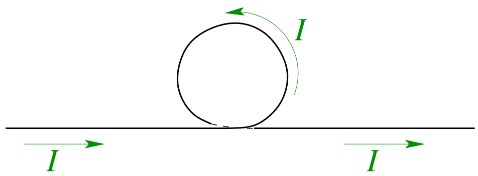
Some partial solutions and/or hints are provided here.

The solutions are possibly incomplete and may not have been proofread carefully, so please use with caution.

– – – – – – – – – – – – – – – ⋆ – – – – – – – – – – – – – –

1. A very long wire carrying current I is straight except for a circular loop of radius R. (An approximate drawing is shown.) Find the magnetic field at the center of the loop.



(Partial) Solution/Hint →

The field due to the straight part is $\dfrac{\mu_0 I}{2\pi R}$ and the field due to the circular loop at the center of the current is $\dfrac{\mu_0 I}{2R}$. Both results have been derived in class. Hence the total field at the center of the loop is

$$\frac{\mu_0 I}{2\pi R} + \frac{\mu_0 I}{2R} = \frac{\mu_0 I}{2\pi R}(1+\pi)$$

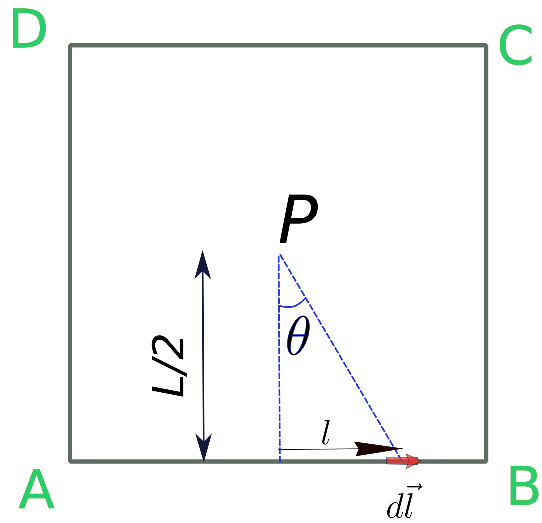
— —

2. Steady electric current I is flowing through a square-shaped loop of wire; let's call the square $ABCD$ with each letter denoting a corner of the square. Each side of the square has length L. We wish to calculate the magnetic field generated at the center of the square, which we denote as point P.

   Let's first concentrate on one of the sides, say AB. Choose an infinitesimal element of this side, of length $|d\mathbf{l}|$. Define $\mathbf{l}$ as its displacement from the center of AB. The line joining this element to the point P makes an angle θ with the line joining the point P to the center of the side AB.

(a) Sketch a top view of the square wire and an element $d\mathbf{l}$, showing clearly the angle θ, the distance l, and the direction of the current. Take the current to flow counter-clockwise.

(Partial) Solution/Hint →



— —

(b) What is the distance between the element and the point P? Write two expressions for this distance: one in terms of L and l, and one in terms of L and θ.

(Partial) Solution/Hint →

Point P is at distance $L/2$ from the midpoint of the wire segment AB, and the element is at distance l from this midpoint.

Considering the right-angled triangle in the figure, we see that the distance to the element is

$$\sqrt{(L/2)^2 + l^2} \;=\; \frac{L/2}{\cos\theta}$$

— —

(c) Using the Biot-Savart law, write down the magnitude of the magnetic field $d\mathbf{B}$ created at P due to the chosen element. Express the magnitude in terms of L and l. (There should be no θ in your expression.)

(Partial) Solution/Hint →

The Biot-Savart law says that the magnetic field element is due to the current in an element of the wire is

$$d\mathbf{B} = \frac{\mu_o I}{4\pi}\frac{d\mathbf{l}\times\hat{r}}{r^2}$$

where $\mathbf{r}$ is the displacement from the wire element in $d\mathbf{l}$ to the point where we are calculating the field, and $\hat{r}$ is a unit vector in the direction of $\mathbf{r}$. In class we also used the more cumbersome notation

$$d\mathbf{B} = \frac{\mu_o I}{4\pi}\frac{d\mathbf{l}\times\widehat{(\mathbf{r}-\mathbf{r}')}}{|\mathbf{r}-\mathbf{r}'|^2}$$

to mean the same thing. Let's use the first notation to avoid clutter. In the present case, $d\mathbf{l}$ makes angle $\frac{\pi}{2}-\theta$ with $\hat{r}$, hence the magnitude of the cross product is

$$|d\mathbf{l}\times\hat{r}| = |d\mathbf{l}|\times 1\times\sin(\frac{\pi}{2}-\theta) = dl\frac{L/2}{\sqrt{(L/2)^2+l^2}}$$

Also, $r^2 = (L/2)^2 + l^2$. Thus the magnitude of the field element is

$$dB = \frac{\mu_o I}{4\pi}\frac{(L/2)dl}{[(L/2)^2+l^2]^{3/2}}$$

— —

(d) What is the direction of $d\mathbf{B}$?

(Partial) Solution/Hint →

Out of the plane of the paper.

— —

(e) Find the magnetic field at P due to the current in the side AB, by integrating your expression for $d\mathbf{B}$ over the displacement l. What are the limits of integration? It might help to know that

$$\int \frac{du}{(u^2+a^2)^{3/2}} = \frac{u}{a^2(u^2+a^2)^{1/2}}$$

(Partial) Solution/Hint →

The limits of integration from A to B (over the side of the square) are $l = -L/2$ to $l = +L/2$. Thus the total field due to the sire segment AB is

$$\begin{aligned} B &= \frac{\mu_o IL}{8\pi}\int_{-L/2}^{+L/2}\frac{dl}{\left[l^2+(L/2)^2\right]^{3/2}} = \frac{\mu_o IL}{8\pi}\left[\frac{l}{(L/2)^2\left[l^2+(L/2)^2\right]^{1/2}}\right] \\ &= \frac{\mu_o I}{2\pi L}\left(\frac{L/2}{\left[(L/2)^2+(L/2)^2\right]^{1/2}} - \frac{(-L/2)}{\left[(L/2)^2+(L/2)^2\right]^{1/2}}\right) \\ &= \frac{\mu_o I}{2\pi L}\sqrt{2} \end{aligned}$$

— —

(f) Express l as a function of θ, and use this equation to find dl as a function of θ and $d\theta$.

(Partial) Solution/Hint →

$$l = \frac{L}{2}\tan\theta \quad \Longrightarrow \quad dl = \frac{L}{2}\sec^2\theta d\theta = \frac{Ld\theta}{2\cos^2\theta}$$

— —

(g) Express the magnitude of $d\mathbf{B}$ in terms of L, θ, and $d\theta$.

(Partial) Solution/Hint →

We found

$$dB = \frac{\mu_o I}{4\pi} \frac{(L/2)dl}{\left[(L/2)^2 + l^2\right]^{3/2}}$$

Noting that

$$(L/2)^2 + l^2 = (L/2)^2(1 + \tan^2\theta) = (L/2)^2 \sec^2\theta\,,$$

we get

$$dB = \frac{\mu_o I}{4\pi} \frac{(L/2)(L/2)\sec^2\theta d\theta}{(L/2)^3 \sec^3\theta} = \frac{\mu_o I}{2\pi L} \frac{d\theta}{\sec\theta} = \frac{\mu_o I}{2\pi L} \cos\theta d\theta$$

— —

(h) Find again the magnetic field at P due to the current in the side AB, by integrating over the angle θ. What are the limits of integration?

(Partial) Solution/Hint →

The drawing should show that the limits of integration, i.e., the angles corresponding to points A and B, are $\theta = -\pi/4$ and $\theta = +\pi/4$. Hence the magnetic field at P due to the current in the side AB is

$$B = \frac{\mu_o I}{2\pi L} \int_{-\pi/4}^{\pi/4} \cos\theta d\theta = \frac{\mu_o I}{2\pi L} \left[\sin\theta\right]_{-\pi/4}^{+\pi/4} = \frac{\mu_o I}{2\pi L} \left(\sin(\tfrac{\pi}{4}) - \sin(-\tfrac{\pi}{4})\right)$$

$$= \frac{\mu_o I}{2\pi L} \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) = \frac{\mu_o I}{2\pi L} \sqrt{2}$$

— —

(i) You have found the magnetic field at P due to the current through one side of the square-shaped wire. What is the total magnetic field at P due to the current-carrying square wire?

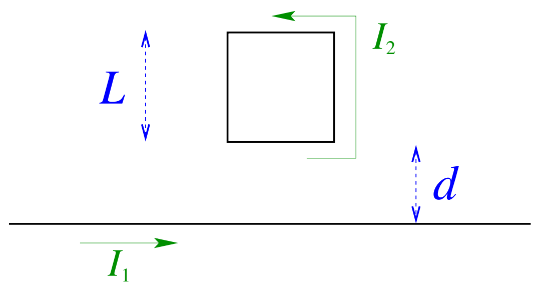
(Partial) Solution/Hint →

By symmetry, each side of the square produces the same magnitude of magnetic field at P. Using the right hand rule (whichever version you use) also shows that each segment creates a field in the same direction, hence the fields due to the four sides add up. The total field is

$$4 \times \frac{\mu_o I}{2\pi L}\sqrt{2} \;=\; \frac{\mu_o I}{\pi L} 2\sqrt{2}$$

— —

3. A square-shaped wire loop (each side of length L) carrying steady current I_2, is placed near an infinitely long straight wire carrying steady current I_1, with the nearest side being at distance d away. (See figure.)



(a) Each side of the square experiences a force due to the magnetic field produced by the long wire. What are the directions of the force on each side? (Either draw the square with arrows showing the forces, or state clearly in words.)

(Partial) Solution/Hint →

The field in the region of the square points out of the plane of the paper. As a result, each side of the square loop is pulled outward (away from the opposide side) by the Lorentz force.

— —

(b) Calculate the total force acting on the square loop due to the current in the long wire. Hint: the forces on two of the sides will cancel by symmetry and thus don't need to be calculated.

(Partial) Solution/Hint →

The sides perpendicular to the wire feel equal and opposite forces, by symmetry. They are also more difficult to calculate. Lucky that they cancel, so we don't have to calculate them.

At the position of the side nearest to the long wire, the magnetic field is $B = \mu_0 I_1/(2\pi d)$, Hence the force on this side is

$$I_2 LB = \frac{\mu_0 I_1 I_2 L}{2\pi d}$$

pointing toward the long wire.

At the position of the side farthest from the long wire, the magnetic field is $B = \dfrac{\mu_0 I_1}{2\pi(d+L)}$. Hence the force on this side is

$$I_2 LB = \frac{\mu_0 I_1 I_2 L}{2\pi(d+L)}$$

acting away from the long wire.

Thus the total force on the square loop is toward the long wire and has magnitude

$$\frac{\mu_0 I_1 I_2 L}{2\pi d} - \frac{\mu_0 I_1 I_2 L}{2\pi(d+L)} = \frac{\mu_0 I_1 I_2 L^2}{2\pi d(d+L)}$$

— —