Some partial solutions and/or hints are provided here.
Not carefully proofread and possibly incomplete — please use responsibly.
If you spot any typo's or errors, please let me know.

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1. An electromagnetic system in vacuum (no charges or currents are present!) is described by the fields

$$\mathbf{E} = E_0 \exp\left[-\frac{(x-wt)^2}{2L^2}\right]\hat{\jmath}$$
$$\mathbf{B} = (E_0/w) \exp\left[-\frac{(x-wt)^2}{2L^2}\right]\hat{k}$$

L and w are positive constants.

$\exp[u]$ is common notation for the exponential function e^u.

   (a) Show that these fields obey Maxwell's third equation (which concerns the curl of the electric field).

   (Partial) Solution/Hint →

$$\nabla \times \mathbf{E} = \frac{\partial E_y}{\partial x}\hat{k} = -\frac{2E_0}{2L^2}(x-wt)\exp\left[-\frac{(x-wt)^2}{2L^2}\right]\hat{k}$$

$$\frac{\partial \mathbf{B}}{\partial t} = \frac{E_0}{w}\left(-\frac{w}{2L^2}\right)(-2(x-wt))\exp\left[-\frac{(x-wt)^2}{2L^2}\right]\hat{k}$$
$$= \frac{E_0}{L^2}(x-wt)\exp\left[-\frac{(x-wt)^2}{2L^2}\right]\hat{k} = -\nabla\times\mathbf{E}$$

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   (b) The fields should also obey Maxwell's fourth equation in vacuum. Find the value of w for which this works.

   (Partial) Solution/Hint →

Fourth law in vacuum: $$\nabla \times \mathbf{B} = \frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}$$

Comparing

$$\nabla \times \mathbf{B} = \frac{E_0}{\omega L^2}(x - wt)\exp\left[-\frac{(x-wt)^2}{2L^2}\right]\hat{\jmath}$$

and

$$\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t} = \frac{1}{c^2}\frac{E_0 w}{L^2}(x - wt)\exp\left[-\frac{(x-wt)^2}{2L^2}\right]\hat{\jmath}$$

it is clear that the fields satisfy the fourth equation only if

$$\frac{1}{\omega} = \frac{w}{c^2} \qquad \Longrightarrow \qquad w = \pm c$$

Since w is given to be positive, $w = c$. Thus traveling solutions of the Maxwell equations are forced to have speed c.

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(c) Look up the properties of the gaussian function (e.g., wikipedia). You should be able to tell the center and the width of a gaussian by looking at its form.

(Partial) Solution/Hint →

The function $\exp\left[-\frac{(x-x_0)^2}{2\sigma^2}\right]$ represents a gaussian centered at x_0 and with width σ.

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(d) Sketch plots of the magnitude of the electric field as a function of the position, at the time instants $t = 0$, $t = 2L/w$, and $t = 4L/w$. (You can think of these as 'snapshots' at different points of time.)

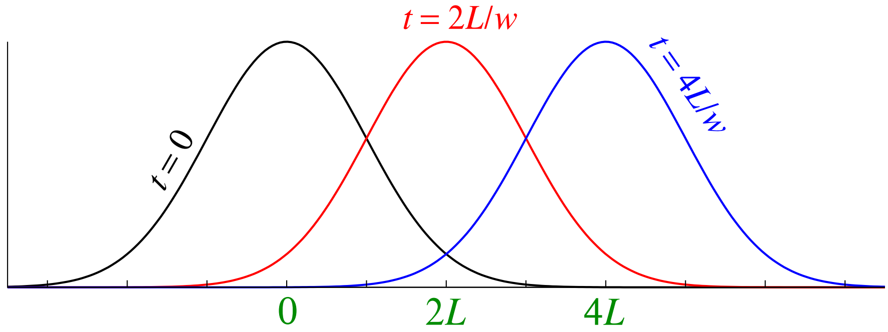
You can use a plotting program for guidance if needed, but please submit a hand-drawn sketch, not a computer printout.

In which direction is our electromagnetic pulse traveling?

(Partial) Solution/Hint →

The plots should show the functions

$$\exp\left[-\frac{x^2}{2L^2}\right], \qquad \exp\left[-\frac{(x-2L)^2}{2L^2}\right], \qquad \exp\left[-\frac{(x-4L)^2}{2L^2}\right].$$



Observing the positions of the three gaussians, we notice that the structure moves in the positive x direction with increasing time. The electromagnetic pulse is thus traveling in the direction of increasing positive x. Of course, one should have been able to guess this from the fact that the fields are functions of $x - ct$.

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(e) Calculate the Poynting vector **S**. In case it has not been treated in class yet: $\mathbf{S} = \dfrac{1}{\mu_0}\mathbf{E} \times \mathbf{B}$.

<u>**(Partial) Solution/Hint →**</u>

$$\mathbf{S} \;=\; \frac{1}{\mu_0}\mathbf{E}{\times}\mathbf{B} \;=\; \frac{1}{\mu_0}\frac{E_0^2}{w}\exp\left[-\frac{(x-wt)^2}{L^2}\right]\hat{\imath} \;=\; \frac{E_0^2}{c\mu_0}\exp\left[-\frac{(x-ct)^2}{L^2}\right]\hat{\imath}$$

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(f) The Poynting vector gives the 'current density' for the flow of electromagnetic energy. Explain whether the calculated direction of **S** makes sense.

<u>**(Partial) Solution/Hint →**</u>

The pulse is traveling in the positive x direction, so the energy it is carrying must also be traveling in this direction; hence it makes sense that **S** points in the $\hat{\imath}$ direction.

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(g) Calculate the energy density for electromagnetic fields,

$$u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$$

for the given fields.

(Partial) Solution/Hint →

$$\begin{aligned} u &= \frac{\epsilon_0 E_0^2}{2} \exp\left[-\frac{(x-wt)^2}{L^2}\right] + \frac{E_0^2}{2w^2\mu_0} \exp\left[-\frac{(x-wt)^2}{L^2}\right] \\ &= \frac{E_0^2}{2}\left(\epsilon_0 + \frac{1}{w^2\mu_0}\right) \exp\left[-\frac{(x-wt)^2}{L^2}\right] \end{aligned}$$

Using the fact that $w^2 = c^2 = 1/(\epsilon_0\mu_0)$, we get

$$\frac{1}{w^2\mu_0} = \epsilon_0 \ ,$$

hence

$$u = \frac{E_0^2}{2}(2\epsilon_0) \exp\left[-\frac{(x-ct)^2}{L^2}\right] = \epsilon_0 E_0^2 \exp\left[-\frac{(x-ct)^2}{L^2}\right]$$

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(h) Conservation of electromagnetic energy in vacuum is encoded in the relation (analogous to the continuity equation for charge)

$$\nabla \cdot \mathbf{S} + \frac{\partial u}{\partial t} \; = \; 0\,.$$

Find out whether this is satisfied for our fields.

(Partial) Solution/Hint →

$$\begin{aligned}\nabla \cdot \mathbf{S} \; &= \; \frac{\partial S_x}{\partial x} \; = \; \frac{E_0^2}{c\mu_0}\frac{\partial}{\partial x}\exp\left[-\frac{(x-ct)^2}{L^2}\right] \\ &= \; \frac{E_0^2}{c\mu_0}\left(-\frac{1}{L^2}\right)2(x-ct)\exp\left[-\frac{(x-ct)^2}{L^2}\right] \\ &= \; -\frac{2E_0^2}{c\mu_0 L^2}(x-ct)\exp\left[-\frac{(x-ct)^2}{L^2}\right]\end{aligned}$$

$$\begin{aligned}\frac{\partial u}{\partial t} \; &= \; \epsilon_0 E_0^2\frac{\partial}{\partial t}\exp\left[-\frac{(x-ct)^2}{L^2}\right] \\ &= \; \epsilon_0 E_0^2\left(-\frac{(-c)}{L^2}\right)2(x-ct)\exp\left[-\frac{(x-ct)^2}{L^2}\right] \\ &= \; \frac{2\epsilon_0 c E_0^2}{L^2}(x-ct)\exp\left[-\frac{(x-ct)^2}{L^2}\right]\end{aligned}$$

Thus

$$\nabla \cdot \mathbf{S} \; + \; \frac{\partial u}{\partial t} \; = \; \frac{2E_0^2}{c\mu_0 L^2}\left(-1 + c^2\epsilon_0\mu_0\right)(x-ct)\exp\left[-\frac{(x-ct)^2}{L^2}\right]$$

Noting that $-1 + c^2\epsilon_0\mu_0 = -1 + 1 = 0$ gives us the desired result.

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2. Using Maxwell's equations in vacuum and the definitions of $\mathbf{S}$ and u, derive the relation $\nabla \cdot \mathbf{S} = -\partial u/\partial t$. You may need the vector identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) \;=\; \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

(Partial) Solution/Hint →

$$\begin{aligned}
\nabla \cdot \mathbf{S} \;&=\; \frac{1}{\mu_0}\nabla \cdot (\mathbf{E} \times \mathbf{B}) \;=\; \frac{1}{\mu_0}\mathbf{B} \cdot (\nabla \times \mathbf{E}) - \frac{1}{\mu_0}\mathbf{E} \cdot (\nabla \times \mathbf{B}) \\
&=\; \frac{1}{\mu_0}\mathbf{B} \cdot \left(-\frac{\partial \mathbf{B}}{\partial t}\right) - \frac{1}{\mu_0}\mathbf{E} \cdot \left(\frac{1}{c^2}\frac{\partial \mathbf{E}}{\partial t}\right) \\
&=\; -\frac{1}{\mu_0}\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} - \frac{1}{\mu_0 c^2}\mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \\
&=\; -\epsilon_0 \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} - \frac{1}{\mu_0}\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t}
\end{aligned}$$

Note that

$$\frac{\partial}{\partial t}(B^2) \;=\; \frac{\partial}{\partial t}(\mathbf{B} \cdot \mathbf{B}) \;=\; \mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} + \frac{\partial \mathbf{B}}{\partial t} \cdot \mathbf{B} \;=\; 2\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t}$$

so that

$$\mathbf{B} \cdot \frac{\partial \mathbf{B}}{\partial t} \;=\; \frac{1}{2}\frac{\partial}{\partial t}(B^2) \qquad \text{and similarly,} \qquad \mathbf{E} \cdot \frac{\partial \mathbf{E}}{\partial t} \;=\; \frac{1}{2}\frac{\partial}{\partial t}(E^2)\,.$$

Therefore

$$\begin{aligned}
\nabla \cdot \mathbf{S} \;&=\; -\frac{\epsilon_0}{2}\frac{\partial}{\partial t}(E^2) - \frac{1}{2\mu_0}\frac{\partial}{\partial t}(B^2) \\
&=\; -\frac{\partial}{\partial t}\left(\frac{\epsilon_0}{2}E^2 + \frac{1}{2\mu_0}B^2\right) \;=\; -\frac{\partial u}{\partial t}
\end{aligned}$$

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3. Consider an electromagnetic wave travelling through empty space described by the electric and magnetic fields

$$\mathbf{E} \;=\; 3\alpha\cos\left(\frac{1}{L}(y-ct)\right)\hat{\imath}\,, \qquad \mathbf{B} \;=\; \mathbf{G}\cos\left(\frac{1}{L}(y-ct)\right)$$

where α and L are positive constants and G is a constant vector.

(a) In which direction is this wave traveling?

(Partial) Solution/Hint $\rightarrow$

The spatial dependence appears as $y-ct$. This means the wave travels in the positive y direction.

If the fields were functions of $y+ct$, it would be a wave traveling in the negative y direction.

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(b) Find the magnitude (in terms of α) and the direction of the constant vector $\mathbf{G}$. You might need to use one of Maxwell's equations.

(Partial) Solution/Hint →

Using Maxwell's third equation:

$$\frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} = -\left[\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right]\hat{k}$$
$$= -\left[0 - 3\alpha\left(-\frac{1}{L}\right)\sin\left(\frac{1}{L}(y-ct)\right)\right]\hat{k}$$
$$= (-3\alpha/L)\sin\left(\frac{1}{L}(y-ct)\right)\hat{k}$$

Integrating gives

$$\mathbf{B} = (-3\alpha/L)\left(\frac{1}{-c/L}\right)(-1)\cos\left(\frac{1}{L}(y-ct)\right)\hat{k} + \begin{bmatrix}\text{time-independent}\\ \text{constant vector}\end{bmatrix}$$
$$= -\frac{3\alpha}{c}\cos\left(\frac{1}{L}(y-ct)\right)\hat{k} + \begin{bmatrix}\text{time-independent}\\ \text{constant vector}\end{bmatrix}$$

Comparing with the given form for the magnetic field, the constant vector (constant of integration) is seen to be zero. In addition, the constant $\mathbf{G}$ is seen to be

$$\mathbf{G} = -\frac{3\alpha}{c}\hat{k}$$

Note that finding a vector means finding both it's magnitude and direction.

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(c) What is the wavelength and the frequency of this wave?

(Partial) Solution/Hint →

Comparing with the standard form

$$\cos\left(\frac{2\pi}{\lambda}(y-ct)\right)$$

the wavelength λ is seen to be related to L according to

$$\frac{2\pi}{\lambda} = \frac{1}{L} \qquad \Longrightarrow \qquad \lambda = 2\pi L$$

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4. An electromagnetic wave has an electric field given by

$$\mathbf{E} = E_0 \cos\left(\frac{2\pi c}{\lambda}t\right) \sin\left(\frac{2\pi z}{\lambda}\right) \hat{\imath}$$

where λ is a positive constant.

(a) Use Maxwell's third equation (Faraday's law in differential form) to calculate the associated magnetic field $\mathbf{B}$. You can assume the time-independent additive term (constant of integration) to be zero.

(Partial) Solution/Hint →

Using Maxwell's third equation,

$$\begin{aligned}
\frac{\partial \mathbf{B}}{\partial t} &= -\nabla \times \mathbf{E} = -\left(\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x}\right)\hat{\jmath} \\
&= -\left(\frac{2\pi E_0}{\lambda}\cos\left(\frac{2\pi c}{\lambda}t\right)\cos\left(\frac{2\pi z}{\lambda}\right) - 0\right)\hat{\jmath} \\
&= -\frac{2\pi E_0}{\lambda}\cos\left(\frac{2\pi c}{\lambda}t\right)\cos\left(\frac{2\pi z}{\lambda}\right)\hat{\jmath}
\end{aligned}$$

Integrating over time gives

$$\mathbf{B} = -\frac{E_0}{c}\sin\left(\frac{2\pi c}{\lambda}t\right)\cos\left(\frac{2\pi z}{\lambda}\right)\hat{\jmath} + \mathbf{G}(\mathbf{r})$$

Here $\mathbf{G}(\mathbf{r})$ is the 'constant of integration'. Since the integration is over time, it can be an arbitrary function of space. As suggested in the problem set, we set it to zero and obtain

$$\mathbf{B} = -\frac{E_0}{c}\sin\left(\frac{2\pi c}{\lambda}t\right)\cos\left(\frac{2\pi z}{\lambda}\right)\hat{\jmath}$$

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(b) Is this wave traveling? In which direction?

(Partial) Solution/Hint →

This is a standing wave — it does not travel in any direction.

If an equation describes a traveling wave in the $+z$ direction, the space- and time-dependence can be combined so that the expression is a function of $z - ct$. Similarly, for a traveling wave in the $-z$ direction, the space and time dependence appear only in the combination $z + ct$.

Here, the dependence of the fields on space and time do not appear in the combination $z - ct$ or $z + ct$.

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(c) Calculate the Poynting vector and explain the direction of energy flow using your result.

(Partial) Solution/Hint →

Taking the cross product of the two fields

$$\begin{aligned} \mathbf{S} &= \frac{1}{\mu_0}\mathbf{E}\times\mathbf{B} \\ &= \frac{E_0^2}{\mu_0 c}\cos\left(\frac{2\pi c}{\lambda}t\right)\sin\left(\frac{2\pi z}{\lambda}\right)\sin\left(\frac{2\pi c}{\lambda}t\right)\cos\left(\frac{2\pi z}{\lambda}\right)\hat{k} \\ &= \frac{E_0^2}{4\mu_0 c}\sin\left(\frac{4\pi c}{\lambda}t\right)\sin\left(\frac{4\pi z}{\lambda}\right)\hat{k} \end{aligned}$$

Because of the factor $\sin\left(\frac{\pi c}{\lambda}t\right)$, The direction of this vector is $+\hat{k}$ half the time and $-\hat{k}$ the other half of the time. The average energy flow is thus zero.

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