MATHEMATICAL PHYSICS

SEMESTER 2

2017–2018

MP352
Special Relativity

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Time allowed: 2 hours

Answer **ALL** questions

1. Consider intertial frames Σ and Σ'. Frame Σ' moves at velocity v with respect to Σ, in the common (positive) x direction. Measurements of an event in the two frames, (ct, x, y, z) and (ct', x', y', z'), are related by the Lorentz transformation

$$ct' = \gamma_v(ct - vx/c) \; ; \quad x' = \gamma_v(x - vt) \; ; \quad y' = y \; ; \quad z' = z$$

where $\gamma_v = (1 - v^2/c^2)^{-1/2}$.

(a) A particle moves with velocity $(-w, 0, 0)$ relative to Σ. Here w is a positive constant.

Write down the four-velocity of the particle as observed from the Σ frame and as observed from the Σ' frame.

[8 marks]

(b) A photon has velocity $\vec{u} = (\frac{3}{5}c, 0, \frac{4}{5}c)$ relative to Σ.

Find the velocity $\vec{u'}$ of the photon relative to Σ'.

Explain how your result is consistent with the constancy of the speed of light.

[18 marks]

(c) A stick is at rest in frame Σ. It lies in the xy plane and makes an angle $\pi/4$ $(= 45°)$ with the x-axis. Using the length contraction formula, find the angle that the stick makes with the x' axis, as observed from the Σ' frame.

[9 marks]

2. Consider the set of 4×4 matrices Λ with real elements which satisfy the relation

$$\Lambda^T g \Lambda = g\,, \quad \text{where} \quad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1}$$

is the metric tensor. These matrices represent Lorentz transformations of spacetime points (ct, x, y, z).

(a) Under what conditions is a matrix of this set orthochronous?
Explain what a non-orthochronous matrix represents physically.

[6 marks]

(b) Show that, if a transformation of spacetime coordinates (ct, x, y, z) preserves the Minkowski norm, then it must satisfiy condition (1).

[13 marks]

(c) The group of matrices satisfying condition (1) is known as $O(1,3)$. What additional conditions are required to obtain the group of physical Lorentz transformations, $SO^{\uparrow}(1,3)$?

[4 marks]

(d) The group $SO^{\uparrow}(1,3)$ is not abelian. Give two example elements of the group which do not commute, and explain what transformations your examples represent.
Which types of pure boosts commute with each other?

[7 marks]

3. (a) Measured in one intertial frame, events A and B have spatial coordinates

$$(x_A,\, y_A,\, z_A) \;=\; (4L,\, -6L,\, 0)\,, \qquad (x_B,\, y_B,\, z_B) \;=\; (7L,\, -2L,\, 0)\,,$$

and temporal coordinates

$$t_A \;=\; 2L/c\,, \qquad t_B \;=\; 12L/c\,,$$

where L is a positive constant.

Calculate the invariant interval (Minkowski interval) between the events. Is this interval timelike, spacelike, or null?

Explain whether there exists a different inertial frame in which the two events occur simultaneously.

[9 marks]

(b) A photon of wavelength λ collides with a stationary electron. After the collision, the photon scatters at an angle θ with respect to the incident direction, and has wavelength λ'. The electron moves with momentum p_e after the collision, in a direction making angle ϕ with the incident direction of the photon.

Write down equations for energy and momentum conservation.

Show that

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta).$$

[17 marks]

(c) On a spacetime diagram (ct versus x diagram), draw the worldline of a photon starting at the origin, and the worldline of an object with velocity c/2 starting at the origin three seconds later.

Sketch the worldline of a particle subject to a constant force in the positive x direction. The particle starts from rest at the origin at time $t = 0$. State in words the initial and asymptotic (late-time) speeds of the particle. Both should be clear from your sketch.

[9 marks]