Due on Monday, April 19th.

If sketches are needed/relevant, please provide them with your solutions.

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1. Consider a magnetostatic situation: currents and fields are time-independent.

   (a) [**4 pts**] The vector potential in a region is given by

   $$\mathbf{A}_1 = \left(-\lambda\frac{z}{2}\right)\hat{\jmath} + \left(\lambda\frac{y}{2}\right)\hat{k}$$

   Find the magnetic field $\mathbf{B}$. Use Ampere's law in differential form to find the current density.

   (b) [**2 pts**] Now consider the vector potentials

   $$\mathbf{A}_2 = (4\lambda)\,\hat{\imath} + (\lambda y)\,\hat{k}\ , \qquad \mathbf{A}_3 = \left(\lambda\frac{y-z}{2}\right)\hat{\jmath} + \left(\lambda\frac{y+z}{2}\right)\hat{k}$$

   Show that both these vector potentials lead to exactly the same magnetic field as the vector potential $\mathbf{A}_1$.

   (c) [**2 pts**] If two choices of the vector potential correspond to the same magnetic field, their difference must have zero curl. Write down the vector function $\mathbf{A}_2 - \mathbf{A}_1$ and show that it indeed has zero curl.

   (d) [**2 pts**] A vector with zero curl can be written as a gradient of some scalar function; thus $\mathbf{A}_2 - \mathbf{A}_1 = \nabla f$. By examining the form of $\mathbf{A}_2 - \mathbf{A}_1$, guess a scalar function f that does this job. Show that the gradient of your function indeed gives $\mathbf{A}_2 - \mathbf{A}_1$.

   (e) [**SELF**] Does $\mathbf{A}_1$ satisfy the Coulomb condition (or Coulomb gauge)? Does $\mathbf{A}_3$ satisfy the Coulomb condition?

2. An infinite wire carrying current I runs along the y axis; the current flows from $y = -\infty$ to $y = +\infty$ through the origin.

   A square loop of wire lies on the xy plane, with the four corners having coordinates (x_0, y_0), $(x_0 + L, y_0)$, $(x_0 + L, y_0 + L)$, and $(x_0, y_0 + L)$.

   A sketch showing a top view of the xy plane might help.

   (a) [**7 pts**] Find the magnetic flux through the square loop. The magnetic field is created by the current through the long wire.

   (b) [**5 pts**] Imagine that the square loop moves away from the long wire with speed v, so that $x_0(t) = vt$ but y_0 and L are constant. Find the EMF generated in the square loop.

(c) [**3 pts**] Imagine instead that the square loop moves in the y-direction (parallel to the long wire) with speed v, so that $y_0(t) = vt$ but x_0 and L are constant. Explain why there is no EMF generated in this situation.

3. An electromagnetic system is described by the time-dependent fields

$$\mathbf{E} = -Cy\cos(\omega t)\hat{k}\ ; \quad \mathbf{B} = B_0\sin(\omega t)\hat{\imath}\ . \qquad \left\{\begin{array}{l}\text{Here } B_0 \text{ and } \omega \text{ are} \\ \text{positive constants.}\end{array}\right.$$

(a) [**5 pts**] Using Maxwell's third equation (which concerns the curl of the electric field), express the constant C as a function of B_0 and ω.

(b) [**5 pts**] Find the current density $\mathbf{J}$. Your answer should contain B_0 and ω, not C. Which of Maxwell's equations are you using?

4. A long solenoid has n turns per unit length and radius R. The current through the solenoid increases with time: $I(t) = \alpha t$. We will use Maxwell's 3rd equation (Faraday's law) in integral form, to calculate the magnitude of the electric field created by electromagnetic induction. The calculation is silmilar to using Ampere's law to calculate the magnetic field due to a thick wire.

We calculated in class the magnetic field inside and outside a solenoid, for steady currents. In the present case the current is time-dependent: Assume that the magnetic field at any instant is given by the steady-state expression, using the instantaneous value of the current.

(a) [**5 pts**] Calculate the magnitude of the electric field induced at a distance $r > R$ from the axis of the solenoid (outside the solenoid).

(b) [**6 pts**] Calculate the magnitude of the electric field induced at a distance $r < R$ from the axis (inside the solenoid).

(c) [**SELF**] You should have clearly sketched the situations; otherwise it would be surprising if you ended up with correct answers for the magnitude. For the geometry/directions chosen your sketch, work out the **direction** of the electric field induced at a distance r from the axis. Use Lenz's law. (If not yet covered in class: look it up!)

5. [**4 pts**] A circular conducting loop with radius R, centered at the origin, rotates around the y-axis, so that the angle $\theta(t)$ between the normal to the loop and the z-axis varies as $\theta(t) = \Omega t$. A constant magnetic field points in the z-direction: $\mathbf{B} = B_0\hat{k}$. Find the EMF induced in the loop.