This problem set is not for submission and will not be marked.

Working through these problems might provide useful exercise.

Remember: if pictures are needed or relevant, you should provide them with your solutions, also in an exam. If you are not drawing relevant pictures, you will probably get the geometry wrong.

− − − − − − − − − − − − − − − ⋆ − − − − − − − − − − − − −

1. Two metallic (perfectly conducting) solid spheres each have radius R. One of them carries charge $+Q$ and is centered at $(0, d/2, 0)$. Assume $R < d/2$, so that the origin is outside the sphere. The other sphere is centered at $(0, -d/2, 0)$ and carries charge $-Q$.

   (a) What is the electric field inside either of the spheres?

   (b) What is the electric field (magnitude and direction) at a point $(x_0, 0, 0)$ on the x axis?

   (c) What is the electric field (magnitude and direction) at a point $(0, y_0, 0)$ on the y axis, which is outside both of the spheres? Approximate this expression for points far from the dipole arrangement, i.e., for $y_0 \gg d$.

   (d) What are the magnitude and the direction of the electric dipole moment?

2. An infinite wire carrying current I_1 runs along the y axis; the current flows from $y = -\infty$ to $y = +\infty$ through the origin.

   A square loop of wire lies in the xy plane, with the four corners having coordinates (x_0, y_0), $(x_0 + L, y_0)$, $(x_0 + L, y_0 + L)$, and $(x_0, y_0 + L)$. Current I_2 circulates around this square loop, counterclockwise when viewed from the top (from the direction of positive z).

   Find separately the forces on each of the four sides of the square loop.

   Find the net force acting on the square loop.

3. A magnetic field of $\mathbf{B} = a \sin(by) e^{bx} \hat{k}$ is produced by a steady electric current. What is the density of that current?

4. A charged sphere has radius a_0 and volume charge density $\rho = \rho_0(r/a_0)$, where r is the distance from the center of the sphere.

   (a) Using Gauss' law (Gauss' dielectric flux theorem), find the electric field inside and outside the sphere.

   (b) What is the total electric flux through any closed surface surrounding the sphere?

   (c) What is the total charge on the sphere?

   (d) Plot the magnitude of the electric field as a function of r. The plot should extend from $r = 0$ to $r = 2a_0$.

   (e) Calculate the electric potential as a function of r, both outside the sphere and inside the sphere. You can take $r = \infty$ as the reference, i.e., the potential at infinite distance can be taken to be zero.

   (f) Plot the electric potential as a function of r. The plot could extend from $r = 0$ to $r = 3a_0$.

5. Find the charge and current distributions if the potentials are

$$V = \alpha c^2 z t \,, \qquad \mathbf{A} = \alpha \left(c^2 t^2 - x^2 \right) \hat{k}$$

   where α is a positive constant.

   Hint: First find the fields from the potentials, then use Maxwell's equations to find the charge density and the current density.

6. In a region where the magnetic field changes with time, $\mathbf{B} = B_0 e^{-2t/t_0} \hat{\imath}$, we consider two square loops.

   One square loop lies in the y-z plane, and has sides of length L_1. Find the emf induced in this loop.

   The other square loop lies in the x-y plane, and has sides of length L_2. Find the emf induced in this loop.

7. The distance between two long parallel wires is d. Currents I and $2I$ flow through the two wires, in opposite directions. Find the magnetic field at a point midway between the wires.

8. An infinite straight wire carrying current I runs along the z axis; the current flows from $z = -\infty$ to $z = +\infty$ through the origin.

   Another infinite straight wire lies parallel to the first (parallel to the z axis) and passes through the point $(d, 0, 0)$. Current $2I$ flows through this wire, in the same direction from $z = -\infty$ to $z = +\infty$.

   Find the total magnetic field created at the point $(0, d, 0)$ on the y axis.

   Find the total magnetic field created at the point $(0, d, 7d)$.

9. In a region of free space, the electromagnetic fields are found to be

$$E_x = 0 \qquad E_y = E_0 \sin(kx + \omega t) \qquad E_z = 0$$
$$B_x = 0 \qquad B_y = 0 \qquad B_z = -B_0 \sin(kx + \omega t)$$

   Use Maxwell's equations in free space to find how B_0 is related to E_0, and how ω is related to k.

10. **Simple Electric Dipole.** Two point charges with charge $+q$ and $-q$ lie on the y-axis, respectively at $(0, d/2, 0)$ and $(0, -d/2.0)$.

    (a) Calculate (or write down from problem set 02) the electric field at the point $(x_0, 0, 0)$ on the x-axis.
    Approximate the electric field for $x_0 \gg d$, expressing your result in terms of the electric dipole moment $\mathbf{p}$. You should be able to write this as a vector equation.

    (b) Consider the point $(0, y_0, 0)$ on the y-axis, with $y_0 > d$. Derive an expression for the electric field at this point.
    Approximate this expression for $y_0 \gg d$ expressing your result in terms of the electric dipole moment $\mathbf{p}$.

    (c) An electric dipole with dipole moment $\mathbf{p}$ at the orign is known to generate the following electric field at point $\mathbf{r}$, provided that $r = |\mathbf{r}|$ is large:

$$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{3(\mathbf{p}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{r^3} = \frac{1}{4\pi\epsilon_0}\left(\frac{3(\mathbf{p}\cdot\mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{p}}{r^3}\right)$$

    where $\hat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{r}$.
    The cases we have considered in this problem involve $\mathbf{r} \perp \mathbf{p}$ and $\mathbf{r} \parallel \mathbf{p}$. Simplify the above expression for these two special cases. Compare with previous results.

11. An electron moving at speed 200 m/s finds itself in a region of uniform magnetic field. The magnetic field points perpendicular to the electron velocity and has strength 10^{-4} Tesla. (You can approximate the electron charge to be $\approx 1.6 \times 10^{-19}$ C and the electron mass to be $\approx 9.11 \times 10^{-31}$ kg.)

    (a) Find the magnitude of the magnetic force experienced by the electron.

    (b) The magnetic force causes cyclotron motion. Find the radius of the circular trajectory.

    (c) How long does it take the electron to traverse the circular orbit 50 times?

    (d) Find the cyclotron frequency.

12. The electric field in a region is given by $\mathbf{E} = (2x + y)\hat{\imath} + (x + 3y)\hat{\jmath}$. We will consider points on the x-y plane.

    (a) Find the difference of electric potentials between two points P and Q with coordinates $(5, 1, 0)$ and $(5, 4, 0)$ respectively.

    (b) Point R has coordinates $(0, 1, 0)$. Find the difference of electric potentials between points P and R.

13. The magnetic field in some region is given by

$$\mathbf{B} = \mu_o \epsilon_0 \alpha \left(3y(x^2 + y^2)\hat{\imath} + Cx(x^2 + y^2)\hat{\jmath} \right)$$

    where α is a positive constant.

    (a) Using Maxwell's second equation (which concerns the divergence of the magnetic field), determine the constant C.

    (b) The electric field in this region is changing with time:

$$\mathbf{E} = -5\alpha \left(x^2 + y^2 \right) t\, \hat{k}$$

    Find the current density $\mathbf{J}$ and the charge density ρ in this region.

14. A flat square plate lies parallel to the x-y plane and is charged with a variable surface charge density, $\sigma = -4\gamma xy$. The four corners of the square plate are at the points $(0, 0, z_0)$, $(L, 0, z_0)$, $(0, L, z_0)$, (L, L, z_0). Find the total electric flux through a surface that completely surrounds this charged plate. (Here γ is a constant.)

15. An electromagnetic system is described by the scalar and vector potentials

$$V = 3E_0 L e^{-z^2/L^2} e^{-2\omega t}\,, \qquad \mathbf{A} = \frac{E_0}{\omega}\sin\left(\frac{\omega}{c}x - \omega t\right)\hat{j}$$

where L, E_0 and ω are positive constants.

(a) Is the Coulomb condition (Coulomb gauge) or the Lorentz condition satisfied by these potentials?

(b) Calculate the electric and magnetic fields.

(c) Check explicitly that Maxwell's third equation (involving the curl of the electric field) is satisfied.

(d) Check explicitly that Maxwell's second equation (involving the divergence of the magnetic field) is satisfied.

(e) Calculate the charge density, the current density, and the displacement current density.

(f) Show explicitly that the continuity equation is satisfied.

(g) Calculate the Poynting vector $\mathbf{S}$.

(h) Calculate the energy density of the fields, $u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0}B^2\right)$.

(i) Show that the energy conservation equation

$$\nabla\cdot\mathbf{S} + \frac{\partial u}{\partial t} + \mathbf{E}\cdot\mathbf{J} = 0$$

is satisfied.

16. A long solenoid has n turns per unit length and carries time-varying current $I(t) = I_0\sin(\omega t)$. Use Faraday's law in integral form to calculate the induced electric field as a function of the distance r from the axis of the solenoid, both *inside* and *outside* the solenoid.

17. Suppose the magnetic field in some region has the form

$$\mathbf{B} = kz\hat{\imath} \qquad (k \text{ is a positive constant}).$$

Find the force on a square loop of side s, lying in the yz plane, centered at the origin, which carries a current I. Two sides of the loop are parallel to the y axis.

18. An infinite wire carrying current I_1 runs along the z axis; the current flows from $z = -\infty$ to $z = +\infty$ through the origin.

    Another infinite wire runs parallel to the first (parallel to the z axis) and goes through the point $(d, 0, 0)$. Current $2I_1$ flows through this wire, in the same direction from negative z to positive z.

    Find the magnetic field at the points $(0, d, 0)$ and $(d, 2d, 0)$, both in the x-y plane.

    Find the magnetic field at the points $(0, d, z_0)$ and $(d, 2d, 2z_0)$, which are not in the x-y plane. No additional calculations should be necessary.

19. The vector potential in some region is given by

$$\mathbf{A} = \left(-\lambda\frac{z}{2}\right)\hat{\jmath} + \left(\lambda\frac{y}{2}\right)\hat{k}$$

    (a) Find the magnetic field $\mathbf{B}$.

    (b) Use Ampere's law in differential form to find the current density $\mathbf{J}$ in this region. Assume a steady-state situation.

    (c) We add ∇f to the vector potential, where f is any scalar function. Explain how the physical system changes due to this transformation.

    (d) Write down or derive a vector potential, different from the one above, which corresponds to the same magnetic field.

20. The magnetic field in some region is uniform: $\mathbf{B} = B_0\hat{k}$. At time $t = 0$, A charged particle (mass m, charge q) is at the origin and has velocity $\mathbf{v} = \alpha\hat{\imath}$. Find the position of the particle, (x, y, z), as a function of time t.

    Describe the motion of the particle in words. Is this cyclotron motion? If so, calculate the cyclotron radius and cyclotron frequency.

21. A long straight wire carries steady current I. The wire cross-section is circular and has radius R. The current density is uniform inside the wire. Using Ampere's law, calculate the magnetic field at distance r from the axis of the wire. Consider separately the cases $r < R$ (inside the wire) and $r > R$ (outside the wire). Plot the magnitude of the magnetic field as a function of r.

22. Find the total electric flux through a closed cylinder containing a line charge along its axis with linear charge density $\lambda = \lambda_0(1 - x/h)$ if the cylinder and the line charge extend from $x = 0$ to $x = h$.

23. The magnetic and electric fields in some region are uniform and point in the same direction:

$$\mathbf{E} = E_0\hat{k}\ , \quad \mathbf{B} = B_0\hat{k}\ .$$

At time $t = 0$, a charged particle (mass m, charge q) is at rest at the origin. Find the position of the particle, (x, y, z), at time t.

Describe the motion of the particle in words.

24. Two concentric spherical metal shells have radii a and b, with $a < b$, They carry charge $+Q$ and $-Q$ respectively.

(a) Find the electric field in the regions $r < a$, $a < r < b$, and $r > b$. Here r is the radial distance from the center of the spheres.

Sketch a plot of the electric field magnitude as a function of r, for the case $b = 2a$. Your plot should start at $r = 0$ and extend beyond $r = b$.

(b) Find the potential difference between the two shells.

(c) Find the electric potential as a function of r. You can make your own choice of reference potential, but it might be natural to take the potential at infinite distance to be zero.

(d) What is the capacitance of this arrangement? Remember: the capacitance is the charge magnitude on the plates divided by the potential difference between the plates.

25. Using Maxwell's equations, show that

$$\nabla \cdot \mathbf{S} + \frac{\partial u}{\partial t} + \mathbf{E} \cdot \mathbf{J} \ = \ 0$$

where $\mathbf{S}$ is the Poynting vector, u is the energy density of electromagnetic fields, and $\mathbf{J}$ is the current density. You may need the vector identity

$$\nabla \cdot (\mathbf{A} \times \mathbf{B}) \ = \ \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$$

26. The magnetic field in some region is found to be uniform: $\mathbf{B} = 5C\hat{k}$. A cylinder of radius r_0 and height h is placed with the cylinder's axis parallel to the z azis.

    Find the magnetic flux through the top, bottom, and curved surfaces of the cylinder.

    Show that the total magnetic flux through the cylinder is zero.

27. Suppose we have an electromagnetic field not in vacuum, but in a medium with conductivity σ. This means that the charge density is still zero, but the current density is given by Ohm's law $\mathbf{J} = \sigma\mathbf{E}$.

    (a) Using $\mathbf{J} = \sigma\mathbf{E}$ and Maxwell's equations, show that the electric and magnetic fields satisfy the equations

    $$\begin{aligned} \nabla^2\mathbf{E} &= \mu_0\epsilon_0\frac{\partial^2\mathbf{E}}{\partial t^2} + \mu_0\sigma\frac{\partial\mathbf{E}}{\partial t} \\ \nabla^2\mathbf{B} &= \mu_0\epsilon_0\frac{\partial^2\mathbf{B}}{\partial t^2} + \mu_0\sigma\frac{\partial\mathbf{B}}{\partial t} \end{aligned}$$

    Remarks: (1) this is similar to the calculation deriving the wave equation in vacuum. (2) We are ignoring the modification of the constants μ_0 and ϵ_0 that may be necessary in a medium.

    (b) Show that the attenuated wave

    $$\mathbf{E} = E_0\, e^{-z/L}\, \sin\left(\frac{\omega}{c}z - \omega t\right)\hat{\imath}$$

    can be a solution to the modified wave equation. Find the attenuation length L (known as the penetration depth) in terms of ω and σ.

    Using Maxwell's third equation, calculate the magnetic field corresponding to this electric field. You are allowed to drop any time-independent additive term that appears.

    What is the direction of propagation of this wave?

28. The electric and magnetic fields in some region are

    $$\mathbf{E} = -\left(\frac{6K_0t}{\epsilon_0}\right)\hat{k}\,, \qquad \mathbf{B} = \mu_0K_0\left(3y\hat{\imath} - 3x\hat{\jmath}\right)\,.$$

    Find the charge density ρ, the displacement current density $\mathbf{J}_D$ and the current density $\mathbf{J}$.

29. A ring of wire with radius R is centered at the origin and lies on the x-y plane, so that it's axis coincides with the z-axis. Current I flows around this circular wire loop.

    (a) Using the Biot-Savart law, calculate the magnetic field at the center of the loop, i.e., at the origin.

    (b) Using the Biot-Savart law, calculate the magnetic field at the point $(0, 0, z_0)$ on the z-axis (a point on the axis of the loop).

30. Remember from a previous problem set: a single rod of length L, placed between $(0, 0, 0)$ and $(0, L, 0)$ and uniformly carrying total charge Q, creates electric fields

    $$\mathbf{E} = \frac{Q}{4\pi\epsilon_0} \frac{1}{y_0(y_0 - L)} \hat{j} \quad \text{at the point } (0, y_0, 0) \text{ on the } y\text{-axis, with } y_0 > L,$$

    and

    $$\mathbf{E} = \frac{1}{4\pi\epsilon_0} \left(\frac{Q}{x_0\sqrt{L^2 + x_0^2}} \right) \hat{i} - \frac{(Q/L)}{4\pi\epsilon_0} \left(\frac{1}{x_0} - \frac{1}{\sqrt{L^2 + x_0^2}} \right) \hat{j}$$

    at the point $(x_0, 0, 0)$ on the x-axis.

    Now consider two thin rods, each of length L, placed along the y-axis, with one end of each at the origin. The rod on the positive y axis, with other end at $(0, L, 0)$, carries a uniformly distributed postive charge Q. The rod on the negative y axis, with other end at $(0, -L, 0)$, carries a uniformly distributed negative charge $-Q$. This arrangement serves as a reasonably complicated electric dipole.

    (a) Using the results for a single rod, find the net electric field at the point $(0, y_0, 0)$ on the y-axis, with $y_0 > L$.

    For $y_0 \gg L$, approximate this expression to obtain a field of the form

    $$E \approx \frac{1}{4\pi\epsilon_0} \frac{2p}{y_0^3}$$

    Identify p in terms of Q and L. This quantity serves as the electric dipole moment for the two-rod configuration.

    (b) Using the results for a single rod, find the net electric field at the point $(x_0, 0, 0)$ on the x-axis.

    For $y_0 \gg L$, approximate this expression and express it in terms of the electric dipole moment.

31. A wire loop of radius R is centered at the origin and lies in the x-y plane, i.e., its axis coincides with the z-axis. Current I flows through this loop.

(a) We considered in class the magnetic field created on the axis, away from the plane of the ring. Write down (from class or textbook) the expression for the magnetic field at the point $(0, 0, z_0)$.

Approximate this expression for the limit $z_0 \gg R$; show that

$$B \approx \frac{\mu_o}{4\pi}\frac{2m}{z_0^3} \qquad \begin{cases} \text{at large distances,} \\ z_0 \gg R \end{cases}$$

Express m in terms of the current and the loop area. The quantity m is the magnetic dipole moment. (A loop of current functions as a magnetic dipole.)

(b) Consider a point on the plane of the loop (x-y plane), at distance x_0 from the center. Let's take this point to be on the x-axis.

Using the Biot-Savart law, it can be shown that the magnetic field produced by the ring at this point (for both $x_0 < R$ and $x_0 > R$) has magnitude

$$B = 2\frac{\mu_0 I}{4\pi}\int_0^\pi \frac{(R - x_0\cos\phi)Rd\phi}{(R^2 + x_0^2 - 2x_0R\cos\phi)^{3/2}}$$

(You could derive this yourself by drawing detailed scketches including all relevant lengths and angles. Don't bother trying to perform this integral; it is not expressible in terms of common functions.)

Consider the limit $x_0 \gg R$. In this limit, we can approximate the integrand so that the integral can be performed. First, neglect the R^2 term in the denominator, which is permissible because $R^2/x_0^2 \ll R/x_0 \ll 1$. Then use the binomial approximation, and again neglect any term with R^2/x_0^2. Performing the integral, show that the field at large distances has the form

$$B \approx \frac{\mu_o}{4\pi}\frac{m}{x_0^3}$$

(c) A magnetic dipole with dipole moment $\mathbf{m}$ at the orign is known to generate the following magnetic field at point $\mathbf{r}$ if $r = |\mathbf{r}|$ is large:

$$\mathbf{B} = \frac{\mu_0}{4\pi}\frac{3(\mathbf{m}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}}{r^3} = \frac{\mu_0}{4\pi}\left(\frac{3(\mathbf{m}\cdot\mathbf{r})\mathbf{r}}{r^5} - \frac{\mathbf{m}}{r^3}\right)$$

where $\hat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{r}$.

The cases we have considered in this problem involve $\mathbf{r} \perp \mathbf{p}$ and $\mathbf{r} \parallel \mathbf{p}$. Simplify the above expression for these two special cases. Compare with previous results.

32. **Ring of Electric Charge.**

A thin ring of radius R is centered at the origin and lies in the x-y plane. (It's position is described by $x^2+y^2 = R^2$, $z = 0$.) It carries positive charge distributed uniformly along its length; the linear charge density is λ.

(a) What is the total charge Q on the ring?

(b) Note that points on the ring can be parametrized with the angle ϕ, such that the coordinates of that point are $(R\cos\phi, R\sin\phi, 0)$. Draw a top view of the ring, and for some point on the ring, show the angle ϕ. If you are comfortable with spherical coordinates, you may recognize ϕ as the azimuthal angle. If you are not familiar with spherical coordinates, do look them up.

(c) Consider the point P on the z-axis with coordinates (0, 0, z_0). We want to calculate the electric field $\mathbf{E}$ at this point. Using symmetry, can you tell which direction the field should point towards?

(d) Consider an infinitesimal element of the ring, between the points specified by angle ϕ and angle $\phi + d\phi$. What is the length of this element? What is the charge carried by this element?

(e) Since $d\phi$ is infinitesimal, this element can be thought of as a point charge. Calculate the electric field at point P due to this element of the ring. Don't bother with all the components: only calculate the component in the direction where the total $\mathbf{E}$ will point in.

(f) Now use the principle of superposition to calculate the total $\mathbf{E}$. This means integrating over ϕ to get the contribution from all parts of the ring. Your top-view picture of the ring should tell you the correct limits of integration.

(g) Find the electric field at the origin (at the center of the ring) using your expression. Explain why you could have expected this result.

(h) Approximate your expression for $z_0 \gg R$, i.e., if the point P is very far away from the ring. Explain why you could have expected this result.

(i) Show an approximate plot of the z-component of the electric field, as a function of z_0. You should include both positive and negative values of z_0.

33. A zinc wire sample at 20°C is found to have resistivity $\rho = 5.9 \times 10^{-8}$ ohm-m.

    (a) Find the conductivity σ.

    (b) An applied electric voltage drives a steady current through the zinc sample. The current density at a point in the sample is found to be $J = 4 \times 10^5 \mathrm{A/m}^2$. Calculate the electric field at that point.

    (c) The wire sample has cross-section area $4\mathrm{mm}^2 = 4 \times 10^{-6}\mathrm{m}^2$ and length $1\mathrm{cm} = 10^{-2}\mathrm{m}$. Find the resistance of the sample.

34. A ring with radius R and linear charge λ spins with frequency ω. This situation can be regarded as a circulating current. Find the field produced at the center of the ring.

35. The magnetic field in some region is uniform: $\mathbf{B} = B_0 \hat{k}$. At time $t = 0$, A charged particle (mass m, charge q) is at the origin and has velocity

$$\mathbf{v} = \alpha \hat{\imath} + w \hat{k}$$

    Find the position of the particle, (x, y, z), as a function of time t.

    Describe the motion of the particle in words. In other words, how would you describe the trajectory of this particle?

36. A solid sphere of radius R carries a total charge Q, uniformly distributed in the volume of the sphere.

    (a) What is the charge density?

    (b) Employ Gauss's dielectric flux theorem to calculate the electric field inside the sphere, at distance $r < R$ from the center of the sphere. Provide a drawing showing relevant surfaces and distances.

    (c) What is the electric field outside the sphere, at distance $r > R$ from the center of the sphere? (Showing a derivation is not necessary.)

    (d) Using your expressions for the electric field, find the electric potential $V(r)$, outside the sphere and inside the sphere.

    (e) Sketch a plot of the electric field magnitude as a function of the distance r from the center. The plot should extend from $r = 0$ to $r = 3R$.

37. The electric and magnetic fields in a region are found to be

$$\mathbf{E} = E_0 \, e^{-x/L} \cos\left(\frac{x}{L} - \frac{ct}{L}\right) \hat{\jmath}$$

$$\mathbf{B} = \frac{E_0}{c} \, e^{-x/L} \left[\cos\left(\frac{x}{L} - \frac{ct}{L}\right) - \sin\left(\frac{x}{L} - \frac{ct}{L}\right)\right] \hat{k}$$

where E_0 and L are positive constants.

(a) Check whether Maxwell's third equation (involving the curl of the electric field) is satisfied.

(b) Check whether Maxwell's second equation (involving the divergence of the magnetic field) is satisfied.

(c) Calculate the charge density, the current density, and the displacement current density.

(d) Show explicitly that the continuity equation is satisfied.

(e) Is the system in vacuum? Is Ohm's law satisfied?

(f) Describe the spatial behavior of the wave.

(g) Calculate the Poynting vector $\mathbf{S}$. Is there a net energy flow in any direction?

(h) Calculate the energy density of the fields, $u = \frac{1}{2}\left(\epsilon_0 E^2 + \frac{1}{\mu_0} B^2\right)$.

(i) Show that the energy conservation equation

$$\nabla \cdot \mathbf{S} + \frac{\partial u}{\partial t} + \mathbf{E} \cdot \mathbf{J} = 0$$

is satisfied.

38. Three point charges are placed symmetrically around the origin along the y axis. A negative charge $-q$ sits at the origin. Tow positive charges, each $+q$, sit at $(0, d, 0)$ and $(0, -d, 0)$.

(a) Find the electric field at the point $(x_0, 0, 0)$ on the x-axis.

(b) Simplify your expression in the limit $x_0 \gg d$ and compare with the electric field due to a single point charge.

39. An infinite wire carrying current I_1 runs along the z axis; the current flows from $z = -\infty$ to $z = +\infty$ through the origin.

    Another infinite wire runs parallel to the x axis, lies in the x-y plane, and goes through the point $(0, -d, 0)$. Current $2I_1$ flows through this wire, from $x = -\infty$ to $x = +\infty$.

    Find the magnetic field created by each wire at the point $(0, 2d, 0)$. Find the magnitude of the total magnetic field at this point.

40. A uniformly charged thin rod of length L lies along the y axis and is centered at the origin, i.e., its ends are at $(0, -L/2, 0)$ and $(0, L/2, 0)$. It carries a total charge Q. Find the force exerted by this charged rod on a point charge q placed on the x axis at $(x_0, 0, 0)$. You can take both Q and q to be positive.

41. A thin rod of length h is placed on the x axis, with one endpoint at the origin and the other endpoint at $(h, 0, 0)$. It is charged non-uniformly: the linear charge density is $\lambda = \lambda_0(1 - x/h)$. Find the electric field created by this rod at the point $(x_0, 0, 0)$. also on the x axis. You may assume $x_0 > h$.

    Find the electric potential at the point $(x_0, 0, 0)$, where $x_0 > h$.

42. An infinite charged plane carries surface charge density σ. Use Gauss' dielectric flux theorem to calculate the electric field at distance z from the plane. Explain very clearly the shape of the closed surface on which you apply the theorem. Plot the electric field magnitude as a function of z.

43. A long straight wire carries steady current I. The wire cross-section is circular and has radius R. The current density inside the wire is $J = J_0(r/R)^2$, where r is the distance from the axis of the wire. Express J_0 as a function of I and R.

44. A long straight wire has circular cross-section and radius R. The current density inside the wire is $J = J_0(r/R)^2$, where r is the distance from the axis of the wire. Using Ampere's law, calculate the magnetic field at distance r from the axis of the wire. Consider separately the cases $r < R$ (inside the wire) and $r > R$ (outside the wire). Plot the magnitude of the magnetic field as a function of r.

45. A wire carrying current I runs down the positive y axis to the origin (from $y = +\infty$ to $y = 0$), and from there out to infinity along the positive x axis (from $x = 0$ to $x = +\infty$). Using the Biot-Savart law, find the magnetic field created at the point $(x_0, y_0, 0)$ in the xy plane. For simplicity, you may assume $x_0 > 0$ and $y_0 > 0$.

    You might need to use

$$\int \frac{du}{(u^2 + a^2)^{3/2}} = \frac{u}{a^2(u^2 + a^2)^{1/2}}$$

46. Three points P, Q and R have coordinates

$$P: \quad (l_0, 5l_0, 0) \qquad\qquad Q: \quad (4l_0, 5l_0, 0) \qquad\qquad R: \quad (l_0, 0, 0)$$

    (a) The electric field is given to be uniform: $\mathbf{E} = (7\ W/l_0)\hat{\imath}$. Find the potential difference V_{PQ} between points P and Q, and the potential difference V_{PR} between points P and R.

    Hint: a sketch showing a top view of the x-y plane might help.

    (b) Now consider the electric field to be given by

$$\mathbf{E} = (2x\ W/l_0^2)\hat{\imath}$$

    Find the potential difference V_{PQ} between points P and Q, and the potential difference V_{PR} between points P and R.

47. The electric field in some region is given by

$$\mathbf{E} = \frac{V_0 y}{x^2 + y^2}\hat{\imath} + \frac{V_0 x}{x^2 + y^2}\hat{\jmath}$$

    Calculate the charge density in this region.

    The magnetic field in this region vanishes at time $t = 0$. Calculate the magnetic field at a later time $t = T$.

48. In an electrostatic system, the electric field is

$$\mathbf{E} = \begin{cases} \frac{\alpha}{R^2} x\hat{\imath} + \frac{\alpha}{R^2} y\hat{\jmath} & \text{when } \sqrt{x^2+y^2} \le R \\ \frac{\alpha x\hat{\imath}}{x^2+y^2} + \frac{\alpha y\hat{\jmath}}{x^2+y^2} & \text{when } \sqrt{x^2+y^2} > R \end{cases}$$

where α and R are positive constants.

(a) Calculate the charge density everywhere.

(b) Imagine a cylindrical surface of length L and radius d, whose axis is the z-axis. If $d < R$, find the total electric flux through the surface.

(c) Imagine a cylindrical surface of length L and radius d, whose axis is the z-axis. If $d > R$, find the total electric flux through the surface.

(d) Calculate the potential difference between the origin $(0, 0, 0)$ and the point $(R, 0, 0)$.

(e) Calculate the potential difference between the origin $(0, 0, 0)$ and the point $(2R, 0, 0)$.

(f) Calculate the potential difference between the points $(0, 0, d)$ and $(R, 0, d)$.

(g) Calculate the potential difference between the points $(d, 0, 0)$ and $(0, d, 0)$, for the case $d < R$, and for the case $d > R$.

49. A positive charge $2q$ is placed at the origin, and two negative charges, $-q$ each, are placed at positions $(0, +d, 0)$ and $(0, -d, 0)$. Calculate the electric field due to this collection of three point charges at the point $(0, x_0, 0)$ on the x-axis.

Approximate the magnitude of the electric field for the case $x_0 \gg d$.

In this limit, the magnitude has the form $G/(x_0)^\alpha$; find G and α.

50. In an electrostatic system, the electric field is

$$\mathbf{E} = \begin{cases} 0 & \text{when } r \leq a_0 \\ \left(\dfrac{W_0}{r^2}\right)\hat{r} & \text{when } r > a_0 \end{cases}$$

Here r is the distance from the origin, and $\hat{r}$ is the radial unit vector in the spherical coordinate system, i.e., points radially away from the origin.

Also, W_0 and a_0 are positive constants.

(a) Calculate the electric potential everywhere, i.e., for both $r > a_0$ and $r \leq a_0$. Take the reference point to be at infinity, i.e., the potential is zero at points infinitely far from the origin.

(b) Plot the electric field magnitude as a function of r. Mark the point $r = a_0$ clearly.

(c) Plot the electric potential as a function of r. Mark the point $r = a_0$ clearly.

(d) What kind of charge distribution would create the electric field given here?