# MP352
# Special Relativity

Time allowed: 2 hours

Answer **ALL** questions

This is a **SAMPLE** exam, roughly reflecting the general structure of the finals for 2017 – 2018.

1. Consider the set of $4 \times 4$ matrices $\Lambda$ with real elements which satisfy the relation

$$\Lambda^T g \Lambda = g \,. \tag{1}$$

These matrices represent Lorentz transformations of spacetime points $(ct, x, y, z)$.

(a) Write down the metric tensor $g$ in the the two common conventions, and mention which you will use in the following.

**[3 marks]**

(b) This set of matrices is denoted as $O(3,1)$. Under what additional conditions do we get the set $SO^{\uparrow}(3,1)$?

**[5 marks]**

(c) Show whether rotations are included in Lorentz transformations, i.e., whether rotation matrices satisfiy condition (1).

**[7 marks]**

(d) Show that the set of matrices which satisfiy condition (1) forms a group under matrix multiplication. Verify all four group properties.

**[20 marks]**

2. Let $\Sigma$ and $\Sigma'$ be inertial frames. Frame $\Sigma'$ moves at velocity $v$ with respect to $\Sigma$, in the common (positive) $x$ direction. Measurements of an event in the two frames, $(ct, x, y, z)$ and $(ct', x', y', z')$, are related by the Lorentz transformation

$$ct' = \gamma_v(ct - vx/c) \; ; \quad x' = \gamma_v(x - vt) \; ; \quad y' = y \; ; \quad z' = z$$

where $\gamma_v = (1 - v^2/c^2)^{-1/2}$.

(a) A particle of mass $m$ has velocity $\vec{u'} = (4v, -3v, 0)$ relative to $\Sigma'$.
Find the velocity of the particle relative to $\Sigma$.
Find the four-momentum of the particle relative to $\Sigma'$.

**[16 marks]**

(b) Represent the $(ct, x)$ axes and the $(ct', x')$ axes on a single spacetime diagram, such that the $ct$ and $x$ axes are perpendicular to each other.
Use the LT to find out how $x'$ units are related to $x$ units on this diagram. (Hint: You could consider the event $(ct', x') = (0, 1)$, find its coordinates in the $\Sigma$ frame, and hence obtain the distance of this point from the origin in $x$ units.)

**[10 marks]**

(c) A rod is at rest in frame $\Sigma$. It lies in the $xy$ plane and makes an angle $\pi/3$ $(= 60°)$ with the $x$-axis. Using the length contraction formula, find the angle that the stick makes with the $x'$ axis, as observed from the $\Sigma'$ frame.

**[9 marks]**

3. (a) Measured in one intertial frame, events $A$ and $B$ have spatial coordinates

$$(x_A, y_A, z_A) = (4L, -6L, 0)\,, \qquad (x_B, y_B, z_B) = (7L, -2L, 0)\,,$$

and temporal coordinates

$$t_A = 2L/c\,, \qquad t_B = 12L/c\,,$$

where $L$ is a positive constant.

Calculate the invariant interval between the events. Is this interval timelike, spacelike, or null?

Explain whether there exists a different inertial frame in which the two events occur simultaneously.

**[10 marks]**

(b) The inertial frame $\tilde{\Sigma}$ is obtained from inertial frame $\Sigma$ by a boost by speed $v$ in the positive $z$ direction, followed by a rotation by angle $\theta$ around the $x$ direction.

Find the matrix that transforms the coordinates of an event in $\Sigma$ to its coordinates in $\tilde{\Sigma}$.

**[10 marks]**

(c) A particle of mass $m$ whose total energy is twice its rest energy collides with an identical particle at rest. If they stick together, what is the mass of the resulting composite particle? What is its velocity?

**[10 marks]**