that you introduce, and any symmetry arguments that you use.

[14 marks]

[(Partial) solution / Hints :]
The geometry implies that the magnetic field is everywhere in the x or the $-x$ direction. In other words, B_y and B_z are zero everywhere; the only nonzero component is B_x.



In addition, using the right hand rule, the field points leftward in the $y > 0$ region, and rightward in the $y < 0$ region, as shown by red arrows in the figure. The magnitude of B_x might depend on y, but is by symmetry independent of z and x.

We use a rectangular Amperean loop Γ, parallel to the x-y plane. The x-direction sides have length L_x, and the y-direction sides have length L_y, and are equally far from the x axis. The z- or x- position of the loop does not matter.

To use Ampere's law, we have to calculate the line integral of the magnetic field counterclockwise around the loop:

$$\oint_\Gamma \mathbf{B} \cdot d\mathbf{l}\,.$$

The contribution from the sides of length L_y is zero. The contribution from the side at $y = +L_y/2$ is

$$\Big|B(y = +L_y/2)\Big| \times L_x$$

The contribution from the side at $y = -L_y/2$ is

$$\Big|B(y = -L_y/2)\Big| \times L_x$$

By symmetry, the fields at these two sides have the same magnitude, thus the total line integral is

$$\oint_\Gamma \mathbf{B}\cdot d\mathbf{l} = \left|B(y=+L_y/2)\right| \times 2L_x$$

By Ampere's law, this is equal to μ_0 times the current piercing the loop. The current piercing the loop is the current flowing through the part of the slab that intersects the loop. Since this part has cross section $2aL_x$, this current is

$$I_{\text{piercing}} = \text{current density} \times \text{cross-section area} = J_0 \times 2aL_x$$

Thus Ampere's law gives

$$\left|B(y=+L_y/2)\right| \times 2L_x = \mu_0 \times J_0 \times 2aL_x$$
$$\implies \quad \left|B(y=+L_y/2)\right| = \mu_0 J_0 a$$

Remarkably, the magnitude is independent of L_y and hence of y. In other words, the magnitude of the magnetic field outside the slab is the same at any distance. To summarize

$$\mathbf{B} \;=\; \begin{cases} -\hat{\imath}\,\mu_0 J_0 a & \text{for } y > 0 \\ +\hat{\imath}\,\mu_0 J_0 a & \text{for } y < 0 \end{cases}$$

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