MP352
Special Relativity

Time allowed: 2 hours

Answer **ALL** questions

This is a **SAMPLE** exam, roughly reflecting the general structure of the finals for 2017 – 2018.

1. Consider the set of 4×4 matrices Λ with real elements which satisfy the relation

$$\Lambda^T g \Lambda = g\,, \quad \text{where} \quad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1}$$

is the metric tensor. These matrices represent Lorentz transformations of spacetime points (ct, x, y, z).

(a) What additional conditions give us the set of physical or restricted Lorentz transformations?

Explain which set is called $O(3,1)$ and which one is called $SO^{\uparrow}(3,1)$.

[6 marks]

(b) Consider the time reversal operator and the spatial reflection operator

$$T = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \qquad P = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}.$$

Explain how these operators divide the set of Lorentz transormations into four disjoint sets.

If these matrices themselves are viewed as transformations, are they members of the Lorentz group?

Are these two transformations orthochronous?

[12 marks]

(c) Show that, if a transformation matrix satisfies condition (1), then the Minkowski norm is preserved under this transformation.

[12 marks]

2. Let Σ and Σ' be inertial frames. Frame Σ' moves at velocity v with respect to Σ, in the common (positive) x direction. Measurements of an event in the two frames, (ct, x, y, z) and (ct', x', y', z'), are related by the Lorentz transformation

$$ct' = \gamma_v(ct - vx/c) \; ; \quad x' = \gamma_v(x - vt) \; ; \quad y' = y \; ; \quad z' = z$$

where $\gamma_v = (1 - v^2/c^2)^{-1/2}$.

(a) A photon moves in the common x, x' direction. Its frequency is f relative to Σ and is f' relative to Σ'.

Write down the four-momentum of the photon in the two frames.

Using the invariance of the norm of four-momenta, express f' in terms of f.

[14 marks]

(b) Represent the (ct, x) axes and the (ct', x') axes on a single spacetime diagram, such that the ct and x axes are perpendicular to each other.

Use the LT to derive the angle between the x' axis and the x axis. (Hint: You could consider the event $(ct', x') = (0, 1)$, and find its coordinates in the Σ frame.)

[10 marks]

(c) Two particles of mass m are observed in Σ to move in opposite directions with speed v, one in the positive x and one in the negative x direction.

In the frame Σ, find the individual four-momenta of the two particles, and the total four-momentum of this system.

Calculate the velocities of the two particles relative to frame Σ'.

In the frame Σ', find the individual four-momenta of the two particles, and the total four-momentum of this system.

Show that the total four-momentum calculated in Σ' can be obtained by the Lorentz transformation from that calculated in Σ.

[11 marks]

3. (a) On a spacetime diagram, draw the worldline of a photon starting at the origin, and the worldline of an object with velocity c/2 starting at the origin two seconds later.

   A particle starting at rest at the origin is accelerated at a constant rate. Draw its worldline on a spacetime diagram. What is the slope of this line at short times, and at very long times?

   [9 marks]

   (b) A particle of mass M is travelling with speed v when it decays into two photons. The photons are seen to emerge at equal angles θ on either side of the original velocity. Show that $v = c\cos\theta$.

   Also find the energy of either photon. Express this energy in terms M and θ.

   Also, calculate the energy of either photon in terms of M, v and θ.

   [16 marks]

   (c) A rocket moves at speed u directly away from earth. It ejects a shuttle at speed v perpendicular to the direction of its motion. Measured with respect to earth, what are the velocity components of the shuttle? Measured with respect to earth, what is the speed of the shuttle?

   [10 marks]