MP352
Special Relativity

Time allowed: 2 hours

Answer **ALL** questions

This is a **SAMPLE** exam, roughly reflecting the general structure of the finals for 2017 – 2018.

1. Consider the set of 4×4 matrices Λ with real elements which satisfy the relation

$$\Lambda^T g \Lambda = g\,, \quad \text{where} \quad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1}$$

is the metric tensor. These matrices represent Lorentz transformations of spacetime points (ct, x, y, z).

(a) Under what conditions is a matrix of this set proper?

Explain what a non-proper matrix represents physically.

[6 marks]

(b) If a matrix satisfies condition (1), show that its inverse satisfies the condition as well.

[8 marks]

(c) Ignoring the y and z directions, write down a two-dimensional version of condition (1). Use this condition to determine the form of an infinitesimal boost in the x-direction.

In other words, find the generator of the group $O(1,1)$ or $SO(1,1)$.

[18 marks]

2. Let Σ and Σ' be inertial frames. Frame Σ' moves at velocity v with respect to Σ, in the common (positive) x direction. Measurements of an event in the two frames, (ct, x, y, z) and (ct', x', y', z'), are related by the Lorentz transformation

$$ct' = \gamma_v(ct - vx/c) \; ; \quad x' = \gamma_v(x - vt) \; ; \quad y' = y \; ; \quad z' = z$$

where $\gamma_v = (1 - v^2/c^2)^{-1/2}$.

(a) A photon leaves the origin of Σ at the time $t = 0$ in a direction which forms an angle of 45° with the x-axis. What is the angle with the x'-axis, as observed in Σ'?

[18 marks]

(b) The rank-2 tensor has components $N^{\alpha\beta}$ in Σ and components $(N')^{\alpha\beta}$ in the Σ'.
Find $(N')^{00}$ and $(N')^{01}$ in terms of the components $N^{\alpha\beta}$.
Hint: the y and z directions (2 and 3 components) play no role.

[12 marks]

(c) Write down the Galilean tranformation relating (ct, x, y, z) and (ct', x', y', z'). Under which limit does the Lorentz transformation reduce to the Galilean transformation?

[5 marks]

3. (a) A Poincaré transformation (Λ, a) involves a Lorentz transformation Λ and a shift by the four-vector a. (A spacetime event x is transformed to $x' = \Lambda x + a$.

   Find out the result of two successive Poincaré transformations, (Λ_1, a_1) and (Λ_2, a_2).

   Are Poincaré transformations commutative?

   [10 marks]

   (b) Explain using equations or inequalities what it means for a four-vector to be time-like, space-like, and light-like.

   Find the four-momentum of a particle with nonzero mass m and velocity $\vec{u} = (c/2, c/2, 0)$. Find out whether this four-vector is time-like, space-like, or light-like.

   [13 marks]

   (c) In the lab frame, two identical balls, each having mass M, collide with equal but opposite velocities of magnitude v. Their collision is perfectly inelastic, so they stick together and form a single body.

   Find the mass of the final body in terms of M and v.

   Inertial frame Σ moves with one of the balls before the collision. Draw the worldlines of all particles as seen from this frame. Indicate the velocities (inverse slopes) of each straight segment.

   [10 marks]