1. Consider a magnetostatic situation: currents and fields are time-independent.

   (a) The vector potential in a region is given by

$$\mathbf{A}_1 = \left(-\lambda\frac{z}{2}\right)\hat{\jmath} + \left(\lambda\frac{y}{2}\right)\hat{k}$$

   Find the magnetic field $\mathbf{B}$. Use Ampere's law in differential form to find the current density.

   <u>**(Partial) Solution/Hint →**</u>

$$\mathbf{B} \;=\; \nabla\times\mathbf{A}_1 \;=\; \left(\frac{\partial A_{1z}}{\partial y} - \frac{\partial A_{1y}}{\partial z}\right)\hat{\imath} \;=\; \left(\frac{\lambda}{2} - \left(-\frac{\lambda}{2}\right)\right)\hat{\imath} \;=\; \lambda\hat{\imath}$$

   Using Ampere's law in differential form ($\nabla\times\mathbf{B} = \mu_0\mathbf{J}$),

$$\mathbf{J} \;=\; \frac{1}{\mu_0}\nabla\times\mathbf{B} \;=\; 0$$

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   (b) Now consider the vector potentials

$$\mathbf{A}_2 = (4\lambda)\,\hat{\imath} + (\lambda y)\,\hat{k}\ , \qquad \mathbf{A}_3 = \left(\lambda\frac{y-z}{2}\right)\hat{\jmath} + \left(\lambda\frac{y+z}{2}\right)\hat{k}$$

   Show that both these vector potentials lead to exactly the same magnetic field as the vector potential $\mathbf{A}_1$.

   <u>**(Partial) Solution/Hint →**</u>

   Both curls give $\mathbf{B} = \lambda\hat{\imath}$.

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(c) If two choices of the vector potential correspond to the same magnetic field, their difference must have zero curl. Write down the vector function $\mathbf{A}_2 - \mathbf{A}_1$ and show that it indeed has zero curl.

(Partial) Solution/Hint →

$$\mathbf{A}_2 - \mathbf{A}_1 \;=\; (4\lambda)\,\hat{\imath} + \left(\lambda\frac{z}{2}\right)\hat{\jmath} + \left(\lambda\frac{y}{2}\right)\hat{k}$$

The curl is

$$\nabla \times (\mathbf{A}_2 - \mathbf{A}_1) \;=\; \left(\frac{\partial A_{1z}}{\partial y} - \frac{\partial A_{1y}}{\partial z}\right)\hat{\imath} \;=\; \left(\frac{\lambda}{2} - \frac{\lambda}{2}\right)\hat{\imath} \;=\; 0$$

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(d) A vector with zero curl can be written as a gradient of some scalar function; thus $\mathbf{A}_2 - \mathbf{A}_1 = \nabla f$. By examining the form of $\mathbf{A}_2 - \mathbf{A}_1$, guess a scalar function f that does this job. Show that the gradient of your function indeed gives $\mathbf{A}_2 - \mathbf{A}_1$.

(Partial) Solution/Hint →

The following function would do the job:

$$f \;=\; 4\lambda x \;+\; \frac{\lambda}{2}yz \;+\; \text{constant}$$

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2. An infinite wire carrying current I runs along the y axis; the current flows from $y = -\infty$ to $y = +\infty$ through the origin.

   A square loop of wire lies on the xy plane, with the four corners having coordinates (x_0, y_0), $(x_0 + L, y_0)$, $(x_0 + L, y_0 + L)$, and $(x_0, y_0 + L)$.

   A sketch showing a top view of the xy plane might help.

   (a) Find the magnetic flux through the square loop. The magnetic field is created by the current through the long wire.

   (Partial) Solution/Hint →

   This won't make much sense unless you sketch the situation. (For such a problem in an exam, you really would be expected to include a sketch of the positions of the wire and the loop.)
   As long as we are considering points on the xy plane, the magnetic field due to the wire has magnitude $\frac{\mu_0 I}{2\pi x}$, because x is perpendicular to the wire. The field is perpendicular to the xy plane, hence perpendicular to the square loop surface; we thus don't have to worry about angles. The flux is

$$\Phi = \int B dS = \int_{y_0}^{y_0+L} dy \int_{x_0}^{x_0+L} dx \frac{\mu_0 I}{2\pi x}$$
$$= \frac{\mu_0 I}{2\pi} L \int_{x_0}^{x_0+L} dx \frac{1}{x}$$
$$= \frac{\mu_0 I L}{2\pi} \ln\left(\frac{x_0 + L}{x_0}\right)$$

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(b) Imagine that the square loop moves away from the long wire with speed v, so that $x_0(t) = vt$ but y_0 and L are constant. Find the EMF generated in the square loop.

(Partial) Solution/Hint →

Now the flux is time-dependent, because x_0 is time-dependent.

$$\Phi(t) \; = \; \frac{\mu_0 IL}{2\pi} \ln\left(1 + \frac{L}{x_0(t)}\right)$$

$$\mathcal{E} \; = \; \Phi'(t) \; = \; \frac{\mu_0 IL}{2\pi}\frac{1}{1+L/x_0}\left(-\frac{L}{x_0^2}\right)x_0'(t) \; = \; -\frac{\mu_0 IL^2}{2\pi}\frac{x_0'}{x_0(x_0+L)}$$

$$\Longrightarrow \qquad |\mathcal{E}| \; = \; \frac{\mu_0 IL^2}{2\pi}\frac{v}{vt(vt+L)}$$

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(c) Imagine instead that the square loop moves in the y-direction (parallel to the long wire) with speed v, so that $y_0(t) = vt$ but x_0 and L are constant. Explain why there is no EMF generated in this situation.

(Partial) Solution/Hint →

Since the flux does not depend on y_0, changing y_0 will not change the flux, thus no emf will be generated.

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3. An electromagnetic system is described by the time-dependent fields

$$\mathbf{E} = -Cy\cos(\omega t)\hat{k}\ ; \quad \mathbf{B} = B_0 \sin(\omega t)\hat{\imath}\ . \qquad \begin{cases} \text{Here } B_0 \text{ and } \omega \text{ are} \\ \text{positive constants.} \end{cases}$$

(a) **[5 pts]** Using Maxwell's third equation (which concerns the curl of the electric field), express the constant C as a function of B_0 and ω.

(Partial) Solution/Hint →

Noting

$$\begin{aligned} \nabla \times \mathbf{E} \ &= \ \hat{\imath}\left(\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z}\right) + \hat{\jmath}\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right) + \hat{k}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right) \\ &= \ \hat{\imath}\frac{\partial E_z}{\partial y} \ = \ -\, C\cos(\omega t)\hat{\imath} \end{aligned}$$

and

$$-\frac{\partial \mathbf{B}}{\partial t} \ = \ -\, B_0\omega\cos(\omega t)\hat{\imath}$$

Maxwell's third equation gives us $C = B_0\omega$.

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(b) Find the current density $\mathbf{J}$. Your answer should contain B_0 and ω, not C. Which of Maxwell's equations are you using?

(Partial) Solution/Hint →

Using Maxwell's fourth equation,

$$\begin{aligned} \mathbf{J} \ = \ \frac{1}{\mu_0}\nabla \times \mathbf{B} \ - \ \epsilon_0\frac{\partial \mathbf{E}}{\partial t} \ &= \ 0 \ - \ \epsilon_0\left(+Cy\omega\sin(\omega t)\hat{k}\right) \\ &= \ -\, \epsilon_0 B_0\omega^2 y\sin(\omega t)\hat{k} \end{aligned}$$

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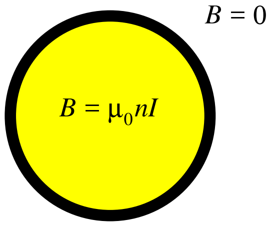
4. A long solenoid has n turns per unit length and radius R. The current through the solenoid increases with time: $I(t) = \alpha t$. We will use Maxwell's 3rd equation (Faraday's law) in integral form, to calculate the magnitude of the electric field created by electromagnetic induction. The calculation is silmilar to using Ampere's law to calculate the magnetic field due to a thick wire.

   We calculated in class the magnetic field inside and outside a solenoid, for steady currents. In the present case the current is time-dependent: Assume that the magnetic field at any instant is given by the steady-state expression, using the instantaneous value of the current.

   (a) Calculate the magnitude of the electric field induced at a distance $r > R$ from the axis of the solenoid (outside the solenoid).

   (Partial) Solution/Hint →



The magnetic field is ZERO outside the solenoid. Inside the solenoid, the field is

$$\mu_0 n I(t) = \mu_0 n \alpha t$$

so that the flux enclosed in the solenoid is

$$\Phi_B = \pi R^2 \mu_0 n \alpha t$$

Consider a circular loop of radius $r > R$; centered on the solenoid axis. By symmetry the electric field is the same everywhere on this loop. The line integral of the electric field is then

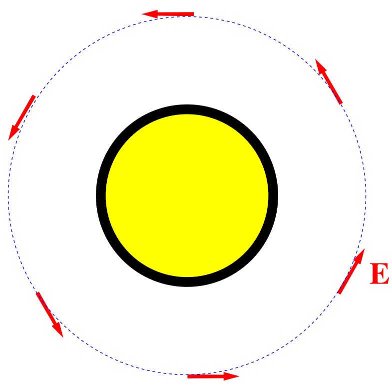
$$\oint \mathbf{E} \cdot d\mathbf{l} = E(2\pi r)$$

Faraday's law in integral form then gives

$$E(2\pi r) = -\frac{d}{dt}\Phi_B = -\pi R^2 \mu_0 n \alpha$$

Thus

$$E = \frac{R^2 \mu_0 n \alpha}{2r}$$



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(b) Calculate the magnitude of the electric field induced at a distance $r < R$ from the axis (inside the solenoid).

(Partial) Solution/Hint →

Now consider a circular loop of radius $r < R$ centered on the solenoid axis (inside the solenoid). Now the flux enclosed is

$$\Phi_B = \pi r^2 \mu_0 n \alpha t$$

while the line integral is still $E(2\pi r)$. Therefore Faraday's law in integral form gives

$$E(2\pi r) = -\frac{d}{dt}\Phi_B = -\pi\mu_0 n\alpha r^2$$

Thus

$$E = \left(\frac{\mu_0 n\alpha}{2}\right) r$$

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5. A circular conducting loop with radius R, centered at the origin, rotates around the y-axis, so that the angle $\theta(t)$ between the normal to the loop and the z-axis varies as $\theta(t) = \Omega t$. A constant magnetic field points in the z-direction: $\mathbf{B} = B_0\hat{k}$. Find the EMF induced in the loop.

(Partial) Solution/Hint →

The magnetic flux through the loop is time-dependent, because the angle between the magnetic field and the normal to the loop is changing with time. The flux is

$$\Phi \;=\; B(\pi R^2)\cos\theta \;=\; \pi B R^2 \cos(\Omega t)$$

$$\mathcal{E} \;=\; -\frac{d\Phi}{dt} \;=\; \pi B R^2 \Omega \sin(\Omega t)$$

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