MATHEMATICAL PHYSICS

SEMESTER 2, REPEAT

2018–2019

MP352
Special Relativity

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Time allowed: 2 hours

Answer **ALL** questions

1. Let Σ and Σ' be inertial frames. Frame Σ' moves at speed v with respect to Σ, in the common (positive) x direction. Measurements of an event in the two frames, (ct, x, y, z) and (ct', x', y', z'), are related by the Lorentz transformation

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma_v & -\gamma_v \frac{v}{c} & 0 & 0 \\ -\gamma_v \frac{v}{c} & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \qquad \text{where } \gamma_v = (1 - v^2/c^2)^{-1/2}.$$

(a) The rapidity ϕ is defined such that $\tanh\phi = v/c$. Express the Lorentz boost transformation above in terms of the rapidity. The speed v, or the quantity γ_v, should not appear in your expression.

(Useful identities: $\tanh\phi = \dfrac{\sinh\phi}{\cosh\phi}$; $\cosh^2\phi - \sinh^2\phi = 1$.)

[8 marks]

(b) A photon has velocity $\vec{u'} = (0, c, 0)$ relative to Σ'.

Find the velocity of the photon relative to Σ.

Calculate the speed of the photon relative to Σ. Explain whether and why your result was expected.

[12 marks]

(c) Represent the (ct, x) axes and the (ct', x') axes on a single spacetime diagram, such that the ct and x axes are perpendicular to each other.

Show two events on this joint diagram which are simultaneous when measured from Σ'. Show which of these events happens earlier according to Σ.

Use the Lorentz transformations to find out how x' units are related to x units on this diagram. (Hint: You could consider the event $(ct', x') = (0, 1)$, find its coordinates in the Σ frame, and hence obtain the distance of this point from the origin in x units.)

[13 marks]

2. Consider the set of 4×4 matrices Λ with real elements which satisfy the relation

$$\Lambda^T g \Lambda = g\,, \quad \text{where} \quad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1}$$

is the metric tensor. These matrices represent Lorentz transformations of spacetime points (ct, x, y, z).

(a) Find the possible values of the determinant of a matrix belonging to this set.

[5 marks]

(b) What additional property must such a matrix satisfy, in order to represent a *proper* Lorentz tranformation?

What does a non-proper Lorentz transformation mean physically?

[6 marks]

(c) If a matrix satisfies condition (1), show that its inverse satisfies the condition as well.

[7 marks]

(d) Ignoring the y and z directions, write down a two-dimensional version of condition (1). Use this condition to determine the form of an infinitesimal boost in the x-direction.

[14 marks]

3. (a) The current density 4-vector J^μ is defined as $(c\rho, \vec{J})$, where ρ is the charge density and $\vec{J}$ is the usual current density or 3-current density. Show that the tensor equation $\partial_\mu J^\mu = 0$ is equivalent to the continuity equation of electromagnetism.

[10 marks]

(b) Explain using equations or inequalities what it means for a four-vector to be time-like, space-like, and light-like.

Find the four-velocity of a particle with nonzero mass m and velocity $\vec{u} = (c/3, c/3, c/3)$. Find out whether this four-vector is time-like, space-like, or light-like.

[14 marks]

(c) In the lab frame, two identical balls, each having mass m_0, collide with equal but opposite velocities of magnitude v. Their collision is perfectly inelastic, so they stick together and form a single body.

Find the mass of the final body in terms of m_0 and v.

Inertial frame Σ moves with one of the balls before the collision. Find the energy of the final body relative to Σ.

[11 marks]