MP352
Special Relativity

Time allowed: 2 hours

Answer **ALL** questions

This is a **SAMPLE** exam, roughly reflecting the general structure of the finals for 2017 – 2018.

1. Consider the set of 4×4 matrices Λ with real elements which satisfy the relation

$$\Lambda^T g \Lambda = g\,, \quad \text{where} \quad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1}$$

is the metric tensor. These matrices represent Lorentz transformations of spacetime points (ct, x, y, z).

(a) Under what conditions is a matrix of this set orthochronous?
Explain what a non-orthochronous matrix represents physically.

[6 marks]

(b) Show that, if a transformation of spacetime coordinates (ct, x, y, z) preserves the Minkowski norm, then it must satisfiy condition (1).

[13 marks]

(c) Explain whether the group of matrices satisfying condition (1) (the Lorentz group) is abelian or not. If it is non-abelian, give two example elements of the group which do not commute.

[11 marks]

2. Let Σ and Σ' be inertial frames. Frame Σ' moves at velocity v with respect to Σ, in the common (positive) x direction. Measurements of an event in the two frames, (ct, x, y, z) and (ct', x', y', z'), are related by the Lorentz transformation

$$ct' = \gamma_v(ct - vx/c) \; ; \quad x' = \gamma_v(x - vt) \; ; \quad y' = y \; ; \quad z' = z$$

where $\gamma_v = (1 - v^2/c^2)^{-1/2}$.

(a) A photon has velocity $\vec{u}' = (\frac{2}{3}c, \frac{2}{3}c, \frac{1}{3}c)$ relative to Σ'.
Find the velocity of the photon relative to Σ.
Explain how your result is consistent with the invariance of the speed of light.

[19 marks]

(b) A body of mass m is at rest in the frame Σ.
Write down its four-velocity in the frame Σ'.
Write down its four-velocity in the frame Σ.
Show that the norm of the four-velocity is the same in the two frames.

[11 marks]

3. (a) Lorentz boosts in the x direction can be represented by a transformation matrix of the form

$$\begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix}$$

where the y and z coordinates have been ignored.

Show that the set of all boosts in the x direction form a group.

Will the set of all boosts in ALL directions also form a group? Explain why, or why not.

[18 marks]

(b) A photon with energy E collides with a stationary mass m. They combine to form one particle. What is the mass of this particle? What is its speed?

[10 marks]

(c) Darragh (D) remains on earth while his twin sister Ciaomhe (C) travels to a distant planet at speed v and then immediately returns, traveling at speed $2v$.

Draw the worldlines of both D and C, as seen from D's frame (earth frame).

Draw the worldlines of both D and C, as seen from the inertial frame that is coincident with Ciaomhe's vehicle during her outward journey. Indicate the speeds (or inverse slopes) of each straight line in your diagram.

[12 marks]