Due on Monday, February 15th, via moodle

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1. Hyperbolic functions: $\sinh(x) = \frac{1}{2}(e^x - e^{-x})$, $\cosh(x) = \frac{1}{2}(e^x + e^{-x})$, and $\tanh(x) = \sinh(x)/\cos(x)$.

   (a) **[3 pts.]** $\sinh(x_1+x_2)$ and $\cosh(x_1+x_2)$ can each be expressed in terms of $\cosh(x_1)$, $\sinh(x_1)$, $\cosh(x_2)$, $\sinh(x_2)$. Look up these expressions, and derive them from the definitions above. Please remember these expressions, or make sure you can derive them at short notice.

   (b) **[2 pts.]** Using the expressions for $\sinh(x_1 + x_2)$ and $\cosh(x_1 + x_2)$, derive an expression for $\tanh(x_1 + x_2)$ in terms of $\tanh(x_1)$ and $\tanh(x_2)$.

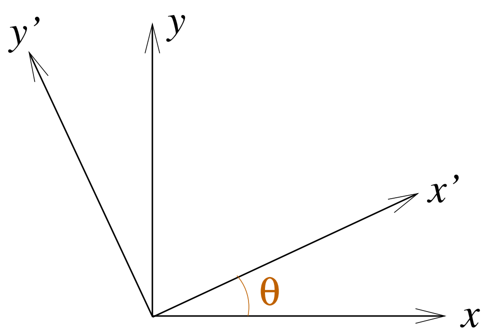
   (c) **[2 pts.]** Show that

$$\cosh^2(x) - \sinh^2(x) = 1$$

2. The rest of this problem set is about Euclidean space observed from two sets of coordinate axes which are rotated with respect to each other. Rotations will turn out to be closely related to the Lorentz transformation.

   We refer to the coordinates as (x, y, z) in the old frame and as (x', y', z') in the new frame.

   We consider a rotation of the axes around the z axis by an angle θ, as shown in the figure. The z and z' axes are identical and point perpendicularly out of the page.
   The unit vectors in the x, y, z directions are $\hat{i}$, $\hat{j}$, $\hat{k}$, and the unit vectors in the x', y', z' directions are $\hat{i}'$, $\hat{j}'$, $\hat{k}'$.



   (a) **[2 pts.]** Calculate $\hat{i}$ in terms of $\hat{i}'$ and $\hat{j}'$.
   Hint: geometrically, you should be able to find the projections (components) of $\hat{i}$ in the x'- and y'- directions.

   (b) **[2 pts.]** Calculate $\hat{j}$ in terms of $\hat{i}'$ and $\hat{j}'$.

(c) **[3 pts.]** The position vector of a point is given by

$$\vec{r} \;=\; x\hat{\imath} + y\hat{\jmath} \;=\; x'\,\hat{\imath}' + y'\,\hat{\jmath}' \tag{1}$$

Provide a neat drawing showing a vector $\vec{r}$ and the four quantities x, y, x', y'.

(d) **[4 pts.]** By substituting in (1) the expressions for $\hat{\imath}$ and $\hat{\jmath}$ in terms of $\hat{\imath}'$ and $\hat{\jmath}'$, and then comparing coefficients of $\hat{\imath}'$ and $\hat{\jmath}'$ on both sides, show that

$$\begin{aligned} x' &= x\cos\theta + y\sin\theta \\ y' &= -\,x\sin\theta + y\cos\theta \end{aligned}$$

(e) **[3 pts.]** Verify the above transformation graphically for the special case of $\theta = \pi/2$, by comparing the coordinates of a point in the two frames. Provide a neat drawing showing the distances representing the coordinates of the point in both frames.

(f) **[3 pts.]** Write the transformation from the coordinates (x, y) to the coordinates (x', y') as a matrix equation:

$$\begin{pmatrix} x' \\ y' \end{pmatrix} \;=\; R(\theta) \begin{pmatrix} x \\ y \end{pmatrix} \tag{2}$$

where $R(\theta)$ is the rotation matrix. Write out the matrix $R(\theta)$ explicitly. What is the size of this matrix?

(g) **[2 pts.]** Let's reinstate the third direction. For the rotation being considered, $z' = z$. Write the transformation from the coordinates (x, y, z) to the coordinates (x', y', z') as a matrix equation:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \;=\; R_z(\theta) \begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{3}$$

where $R_z(\theta)$ is the rotation matrix. Write out the matrix $R_z(\theta)$ explicitly. What is the size of this matrix?

The notation is a bit more sophisticated now: $R_z(\theta)$ indicates a counterclockwise rotation by angle θ around the z axis.

(h) **[4 pts.]** Show that $R_z(\theta)R_z(-\theta) = I$ and explain why this makes sense.

(i) [**4 pts.**] Show that

$$R_z(\theta)R_z(\phi) \;=\; R_z(\theta+\phi)$$

and interpret this in terms of successive rotations.

(j) [**4 pts.**] Show that the magnitude of the position vector is invariant under the rotation operator, i.e., show that

$$x^2+y^2+z^2 = x'^2+y'^2+z'^2 \qquad \text{or} \qquad r^2 = r'^2$$

Explain why this makes sense physically.

(k) [**2 pts.**] Look up and report the definition of an orthogonal matrix.

(l) [**3 pts.**] Show that the matrix $R_z(\theta)$ is orthogonal.

3. [**7 pts.**] Consider an arbitrary rotation of the coordinate frame, not necessarily around the z axis:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} \;=\; R\begin{pmatrix} x \\ y \\ z \end{pmatrix} \tag{4}$$

Show that if $r^2 = r'^2$, then the matrix R must be orthogonal.