MATHEMATICAL PHYSICS

SEMESTER 2, MAY EXAM

2019–2020

MP352

Special Relativity

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Time allowed: 2 hours

Answer **ALL** questions

Exam Duration: 2 hours

You have three hours to complete this examination. This includes the time taken to download the question paper, answer the question, scan your solutions, and uploading the resulting PDF to the Moodle Examination page E:MP352. You should not spend more than two hours doing the examination itself as this may mean that you do not have time to submit your work.

This is an unsupervised examination. You may consult your notes or textbooks as you wish, but you must show all your working to get full credit for correct answers. You must not collude with other students. Submission of work through your Moodle account will be taken as an assertion that the work was done by yourself.

If you encounter any technical problems with the download or upload please contact the MP352 lecturer (Masud Haque) at

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or email

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or by messaging the lecturer on Moodle or on MS Teams.

In case Dr. Haque is unreachable for some reason, you can try contacting Prof. Coles for urgent questions: Peter.Coles@mu.ie

The deadline for submitting your answers is **17:15** on Thursday 28th May 2020. Unless you have been granted an adjustment, a late submission will be accepted only at the discretion of the Examiner(s) if you have not notified the Examiner that you have encountered problems.

1. Let Σ and Σ' be inertial frames. Frame Σ' moves at speed v with respect to Σ, in the common (positive) x direction. Measurements of an event in the two frames, (ct, x, y, z) and (ct', x', y', z'), are related by the Lorentz transformation

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \gamma & -\gamma\frac{v}{c} & 0 & 0 \\ -\gamma\frac{v}{c} & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix} \quad \text{where} \quad \gamma = \gamma_v = \sqrt{1 - v^2/c^2}.$$

(a) A photon has velocity $\vec{u'} = (0, 0, c)$ relative to Σ'.
Find the velocity of the photon relative to Σ.
Calculate the speed of the photon relative to Σ. Explain whether and why your result was expected.

[12 marks]

(b) The photon in the previous question has frequency f', as observed from Σ'. Write down the 4-momentum of the photon relative to Σ'.
Recall that all 4-vectors, including 4-momenta, transform the same way under Lorentz transformations. Use the Lorentz transformation to find the 4-momentum relative to Σ.
Use your result to find the energy, the 3-momentum, and the frequency of the photon as observed from Σ.

[14 marks]

(c) Represent the (ct, x) axes and the (ct', x') axes on a single spacetime diagram, such that the ct and x axes are perpendicular to each other.
Use the Lorentz transformations to find out how x' units are related to x units on this diagram. (Hint: You could consider the event $(ct', x') = (0, 1)$, find its coordinates in the Σ frame, and hence obtain the distance of this point from the origin in x units.)

[9 marks]

2. (a) A neutral pion of rest mass m, while moving in the positive x direction, decays into two photons. One of the photons is emitted in the same direction as the original pion, and the other in the opposite direction. If the first photon has four times the energy of the second, find the original speed of the pion.

[12 marks]

(b) Find the four-velocity of a particle with 3-velocity $\vec{u} = (c/2, c/2, 0)$. Find out whether this four-vector is time-like, space-like, or light-like.

Explain why, if a four-vector is time-like in one inertial frame, it has to be time-like in all inertial frames.

[12 marks]

(c) Relative to the lab frame, Aoife has 4-velocity U^μ. Two events, having spacetime coordinates X^μ and Y^μ relative to the lab frame, are observed to be simultaneous by Aoife. Show that the Minkowski inner product

$$g_{\mu\nu} U^\mu \left(X^\nu - Y^\nu\right) \qquad \text{or} \qquad U_\mu \left(X^\mu - Y^\mu\right)$$

must vanish in any inertial frame.

In which situation could you expect the two events to be simultaneous according to the lab frame as well?

[11 marks]

3. Consider the set of 4×4 matrices Λ with real elements which satisfy the relation

$$\Lambda^T g \Lambda = g\,, \quad \text{where} \quad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \tag{1}$$

is the metric tensor. These matrices represent Lorentz transformations of spacetime points (ct, x, y, z).

(a) The Minkowski norm of a 4-vector A is $A^T g A$.
Show that the Minkowski norm of a vector is preserved under a Lorentz transformation.

[6 marks]

(b) Write down the 4×4 matrix transforming spacetime coordinates (ct, x, y, z) of an event under a Lorentz boost in the y direction, with rapidity ϕ. Show that this matrix satisfies condition (1).
Possibly useful identity: $\cosh^2\phi - \sinh^2\phi = 1$.

[8 marks]

(c) Ignoring the y and z directions, write down a two-dimensional version of condition (1). Use this condition to determine the form of an infinitesimal boost in the x-direction.

[16 marks]