Some partial solutions and/or hints are provided here.
Not carefully proofread and possibly incomplete — please use responsibly.

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1. Divergence of curl:

   (a) Using Cartesian coordinates, show that the divergence of the curl of any vector $\mathbf{v}$ is zero.

   (Partial) Solution/Hint →

$$\nabla\cdot(\nabla\times\mathbf{A}) = \nabla\cdot\left[\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)\hat{\imath}+\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)\hat{\jmath}+\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)\hat{k}\right]$$
$$= \frac{\partial}{\partial x}\left(\frac{\partial A_z}{\partial y}-\frac{\partial A_y}{\partial z}\right)+\frac{\partial}{\partial y}\left(\frac{\partial A_x}{\partial z}-\frac{\partial A_z}{\partial x}\right)+\frac{\partial}{\partial z}\left(\frac{\partial A_y}{\partial x}-\frac{\partial A_x}{\partial y}\right)$$
$$= \left(\frac{\partial^2 A_x}{\partial y\partial z}-\frac{\partial^2 A_x}{\partial z\partial y}\right)+\left(\frac{\partial^2 A_y}{\partial z\partial x}-\frac{\partial^2 A_y}{\partial x\partial z}\right)+\left(\frac{\partial^2 A_z}{\partial x\partial y}-\frac{\partial^2 A_z}{\partial y\partial x}\right) = 0$$

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   (b) Maxwell's fourth equation is: $\nabla\times\mathbf{B} = \mu_0\mathbf{J}+\mu_0\epsilon_0\dfrac{\partial\mathbf{E}}{\partial t}$.
   The divergence of the left side is zero; therefore the right side also needs to have zero divergence. Show that the divergence of the right side is indeed zero. You will need to use the continuity equation and Maxwell's first equation. Point out clearly the steps where you use these.

   (Partial) Solution/Hint →

   The divergence of the right hand side:

$$\nabla\cdot\left(\mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial\mathbf{E}}{\partial t}\right) = \mu_0\nabla\cdot\mathbf{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}(\nabla\cdot\mathbf{E})$$
$$= \mu_0\nabla\cdot\mathbf{J} + \mu_0\epsilon_0\frac{\partial}{\partial t}\left(\frac{\rho}{\epsilon_0}\right) \qquad \text{used Maxwell's first equation}$$
$$= \mu_0\left(\nabla\cdot\mathbf{J} + \frac{\partial\rho}{\partial t}\right) = \mu_0\times 0 \qquad \text{used the continuity equation}$$
$$= 0$$

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2. Steady current I flows through an infinitely long straight wire placed along the z azis, from $z = -\infty$ to $z = +\infty$ through the origin.

   We will use cylindrical coordinates, (r, ϕ, z), in this problem. The unit vectors are denoted as $\hat{r}$, $\hat{\phi}$, $\hat{z}$. (They are often denoted as $\mathbf{e}_r$, $\mathbf{e}_\phi$, $\mathbf{e}_z$.) You will need to recall or learn what these coordinates and unit vectors mean. Note that the directions of the $\hat{r}$ and $\hat{\phi}$ vectors depend on the angle ϕ.

   (a) Write down an expression for the magnetic field vector $\mathbf{B}$ created by the current I at point (r, ϕ, z). This should be a vector equation. What are the components B_r, B_ϕ and B_z?

   (Partial) Solution/Hint →

   A 3D situation!! As always, solving such problems will only work if you sketch figures showing various directions and distances. It's best to draw the situation from a couple of different perspectives (top view, side view, view of the x-y plane, etc.) — this is useful for building a complete mental picture of the situation.

   If you sketch the situation showing directions of $\hat{r}$, $\hat{\phi}$, $\hat{z}$ and the direction of $\mathbf{B}$, you should see that $\mathbf{B}$ points in the same direction as the unit vector $\hat{\phi}$. Also, the perpendicular distance from the point (r, ϕ, z) to the wire is exactly r, so that the magnitude of the current at this point is exactly $\mu_0 I/(2\pi r)$. Hence

$$\mathbf{B} \;=\; \frac{\mu_0 I}{2\pi r}\hat{\phi}$$

   The components are

$$B_r = 0\,; \quad B_\phi = \frac{\mu_0 I}{2\pi r}\,; \quad B_z = 0$$

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(b) Use your expression for **B** to calculate the vector potential **A**. You might need to recall that the curl of a vector **v** in cylindrical coordinates is

$$\nabla \times \mathbf{v} \;=\; \left(\frac{1}{r}\frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z}\right)\hat{r} + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r}\right)\hat{\phi} + \frac{1}{r}\left(\frac{\partial (r v_\phi)}{\partial r} - \frac{\partial v_r}{\partial \phi}\right)\hat{z}$$

Also, you can assume that **A** has no z-dependence, which would be consistent with the symmetry of the problem.

(Partial) Solution/Hint $\rightarrow$

$\mathbf{B} = \nabla \times \mathbf{A}$ only has a component in the $\hat{\phi}$ direction.

$$B_\phi \;=\; \left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right)$$

Since there is no z-dependence, the term with the z-derivative must vanish. Hence

$$B_\phi \;=\; -\frac{dA_z}{dr}$$

$$\implies \quad A_z \;=\; -\int B_\phi dr \;=\; -\frac{\mu_0 I}{2\pi}\int \frac{dr}{r} \;=\; -\frac{\mu_0 I}{2\pi}\ln r$$

Thus

$$\mathbf{A} \;=\; -\left(\frac{\mu_0 I}{2\pi}\ln r\right)\hat{z}$$

Note that there might be an arbitrary constant of integration added.

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3. We consider now a *thick* wire, of circular cross-section, with radius R. The wire is infinitely long, and its axis is placed along the z azis. Steady current I flows through the wire from $z = -\infty$ to $z = +\infty$. We will continue to use cylindrical coordinates.

   First consider the current I to be uniformly distributed over the circular cross-section, so that the current density is

$$\mathbf{J} = \left(\frac{I}{\pi R^2}\right)\hat{z} \quad \text{for} \quad r \leq R \qquad \text{and} \qquad \mathbf{J} = 0 \quad \text{for} \quad r > R$$

   In class, we used Amperean loops to show that the magnetic field is

$$\mathbf{B} = \left(\frac{\mu_0 I}{2\pi}\frac{r}{R^2}\right)\hat{\phi} \quad \text{for} \quad r \leq R \qquad \text{and} \qquad \mathbf{B} = \left(\frac{\mu_0 I}{2\pi}\frac{1}{r}\right)\hat{\phi} \quad \text{for} \quad r > R$$

   (a) Show that Ampere's law in differential form ($\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$) is satisfied outside the wire ($r > R$). Use cylindrical coordinates.

   (Partial) Solution/Hint →

   Using $\mathbf{B} = \left(\frac{\mu_0 I}{2\pi}\frac{1}{r}\right)\hat{\phi}$ and the expression for the curl in cylindrical coordinates, we obtain

$$\nabla \times \mathbf{B} \;=\; -\frac{\partial v_\phi}{\partial z}\hat{r} \;+\; \frac{1}{r}\frac{\partial(rv_\phi)}{\partial r}\hat{z} \;=\; -\,0\hat{r} \;+\; \frac{1}{r}\frac{\partial}{\partial r}(\mu_0 I/2\pi)\hat{z} \;=\; 0$$

   Since the current density is also zero outside the cylinder, Ampere's law in differential form ($\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$) is satisfied.

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   (b) Show that Ampere's law in differential form is satisfied inside the wire ($r < R$). Use cylindrical coordinates.

   (Partial) Solution/Hint →

   Using $B_\phi = \left(\frac{\mu_0 I}{2\pi}\frac{r}{R^2}\right)$ for the inside region,

$$\begin{aligned}
\nabla \times \mathbf{B} \;&=\; -\frac{\partial v_\phi}{\partial z}\hat{r} \;+\; \frac{1}{r}\frac{\partial(rv_\phi)}{\partial r}\hat{z} \;=\; -\,0\hat{r} \;+\; \frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\mu_0 I}{2\pi}\frac{r^2}{R^2}\right)\hat{z} \\
&=\; \frac{\mu_0 I}{2\pi R^2}\frac{1}{r}\frac{d}{dr}(r^2)\hat{z} \;=\; \frac{\mu_0 I}{2\pi R^2}2\hat{z} \;=\; \mu_0\frac{I}{\pi R^2}\hat{z} \;=\; \mu_0 \mathbf{J}
\end{aligned}$$

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(c) Express **B** inside the wire ($r < R$) in Cartesian coordinates.

(Partial) Solution/Hint →

A drawing would show that

$$B_x = B_r \cos\phi - B_\phi \sin\phi\ , \quad B_y = B_r \sin\phi + B_\phi \cos\phi\ ,$$
$$r = \sqrt{x^2+y^2}\ , \quad \cos\phi = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}\ , \quad \sin\phi = \frac{y}{\sqrt{x^2+y^2}} = \frac{y}{r}\ .$$

Thus using $B_\phi = \dfrac{\mu_0 I}{2\pi}\dfrac{r}{R^2}$, $B_r = 0$, we get

$$B_x = -\frac{\mu_0 I}{2\pi}\frac{r}{R^2}\frac{y}{\sqrt{x^2+y^2}} = \frac{\mu_0 I}{2\pi R^2}(-y)\ , \quad B_y = \frac{\mu_0 I}{2\pi R^2}x$$

And also obviously $B_z = 0$.

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(d) Using Cartesian coordinates, show that Ampere's law in differential form is satisfied inside the wire ($r < R$).

(Partial) Solution/Hint →

The curl is

$$\begin{aligned}\nabla\times\mathbf{B} &= \left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\hat{k} = \frac{\mu_0 I}{2\pi R^2}\left(\frac{\partial}{\partial x}(x) - \frac{\partial}{\partial y}(-y)\right)\hat{k} \\ &= \frac{\mu_0 I}{2\pi R^2}2\hat{k} = \mu_0\left(\frac{I}{\pi R^2}\hat{k}\right) = \mu_0\mathbf{J}\end{aligned}$$

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4. **[SELF]** We continue with the thick-wire geometry of the previous problem.

   Now imagine that the current density is larger near the outer region of the wire cross-section than at the center: $J = kr$, so that $I = \left(\frac{2}{3}\pi R^3\right) k$. (You might want to derive this for yourself before working on the problems below. You may have done something similar for problem set 04.) We thus have

$$\mathbf{J} = kr\hat{z} = \left(\frac{I}{\frac{2}{3}\pi R^3} r\right)\hat{z} \quad \text{for} \quad r \le R \qquad \text{and} \qquad \mathbf{J} = 0 \quad \text{for} \quad r > R$$

   Comments →

   It would be extremely unwise to attempt this problem without making multiple sketches of the geometry.

   You should be able to derive $I = \left(\frac{2}{3}\pi R^3\right) k$ using the relationship

$$I = \int_\Sigma \mathbf{J} \cdot d\mathbf{S}$$

   between the current I through a surface Σ and the current density $\mathbf{J}$.

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   (a) **[SELF]** Construct an Amperean loop and use Ampere's law in integral form, to find the magnetic field outside the wire ($r > R$). Express $\mathbf{B}$ in cylindrical coordinates. Express your answer first in terms of I, and then in terms of k.

   (Partial) Solution/Hint →

   We use an Amperean loop in the shape of a circle whose center lies at the axis of the current-carrying wire, with radius r larger than the wire radius R. The plane of the circle is perpendicular to the wire axis. (Please do sketch the loop geometry yourself.)

   The current through the loop is the entire current flowing through the wire, $I = \left(\frac{2}{3}\pi R^3\right) k$. Hence the calculation using Ampere's law is the same as that for a thin long wire.

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I \quad \Longrightarrow \quad B(2\pi r) = \mu_0 I$$

To evaluate the line integral, the usual symmetry arguments have been used: the field $\mathbf{B}$ points tangential to the Amperean loop, hence in the same direction as $d\mathbf{l}$, and its magnitude is the same everywhere on the loop. Hence the line integral is $|\mathbf{B}| = B$ times the circumference $(2\pi r)$ of the loop. These symmetry arguments work because the current density is cylindrically symmetric, J is a function of the cylindrical distance r only.

Thus we obtain $B = \mu_0 I/(2\pi r)$ exactly as for a thin wire. In terms of k,

$$B = \frac{\mu_0 I}{2\pi r} = \frac{\mu_0 \left(\frac{2}{3}\pi R^3\right) k}{2\pi r} = \frac{\mu_0 R^3 k}{3r}$$

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(b) **[SELF]** Use an Amperean loop and Ampere's law in integral form, to find the magnetic field inside the wire $(r < R)$. Express $\mathbf{B}$ in cylindrical coordinates. Express your answer in terms of I and in terms of k.

(Partial) Solution/Hint $\rightarrow$

Now the circular Amperean loop is inside the thick cylindrical wire. Using Ampere's law,

$$\oint \mathbf{B} \cdot d\mathbf{l} = \mu_0 I_{\text{encl.}} \qquad \Longrightarrow \qquad B(2\pi r) = \mu_0 I_{\text{encl.}}$$

However the current $I_{\text{encl.}}$ piercing through the loop is now no longer the total current through the wire. To find $I_{\text{encl.}}$, we need to integrate over the plane surface enclosed by the loop.

$$I_{\text{encl.}} = \int \mathbf{J} \cdot d\mathbf{S} = \int_0^r J(r_1)(2\pi r_1) dr_1$$

We have used the fact that the current density $\mathbf{J}$ points in the same direction as the surface element $d\mathbf{S}$, and also that $\mathbf{J}$ is cylindrically symmetric, does not depend on the angle. To avoid notation conflict, the integration variable (cylindrical distance) is called r_1 instead of the usual r. The symbol r here already means the radius of the loop. Take care to avoid having the same symbols meaning two different things!

Hence

$$I_{\text{encl.}} = \int_0^r k r_1 (2\pi r_1) dr_1 = 2\pi k \int_0^r r_1^2 dr_1 = \frac{2}{3}\pi k r^3$$

Subsitituting into Ampere's law, we obtain

$$B(2\pi r) = \mu_0 \frac{2}{3}\pi k r^3 \quad \Longrightarrow \quad B = \frac{1}{3}\mu_0 k r^2$$

Thus the magnetic field is zero at the axis (does that make physical sense?), and grows quadratically with the distance as the cylindrical distance is increased.

We have found expressions for the magnetic field for $r < R$ and for $r > R$.
Surely the two expressions should match at the boundary, $r = R$?
Check whether they do match at $r = R$.
You could also try plotting the magnitude of the current as a function of r.

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(c) **[SELF]** Show that your expression for $\mathbf{B}$ inside the wire ($r < R$) satisfies the differential form of Ampere's law.

(Partial) Solution/Hint →

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5. (Applying Maxwell's equations.) In some region, the electric field changes with time but the magnetic field does not:

$$\mathbf{E} = -\left(\frac{2K_0 t}{\epsilon_0}\right)\hat{k}\ ; \qquad \mathbf{B} = \mu_0 K_0\left(3y\hat{\imath} - 3x\hat{\jmath}\right)\ .$$

(a) Find the charge density in the region.

(Partial) Solution/Hint →

Using the first equation

$$\rho = \epsilon_0 \nabla\cdot\mathbf{E} = \epsilon_0\frac{\partial}{\partial z}\left(-\frac{2K_0 t}{\epsilon_0}\right) = 0$$

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(b) Find the current density in the region.

(Partial) Solution/Hint →

The current density is obtained from the fourth equation

$$\begin{aligned}\mathbf{J} &= \frac{1}{\mu_0}\nabla\times\mathbf{B} - \epsilon_0\frac{\partial\mathbf{E}}{\partial t} \\ &= \frac{1}{\mu_0}\left(\mu_0 K_0(-3-3)\hat{k}\right) \ - \ \left(-2K_0\hat{k}\right) \ = \ -4K_0\hat{k}\end{aligned}$$

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(c) Show that the continuity equation is satisfied.

(Partial) Solution/Hint →

From the current density that we have calculated, we see that

$$\nabla\cdot\mathbf{J} \ = \ \frac{\partial}{\partial z}(-4K_0) \ = \ 0$$

From the charge density that we have calculated, we find

$$\frac{\partial\rho}{\partial t} \ = \ \frac{\partial}{\partial t}(0) \ = \ 0$$

Thus the continuity equation ($\nabla\cdot\mathbf{J} = -\frac{\partial\rho}{\partial t}$) is satisfied, as both sides vanish.

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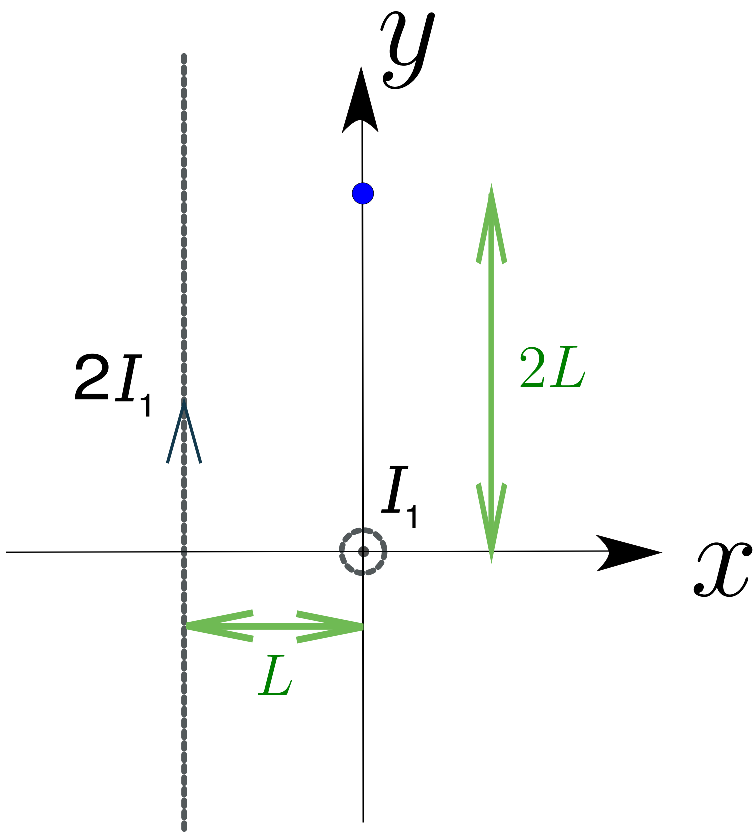
6. An infinite wire carrying current I_1 runs along the z axis; the current flows from $z = -\infty$ to $z = +\infty$ through the origin.

   Another infinite wire runs parallel to the y axis, lies in the x-y plane, and goes through the point $(-L, 0, 0)$. Current $2I_1$ flows through this wire, from $y = -\infty$ to $y = +\infty$.

   Find the magnetic field created by each wire at the point $(0, 2L, 0)$ on the y-axis. Find the magnitude of the total magnetic field at this point.

   (Partial) Solution/Hint →

   The situation is sketched below. The point $(0, 2L, 0)$ is shown with a blue dot.



   In the first wire (running along the z axis), the direction of current is perpendicular to the page, pointing outward from the plane of the paper (toward the reader). Why outward toward the reader, and not inward into the page? Because the z axis points outward from the plane of the paper, and the question specifies that the current flows from $z = -\infty$ to $z = +\infty$ through the origin.

Why is the positive z direction pointing out of the paper?
(Could you have chosen it to point inward?)

We don't have this freedom because we always use *right-handed* coordinate systems, i.e., $\hat{\imath} \times \hat{\jmath}$ should be $\hat{k}$, not $-\hat{k}$. So, once the positive x and positive y directions are specified, the positive z direction becomes specified as well and there is no freedom in choosing between the positive z and negative z directions.

At the point $(0, 2L, 0)$, the first current (along z axis) creates magnetic field in the negative x direction, of magnitude

$$B_1 = \frac{\mu_0 I_1}{2\pi(2L)} = \frac{\mu_0 I_1}{4\pi L}.$$

(Make sure you know why this points in the $-\hat{\imath}$ direction.)

The other current (parallel to y axis) creates magnetic field in the negative z direction, of magnitude

$$B_2 = \frac{\mu_0 (2I_1)}{2\pi(L)} = \frac{\mu_0 I_1}{\pi L}$$

(Again, make sure you are able by yourself to deduce the direction of this magnetic field.)

The total magnetic field is thus

$$B_1(-\hat{\imath}) + B_2(-\hat{k}) = -\frac{\mu_0 I_1}{4\pi L}\hat{\imath} - \frac{\mu_0 I_1}{\pi L}\hat{k}$$

Exercise: Looking at this result and at the sketch on the previous page, try visualizing the direction of the total magnetic field.

The magnitude of this total magnetic field is

$$\sqrt{\left(\frac{\mu_0 I_1}{4\pi L}\right)^2 + \left(\frac{\mu_0 I_1}{\pi L}\right)^2} = \frac{\mu_0 I_1}{\pi L}\frac{\sqrt{17}}{4}$$

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