Due on Monday, February 22nd.

Some amount of research (looking up and reading up, wikipedia or textbooks) might be necessary for this problem set.

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1. **2D rotations.**

In the previous problem set we found that rotations of the coordinate frame around the z axis are described by the matrices

$$R_z(\theta) \;=\; \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

If we restrict to rotations around the z axis, we can think of these as 2D rotation matrices. This is a one-parameter family of matrices, parametrized by the rotation angle θ.

(a) **[5 pts.]** Is multiplication of these 2D rotation matrices commutative? Demonstrate using matrix multiplication, and then explain physically why. (If needed, look up what 'commutative' means.)

(b) **[4 pts.]** Look up and report the definition of a GROUP in mathematics (basic abstract algebra).

(c) **[6 pts.]** Show that the 2D rotation matrices form a group under multiplication. Make sure you explicitly demonstrate all aspects of the definition of a group (closure, associativity, identity and inverse.)

(d) **[3 pts.]** Explain whether the group of 2D rotations is abelian or non-abelian. (If necessary, look up what these mean.)

(e) **[2 pts.]** We have found that, under 2D rotation of the coordinate axes, the components of the displacement **r** transform via the matrix given above. Argue why the components of any vector (e.g., velocity, force, angular momentum,...) would transform the same way under the same rotation.

2. **[6 pts.]** What is the determinant of an orthogonal matrix?

   What do positive and negative determinant physically mean for rotation matrices?

   Hint: First find out (or recall) whether the determinant of the product of two matrices is the product of their determinants. Also: what is the determinant of the transpose of a matrix?

3. **Rotations in 3D.**

   (a) **[4 pts.]** By analogy, write the transformation matrix for counterclockwise rotation around the x-axis (rotation of the y-z plane). In other words, write out the 3×3 matrix $R_x(\phi)$.

   (b) **[4 pts.]** Show that $R_z(\theta)$ and $R_x(\phi)$ do not commute.

   (c) **[2 pts.]** The set of all possible 3D rotations (by any angle, around any axis) is known to be a group. Explain whether this group is abelian or non-abelian.

   (d) **[4+3 pts.]** An arbitrary 3D rotation matrix can be constructed by successive rotations around the x-, y- and z- axes. Using this fact, argue that all 3D rotation matrices are orthogonal. Using the same fact, argue that all rotation matrices have unit determinant.

   Hint: First find out whether the product of two orthogonal matrices also orthogonal.

4. (a) **[5 pts.]** If $\mathbf{A}$ and $\mathbf{B}$ are two vectors, show that the scalar product $\mathbf{A} \cdot \mathbf{B}$ is invariant under rotation of coordinate frames, Use the fact that the rotation matrix R is orthogonal.

   (b) **[2 pts.]** We've been using the rotation matrices to describe the rotation of coordinate axes. Is this an *active transformation* or a *passive transformation*? Explain why.