1. Consider a capacitor consisting of two parallel metallic plates, each of area A. We showed in an earlier problem set that, when the plates carry opposite charges $+Q$ and $-Q$, the electric field in the region between the plates has magnitude $E = Q/(A\epsilon_0)$.

   Consider now the situation that the capacitor is *discharging*, i.e., the magnitude of the charge on each plate is decreasing, as $Q(t) = Q_0 e^{-t/T_0}$. Find the displacement current density in the region between the two plates.

   (Partial) Solution/Hint →

   The electric field magnitude is

   $$E(t) \;=\; \frac{Q(t)}{A\epsilon_0} \;=\; \frac{Q_0}{A\epsilon_0}e^{-t/T_0}$$

   Introducing the unit vector $\hat{n}$ pointing in the direction of the electric field, we can write

   $$\mathbf{E}(t) \;=\; \frac{Q_0}{A\epsilon_0}e^{-t/T_0}\;\hat{n}$$

   $$\Longrightarrow \quad \mathbf{J}_D \;=\; \epsilon_0\frac{\partial \mathbf{E}}{\partial t} \;=\; \frac{Q_0}{A}\left(-\frac{1}{T_0}e^{-t/T_0}\right)\hat{n} \;=\; -\frac{Q_0}{AT_0}e^{-t/T_0}\;\hat{n}$$

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2. An electromagnetic field is described by the scalar potential V and vector potential $\mathbf{A}$, given by

$$V = \frac{2B_0}{\mu_0\epsilon_0}t\,; \quad \mathbf{A} = B_0\left[x\hat{\imath} + (y\sin\omega t - 3z)\hat{k}\right]$$

where B_0 and ω are positive constants.

(a) Find the electric and magnetic fields in this system.

(Partial) Solution/Hint →

$$\mathbf{E} \;=\; -\nabla V - \frac{\partial\mathbf{A}}{\partial t} \;=\; -0 - B_0\left(\omega y\cos\omega t\right)\hat{k} \;=\; -B_0\omega y\cos\omega t\,\hat{k}$$

$$\mathbf{B} \;=\; \nabla\times\mathbf{A} \;=\; \left(\partial_y A_z - \partial_z A_y\right)\hat{\imath} \;+\; 0\hat{\jmath} \;+\; 0\hat{k} \;=\; B_0\sin\omega t\,\hat{\imath}$$

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(b) Show explicitly that Maxwell's third equation is satisfied.

(Partial) Solution/Hint →

$$\begin{aligned}\nabla\times\mathbf{E} \;&=\; \left(\partial_y E_z - \partial_z E_y\right)\hat{\imath} \;+\; 0\hat{\jmath} \;+\; 0\hat{k} \;=\; -B_0\omega\cos\omega t\,\hat{\imath} \\ &=\; -\frac{\partial}{\partial t}\left(B_0\sin\omega t\,\hat{\imath}\right) \;=\; -\frac{\partial\mathbf{B}}{\partial t}\end{aligned}$$

Thus Maxwell's third equation is satisfied.

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(c) If **E** and **B** are derived from potentials, then the 3rd equation is expected to be satisfied automatically. Why? Which of Maxwell's other equations is satisfied automatically?

(Partial) Solution/Hint →

The fields are expressed in terms of the potentials

$$\mathbf{E} \;=\; -\nabla V - \frac{\partial \mathbf{A}}{\partial t}, \qquad \mathbf{B} \;=\; \nabla \times \mathbf{A}$$

These imply Maxwell's 3rd equation (Faraday's law):

$$\begin{aligned}\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \;&=\; \nabla \times \left(-\nabla V - \frac{\partial \mathbf{A}}{\partial t}\right) + \frac{\partial}{\partial t}\left(\nabla \times \mathbf{A}\right) \\ &=\; -\nabla \times (\nabla V) - \nabla \times \left(\frac{\partial \mathbf{A}}{\partial t}\right) + \nabla \times \left(\frac{\partial \mathbf{A}}{\partial t}\right)\end{aligned}$$

In the last term, the order of the two operations (curl and time-derivative) have been switched. Thus

$$\nabla \times \mathbf{E} + \frac{\partial \mathbf{B}}{\partial t} \;=\; -\nabla \times (\nabla V) \;=\; 0$$

because the curl of a gradient is always zero.

Expressing the fields in terms of the potentials also leads to the 2nd equation (zero divergence of **B**):

$$\nabla \cdot \mathbf{B} \;=\; \nabla \cdot (\nabla \times \mathbf{A}) \;=\; 0$$

because the divergence of a curl is always zero.

Thus, if **E** and **B** are derived from potentials, then Maxwell's 3rd and 2nd equations are satisfied automatically.

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(d) Find the displacement current density $\mathbf{J}_D$ and the current density $\mathbf{J}$.

(Partial) Solution/Hint →

Displacement current density:

$$\mathbf{J}_D \;=\; \epsilon_0 \frac{\partial \mathbf{E}}{\partial t} \;=\; \epsilon_0 B_0 \omega^2 y \sin \omega t \, \hat{k}$$

For the current density, we use Maxwell's fourth equation:

$$\mathbf{J} \;=\; \frac{1}{\mu_0} \nabla \times \mathbf{B} \;-\; \mathbf{J}_D \;=\; 0 \;-\; \epsilon_0 B_0 \omega^2 y \sin \omega t \, \hat{k} \;=\; -\; \epsilon_0 B_0 \omega^2 y \sin \omega t \, \hat{k}$$

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(e) Show whether or not the continuity equation is satisfied.

(Partial) Solution/Hint →

The continuity equation is $\nabla \cdot \mathbf{J} = -\dfrac{\partial \rho}{\partial t}$.
The current density has only a z-component, which is not z-dependent. Hence it has zero divergence:

$$\nabla \cdot \mathbf{J} \;=\; \frac{\partial J_z}{\partial z} \;=\; 0$$

To find the charge density ρ, we use Maxwell's first equation:

$$\rho \;=\; \epsilon_0 \nabla \cdot \mathbf{E} \;=\; 0$$

The divergence of the electric field is zero for the same reason: it has only a z-component, which is not z-dependent. This leads to

$$-\frac{\partial \rho}{\partial t} \;=\; 0 \;=\; \nabla \cdot \mathbf{J}$$

i.e., the continuity equation is satisfied.

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3. (a) Maxwell's first equation in differential form is an expression for the divergence of the electric field. Rewrite this equation in terms of the scalar and vector potentials, V and $\mathbf{A}$.

(Partial) Solution/Hint →

Using $\mathbf{E} \;=\; -\nabla V - \frac{\partial \mathbf{A}}{\partial t}$, we get

$$\nabla \cdot \mathbf{E} \;=\; -\nabla \cdot (\nabla V) \;-\; \nabla \cdot \left(\frac{\partial \mathbf{A}}{\partial t}\right) \;=\; -\nabla^2 V \;-\; \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A})$$

Thus Maxwell's first equation becomes

$$\nabla^2 V \;+\; \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) \;=\; -\;\frac{\rho}{\epsilon_0}$$

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(b) Write down the Coulomb gauge condition for the vector potential. Use this condition to express Maxwell's first equation as an equation for the scalar potential V.

(Partial) Solution/Hint →

Coulomb gauge condition: $\nabla \cdot \mathbf{A} = 0$.
The first equation becomes

$$\nabla^2 V \;=\; -\;\frac{\rho}{\epsilon_0}$$

which is known as Poisson's equation.

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4. A square loop with sides of length L lies in the x-y plane in a region in which the magnetic field points in the z-direction and changes over time as

$$\mathbf{B}(t) = B_0 e^{-5t/t_0} \hat{k}$$

   (a) Find the magnitude of the EMF induced in the wire, and sketch a plot of the EMF as a function of time.

   (Partial) Solution/Hint →

   Magnetic flux: $\Phi_B = L^2 B_0 e^{-5t/t_0}$
   EMF:

$$|\mathcal{E}| = \left| \frac{d\Phi_B}{dt} \right| = \frac{5L^2 B_0}{t_0} e^{-5t/t_0}$$

   Plot: should show an exponentially decaying function of time.

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   (b) You are looking from 'above' onto the x-y plane, so that the positive z-axis points out of the paper toward you. What is the direction of the induced current in the loop? (Counterclockwise or clockwise?) You will only get credits for your answer if you justify it clearly using Lenz's law.

   (Partial) Solution/Hint →

   The induced current will oppose the change causing the induction. The change is due to the decrease of $\mathbf{B}$ in positive z-direction. Therefore, the induced current will produce a field in the positive z direction, to bolster up the field which is decreasing. Thus the current needs to be counterclockwise when viewed from above, as this creates a magnetic field in the positive z direction.

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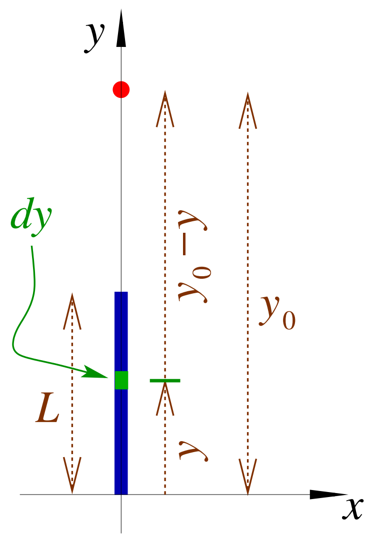
5. A thin glass rod of length L lies along the y axis with one end at the origin and the other end at $(0, L, 0)$. The rod carries a uniformly distributed postive charge Q.

   (a) Consider the point $(0, y_0, 0)$, also on the y axis, with $y_0 > L$. Calculate the electric field generated at this point. You will have to first consider an infinitesimal element of the rod, and then integrate over appropriate limits.

   (Partial) Solution/Hint →

Consider an infinitesimal slice of the rod at distance y from the origin, of width dy. The amount of charge in this small piece is $(Q/L)dy$, and is it at distance $y_0 - y$ from the point $(0, y_0, 0)$. The electric field due to this piece of charge is

$$
\begin{aligned}
d\mathbf{E} &= \frac{(Q/L)dy}{4\pi\epsilon_0} \frac{\hat{j}}{(y_0 - y)^2} \\
&= \frac{(Q/L)\hat{j}}{4\pi\epsilon_0} \frac{dy}{(y - y_0)^2}
\end{aligned}
$$



Integrating from $y = 0$ to $y = L$ gives the total electric field created at the point $(0, y_0, 0)$:

$$
\begin{aligned}
\mathbf{E} &= \frac{(Q/L)\hat{j}}{4\pi\epsilon_0} \int_0^L \frac{dy}{(y - y_0)^2} = \frac{(Q/L)\hat{j}}{4\pi\epsilon_0} \left[\frac{-1}{(y - y_0)} \right]_0^L \\
&= \frac{(Q/L)\hat{j}}{4\pi\epsilon_0} \left(\frac{1}{y_0 - L} - \frac{1}{y_0} \right) = \frac{Q}{4\pi\epsilon_0} \frac{1}{y_0(y_0 - L)} \hat{j}
\end{aligned}
$$

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   (b) Consider the limit $y_0 \gg L$ and approximate your result for this limit. Explain why you could have expected this result for the electric field far from the rod, by comparing with the field due to a point charge.

(Partial) Solution/Hint →

At large distances, $y_0 \gg L$, this simplifies to

$$\mathbf{E} \approx \frac{Q}{4\pi\epsilon_0}\frac{1}{y_0^2}\,\hat{j}$$

This is the electric field due to a point charge Q at distance y_0. This makes sense because, at large enough distance, the structure of the charge-carrying stick will not be important and it will look like a small ($\approx$point) object.

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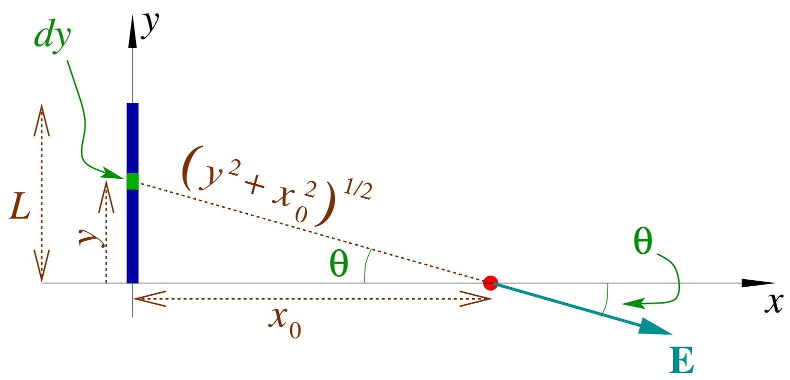
(c) Consider the point $(x_0, 0, 0)$ on the x axis, with $x_0 > L$. Calculate the electric field generated at this point.

In this case you will have to consider both components of the field created by the infinitesimal element, and integrate separately. You might need the integrals

$$\int \frac{du}{(u^2+a^2)^{3/2}} = \frac{u}{a^2\sqrt{u^2+a^2}}\,; \qquad \int \frac{udu}{(u^2+a^2)^{3/2}} = -\frac{1}{\sqrt{u^2+a^2}}\,.$$

(Partial) Solution/Hint →

Consider the same infinitesimal point as before. From the figure: the element is at distance $\sqrt{y^2+x_0^2}$ from the point $(x_0, 0, 0)$ where we need to calculate the field. The field has both x and y components.



The electric field created by the small element has the magnitude

$$dE = \frac{(Q/L)dy}{4\pi\epsilon_0}\frac{1}{y^2+x_0^2}$$

The x- and y- components of this field element are

$$\begin{aligned} dE_x &= dE\cos\theta = \frac{(Q/L)dy}{4\pi\epsilon_0}\frac{1}{y^2+x_0^2}\frac{x_0}{\sqrt{y^2+x_0^2}} = \frac{(Q/L)x_0}{4\pi\epsilon_0}\frac{dy}{(y^2+x_0^2)^{3/2}} \\ dE_y &= -dE\sin\theta = -\frac{(Q/L)dy}{4\pi\epsilon_0}\frac{1}{y^2+x_0^2}\frac{y}{\sqrt{y^2+x_0^2}} = -\frac{(Q/L)}{4\pi\epsilon_0}\frac{ydy}{(y^2+x_0^2)^{3/2}} \end{aligned}$$

The components of the total electric field are found by integrating the two expressions above:

$$\begin{aligned} E_x &= \int dE_x = \frac{(Q/L)x_0}{4\pi\epsilon_0}\int_0^L \frac{dy}{(y^2+x_0^2)^{3/2}} \\ &= \frac{(Q/L)x_0}{4\pi\epsilon_0}\left[\frac{y}{x_0^2\sqrt{y^2+x_0^2}}\right]_0^L \\ &= \frac{Q/(Lx_0)}{4\pi\epsilon_0}\left(\frac{L}{\sqrt{L^2+x_0^2}} - \frac{0}{\sqrt{0^2+x_0^2}}\right) \\ &= \frac{1}{4\pi\epsilon_0}\left(\frac{Q}{x_0\sqrt{L^2+x_0^2}}\right) \end{aligned}$$

and

$$\begin{aligned} E_y &= \int dE_y = -\frac{(Q/L)}{4\pi\epsilon_0}\int_0^L \frac{ydy}{(y^2+x_0^2)^{3/2}} \\ &= -\frac{(Q/L)}{4\pi\epsilon_0}\left[-\frac{1}{\sqrt{y^2+x_0^2}}\right]_0^L \\ &= -\frac{(Q/L)}{4\pi\epsilon_0}\left(\frac{1}{x_0} - \frac{1}{\sqrt{L^2+x_0^2}}\right) \end{aligned}$$

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