Due on Monday, March 1st.

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In this problem set, we consider only 'standard' Lorentz boosts, i.e, we only consider frames that are coincident at the common zero time, and have relative velocities in the common x direction.

Frame $\tilde{\Sigma}$ moves with velocity $v\hat{i}$ with respect to frame Σ. Here $\hat{i}$ is the unit vector along the common x, $\tilde{x}$ direction. Measurements of the same event from the two frames, (x, y, z, t) and $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$, are related by the LT

$$\tilde{x} = \gamma_v(x - vt) \ ; \quad \tilde{y} = y \ ; \quad \tilde{z} = z \ ; \quad \tilde{t} = \gamma_v(t - vx/c^2) \ .$$

1. (a) **[5 pts.]** Using the LT, derive algebraically the invariance

$$c^2\tilde{t}^2 - \tilde{x}^2 - \tilde{y}^2 - \tilde{z}^2 \ = \ c^2t^2 - x^2 - y^2 - z^2.$$

(In class we derived the LT from the invariance.)

(b) **[1 pt.]** Write down the inverse LT, i.e., (x, y, z, t) in terms of $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$, using the fact that frame Σ moves with velocity $-v\hat{i}$ with respect to frame $\tilde{\Sigma}$.

(c) **[5 pts.]** Invert the LT algebraically to find the inverse LT, i.e., determine (x, y, z, t) in terms of $(\tilde{x}, \tilde{y}, \tilde{z}, \tilde{t})$ algebraically.

(d) **[3 pts.]** Two events have coordinates

$$(x_1, 0, 0, t_1) \quad \text{and} \quad (x_2, 0, 0, t_2) \quad \text{in the } \Sigma \text{ frame;}$$
$$(\tilde{x}_1, 0, 0, \tilde{t}_1) \quad \text{and} \quad (\tilde{x}_2, 0, 0, \tilde{t}_2) \quad \text{in the } \tilde{\Sigma} \text{ frame.}$$

If $\tilde{x}_1 = \tilde{x}_2 = \tilde{X}$, what is the proper time interval between the two events? Explain why.

Relate the time interval between the two events as measured in frame Σ with the time interval as measured in frame $\tilde{\Sigma}$.

(e) **[6 pts.]** In the Σ frame, the points A and B are fixed and have spatial coordinates

$$(x_A = L, y_A = 0, z_A = 0) \quad \text{and} \quad (x_B = 4L, y_B = 4L, z_B = 0) \ .$$

Find the time it takes for a light pulse emitted from point A to reach point B, as measured in frame Σ and as measured in frame $\tilde{\Sigma}$. If you use the LT, specify clearly which events you are using it for.

Is the time measured in one of the frames a *proper* time interval? Explain, which frame, or why not.

(f) **[5 pts.]** In the Σ frame, two events occur at times $t_1 = \dfrac{L}{c}$ and $t_2 = \dfrac{L}{2c}$, and have spatial coordinates

$$(x_1 = 3L, y_1 = 0, z_1 = 0) \quad \text{and} \quad (x_2 = L, y_2 = d, z_2 = 0)$$

respectively. What is the speed v if the two events are found to be simultaneous in frame $\tilde{\Sigma}$?

2. Let's omit the perpendicular directions (y, y', z, z').

   Using the rapidity $\phi = \tanh^{-1}(v/c)$, the LT can be written as

$$\begin{pmatrix} c\tilde{t} \\ \tilde{x} \end{pmatrix} = \begin{pmatrix} \cosh\phi & -\sinh\phi \\ -\sinh\phi & \cosh\phi \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix} = \Lambda(\phi) \begin{pmatrix} ct \\ x \end{pmatrix}$$

   (a) **[3 pts.]** Show that two successive LT's, with rapidities ϕ_1 and ϕ_2, result in a LT of rapidtiy $\phi_1 + \phi_2$. In other words, show via matrix multiplication that $\Lambda(\phi_1)\Lambda(\phi_2) = \Lambda(\phi_1 + \phi_2)$.

   (b) **[4 pts.]** The velocities corresponding to rapidities ϕ_1 and ϕ_2 are v_1 and v_2 respectively. Find the velocity corresponding to $\phi_1 + \phi_2$, in terms of v_1 and v_2. How should we interpret this velocity?

   (c) **[3 pts.]** The LT for rapidity $-\phi$ should be the inverse of the LT for rapidity ϕ, i.e., we should have $\Lambda(-\phi) = [\Lambda(\phi)]^{-1}$. Show by matrix multiplication that $\Lambda(-\phi)\Lambda(\phi)$ is indeed the identity matrix.

   (d) **[6 pts.]** Show that the set of Lorentz boosts in the x direction, represented by the matrices Λ, form a group under successive application (i.e., under matrix multiplication).

   (e) **[2 pts.]** Is this group (Lorentz boosts in the same direction) abelian or non-abelian? Explain why.

   (f) **[4 pts.]** Remember that rotation matrices are orthogonal. Check whether the Λ matrices are orthogonal. Explain why, or why not, in terms of an invariant.

   (g) **[3 pts.]** Remember that rotation matrices have unit determinant. Check whether the Lorentz boost matrices Λ have unit determinant.