Due on Monday, February 8th, via Moodle

Problems marked [**SELF**] are for self-study and additional exercise; will not be marked.

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The relativistic factor

$$\gamma(v) \;=\; \frac{1}{\sqrt{1-(v/c)^2}}$$

is central in special relativity. It is a function of the speed (magnitude of velocity) v. It's vital to know the properties of $\gamma(v)$ in some detail.

If tired of carrying along factors of c, you could use the common notation $\beta = \frac{v}{c}$. If there are multiple speeds involved, such as u and v, you could use $\beta_u = \frac{u}{c}$ and $\beta_v = \frac{v}{c}$. But please express final results in terms of u, v and c again.

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1. (a) [**2 pts.**] Show that

$$\frac{1}{\gamma^2} + \frac{v^2}{c^2} = 1 \qquad \begin{cases}\text{Here } \gamma \text{ is used as} \\ \text{shorthand for } \gamma(v)\end{cases}$$

(b) [**2 pts.**] Show that

$$\gamma(v) \times (c+v) = c\sqrt{\frac{c+v}{c-v}}$$

(c) [**6 pts.**] Show that

$$\frac{d\gamma}{dt} = \frac{1}{c^2}\gamma^3 v\dot{v} \qquad \text{and} \qquad \frac{d\gamma}{dt} = \frac{1}{c^2}\gamma^3 \vec{v}\cdot\vec{a}$$

where we use the shorthands $\gamma = \gamma(v)$ and $\dot{v} = \frac{dv}{dt}$. Note that $\dot{v}$ is the derivative of the speed, not the derivative of the 3-vector $\vec{v}$, i.e., $\dot{v}$ is not the acceleration vector $\vec{a} = \frac{d\vec{v}}{dt}$.

(d) [**5 pts.**] If $w = (u+v)/(1+uv/c^2)$, derive a relation of the form

$$\gamma(w) \;=\; \gamma(u)\gamma(v) \times \text{something}$$

and state explicitly what this something is.

2. **[3 pts.]** Present a neat hand-drawn plot of $\gamma(v)$ as a function of v or v/c. (As usual, you can use a computer plotting programme to figure out what this looks like, but please submit a hand-drawn plot.)

   Looking at your curve, I should be able to immediately see the value of $\gamma(v)$ at zero speed, the slope at zero speed, and at which speed, if any, $\gamma(v)$ diverges.

3. The binomial series

$$(1+x)^\epsilon = 1 + \epsilon x + \frac{1}{2!}\epsilon(\epsilon-1)x^2 + \frac{1}{3!}\epsilon(\epsilon-1)(\epsilon-2)x^3 + ...$$

   holds for all $|x| < 1$. Please find a way to remember this, so that you can use it without having to look it up. (Look up 'binomial series' online.)

   (a) **[3 pts.]** Derive the binomial expansion using the Taylor expansion. State clearly which function you are expanding and around what point.

   (b) **[2 pts.]** Expand $(P+Q)^\epsilon$, finst assuming $|Q| < |P|$, and then for the case $|P| < |Q|$.

   (c) **[3 pts.]** Use the binomial expansion to find good approximations for $(1001)^{1/3}$ and $(15)^{1/4}$, without using a calculator.

   (d) **[SELF]** For which values of ϵ does the series terminate after a finite number of terms?

   (e) **[4 pts.]** Expand $\gamma(v)$ in powers of v/c, including at least up to order $(v/c)^6$. This expansion is useful for small v, when γ is close to 1.

   (f) **[3 pts.]** If you truncate the series, you get an approximant for $\gamma(v)$. Plot $\gamma(v)$ as a function of v/c, together with the approximants obtained by keeping the first one, two, three or four terms of the series. This should show how successive approximants do a better and better job of approximating our $\gamma(v)$ function.

   Please submit only a neat hand-drawn plot, no computer printouts.

   (g) **[4 pts.]** Express v as a function of $1/\gamma$. Expand using the binomial expansion, at least up to order $1/\gamma^6$. For which values of v is this expansion useful?

4. **[3 pts.]** A low-flying earth satellite travels at about 8000 m/s. What is the factor γ for this speed?

5. We will be dealing a lot with *hyperbolic functions*.

   (a) **[3 pts.]** Look up (or recall) and report the definitions of the hyperbolic functions $\cosh(x)$, $\sinh(x)$ and $\tanh(x)$, in terms of the exponential functions e^x and e^{-x}.

   (b) **[3 pts.]** Provide neat hand-drawn plots of the three hyperbolic functions.

   (You can use a computer program if you need to, but you really should be able to figure out how these functions look like, from the definitions in terms of e^x and e^{-x}. I suggest trying first without a computer program or textbook.)

   (c) **[4 pts.]** Express the corresponding three trigonometric functions in terms of e^{ix} and e^{-ix}. Are the relations

   $$\sinh(x) = c_1 \sin(ix) \qquad \text{and} \qquad \sinh(ix) = c_2 \sin(x)$$

   correct for some value of c_1 and c_2? Which values?