MATHEMATICAL PHYSICS

SEMESTER 2, REPEAT EXAM

2019–2020

MP352

Special Relativity

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Time allowed: 2 hours

Answer **ALL** questions

1. Consider inertial frames Σ and Σ'. Frame Σ' moves at speed v with respect to Σ, in the common (positive) x direction. Measurements of an event in the two frames, (ct, x, y, z) and (ct', x', y', z'), are related by the Lorentz transformation

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

where

$$\gamma = \gamma_v = \frac{1}{\sqrt{1 - v^2/c^2}}.$$

(a) A particle moves with velocity $(w, 0, 0)$ relative to Σ. Here w is a positive constant, and $w > v$.

Write down the four-velocity of the particle as observed from the Σ frame and as observed from the Σ' frame.

[8 marks]

(b) A photon has velocity $\vec{u} = (\frac{3}{5}c, 0, \frac{4}{5}c)$ relative to Σ.

Find the velocity $\vec{u'}$ of the photon relative to Σ'.

Explain how your result is consistent with the constancy of the speed of light.

[18 marks]

(c) The rank-2 contravariant tensor $T^{\mu\nu}$ is antisymmetric.

Find the 01 component of this tensor in the Σ' frame (i.e., find $(T')^{01}$) in terms of the components in the Σ frame.

Show that $(T')^{01}$ can be expressed in terms of T^{01} alone.

[14 marks]

2. (a) Measured in one inertial frame, events A and B have spatial coordinates

$$(x_A, y_A, z_A) \; = \; (4L, -3L, 0) \, , \qquad (x_B, y_B, z_B) \; = \; (7L, 0, 2L) \, ,$$

and temporal coordinates

$$t_A \; = \; 5L/c \, , \qquad t_B \; = \; 14L/c \, ,$$

where L is a positive constant.

Calculate the invariant interval (Minkowski interval) between the events. Is this interval timelike, spacelike, or null?

Explain whether there exists a different inertial frame in which the two events occur simultaneously.

[9 marks]

(b) Particle A, while at rest in the lab, decays into particle B and a photon. The rest masses of A and B are m_A and m_B respectively.

Show that, after the decay, the particle B and the photon each have momentum of magnitude

$$\frac{m_A^2 - m_B^2}{2m_A} c$$

as measured in the lab frame.

[12 marks]

(c) On a spacetime diagram (ct versus x diagram), draw the worldline of a photon starting at the origin, and the worldline of an object with velocity c/2 starting at the origin five seconds later.

Sketch the worldline of a particle subject to a constant force in the positive x direction. The particle starts from rest at the origin at time $t = 0$. State in words the initial and asymptotic (late-time) speeds of the particle. Both should be clear from your sketch.

[9 marks]

3. Consider the set of 4×4 matrices Λ with real elements which satisfy the relation

$$\Lambda^T g \Lambda = g\,, \quad \text{where} \quad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1}$$

is the metric tensor. These matrices represent Lorentz transformations of spacetime points (ct, x, y, z).

(a) Under what conditions is a matrix of this set orthochronous?

Explain what a non-orthochronous matrix represents physically.

[6 marks]

(b) Show that, if a transformation of spacetime coordinates (ct, x, y, z) preserves the Minkowski norm, then it must satisfiy condition (1).

[13 marks]

(c) The group of matrices satisfying condition (1) is known as $O(1, 3)$. What additional conditions are required to obtain the group of physical Lorentz transformations, $SO^{\uparrow}(1, 3)$?

[4 marks]

(d) The group $SO^{\uparrow}(1, 3)$ is not abelian. Give two example elements of the group which do not commute, and explain what transformations your examples represent.

Which types of pure boosts commute with each other?

[7 marks]