Due on Monday, April 12th.

If pictures are needed/relevant, please provide them with your solutions.

Questions marked [**SELF**] are for your practice and will not be marked; no need to submit those.

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1. Divergence of curl:

   (a) [**SELF**] Using Cartesian coordinates, show that the divergence of the curl of any vector $\mathbf{v}$ is zero.

   (b) [**5 pts**] Maxwell's fourth equation is: $\nabla \times \mathbf{B} = \mu_0 \mathbf{J} + \mu_0 \epsilon_0 \frac{\partial \mathbf{E}}{\partial t}$.
   The divergence of the left side is zero; therefore the right side also needs to have zero divergence. Show that the divergence of the right side is indeed zero. You will need to use the continuity equation and Maxwell's first equation. Point out clearly the steps where you use these.

2. Steady current I flows through an infinitely long straight wire placed along the z azis, from $z = -\infty$ to $z = +\infty$ through the origin.

   We will use cylindrical coordinates, (r, ϕ, z), in this problem. The unit vectors are denoted as $\hat{r}$, $\hat{\phi}$, $\hat{z}$. ($\mathbf{e}_r$, $\mathbf{e}_\phi$, $\mathbf{e}_z$ is also common notation.) You will need to recall or learn what these coordinates and unit vectors mean. Note that the directions of the $\hat{r}$ and $\hat{\phi}$ vectors depend on the angle ϕ.

   (a) [**4 pts**] Write down an expression for the magnetic field vector $\mathbf{B}$ created by the current I at point (r, ϕ, z). This should be a vector equation. What are the components B_r, B_ϕ and B_z?

   (b) [**6 pts**] Use your expression for $\mathbf{B}$ to calculate the vector potential $\mathbf{A}$. You might need to recall that the curl of a vector $\mathbf{v}$ in cylindrical coordinates is

   $$\nabla \times \mathbf{v} = \left(\frac{1}{r} \frac{\partial v_z}{\partial \phi} - \frac{\partial v_\phi}{\partial z} \right) \hat{r} + \left(\frac{\partial v_r}{\partial z} - \frac{\partial v_z}{\partial r} \right) \hat{\phi} + \frac{1}{r} \left(\frac{\partial (r v_\phi)}{\partial r} - \frac{\partial v_r}{\partial \phi} \right) \hat{z}$$

   Also, you can assume that $\mathbf{A}$ has no z-dependence, which would be consistent with the symmetry of the problem.

3. We consider now a *thick* wire, of circular cross-section, with radius R. The wire is infinitely long, and its axis is placed along the z azis. Steady current I flows through the wire from $z = -\infty$ to $z = +\infty$. We will continue to use cylindrical coordinates.

   First consider the current I to be uniformly distributed over the circular cross-section, so that the current density is

   $$\mathbf{J} = \left(\frac{I}{\pi R^2}\right)\hat{z} \quad \text{for} \quad r \leq R \qquad \text{and} \qquad \mathbf{J} = 0 \quad \text{for} \quad r > R$$

   In class, we used Amperean loops to show that the magnetic field is

   $$\mathbf{B} = \left(\frac{\mu_0 I}{2\pi}\frac{r}{R^2}\right)\hat{\phi} \quad \text{for} \quad r \leq R \qquad \text{and} \qquad \mathbf{B} = \left(\frac{\mu_0 I}{2\pi}\frac{1}{r}\right)\hat{\phi} \quad \text{for} \quad r > R$$

   (a) [**4 pts**] Show that Ampere's law in differential form ($\nabla \times \mathbf{B} = \mu_0 \mathbf{J}$) is satisfied outside the wire ($r > R$). Use cylindrical coordinates.

   (b) [**4 pts**] Show that Ampere's law in differential form is satisfied inside the wire ($r < R$). Use cylindrical coordinates.

   (c) [**4 pts**] Express $\mathbf{B}$ inside the wire ($r < R$) in Cartesian coordinates.
   A drawing might help you to find the Cartesian components of $\mathbf{B}$, or you can do this algebraically.

   (d) [**SELF**] Using Cartesian coordinates, show that Ampere's law in differential form is satisfied inside the wire ($r < R$).

4. [**SELF**] We continue with the thick-wire geometry of the previous problem.

   Now imagine that the current density is larger near the outer region of the wire cross-section than at the center: $J = kr$, so that $I = \left(\frac{2}{3}\pi R^3\right)k$. (You might want to derive this for yourself before working on the problems below. You may have done something similar for problem set 04.) We thus have

   $$\mathbf{J} = kr\hat{z} = \left(\frac{I}{\frac{2}{3}\pi R^3}r\right)\hat{z} \quad \text{for} \quad r \leq R \qquad \text{and} \qquad \mathbf{J} = 0 \quad \text{for} \quad r > R$$

   (a) [**SELF**] Construct an Amperean loop and use Ampere's law in integral form, to find the magnetic field outside the wire ($r > R$). Express $\mathbf{B}$ in cylindrical coordinates. Express your answer first in terms of I, and then in terms of k.

(b) **[SELF]** Use an Amperean loop and Ampere's law in integral form, to find the magnetic field inside the wire ($r < R$). Express $\mathbf{B}$ in cylindrical coordinates. Express your answer in terms of I and in terms of k.

(c) **[SELF]** Show that your expression for $\mathbf{B}$ inside the wire ($r < R$) satisfies the differential form of Ampere's law.

5. (Applying Maxwell's equations.) In some region, the electric field changes with time but the magnetic field does not:

$$\mathbf{E} = -\left(\frac{2K_0 t}{\epsilon_0}\right)\hat{k} \; ; \qquad \mathbf{B} = \mu_0 K_0 \left(3y\hat{\imath} - 3x\hat{\jmath}\right) \; .$$

Here K_0 is a positive constant.

(a) **[4 pts]** Find the charge density in the region.

(b) **[7 pts]** Find the current density in the region.

(c) **[3 pts]** Show that the continuity equation is satisfied.

6. **[9 pts]** An infinite wire carrying current I_1 runs along the z axis; the current flows from $z = -\infty$ to $z = +\infty$ through the origin.

Another infinite wire runs parallel to the y axis, lies in the x-y plane, and goes through the point $(-L, 0, 0)$. Current $2I_1$ flows through this wire, from $y = -\infty$ to $y = +\infty$.

Find the magnetic field created by each wire at the point $(0, 2L, 0)$ on the y-axis. Find the magnitude of the total magnetic field at this point.