Some solutions/hints for problem set 05.

I didn't proofread carefully, so please be careful about typos.

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1. Consider inertial frames Σ and Σ' that are aligned at time $t = t' = 0$. The relative velocity of Σ' with respect to Σ is $\vec{v}$, not necessarily aligned with one of the axes. The transformation from $(t, \vec{r})$ to $(t', \vec{r}')$ is

$$t' \;=\; \gamma_v \left(t - \frac{\vec{v}\cdot\vec{r}}{c^2} \right) \;; \qquad \vec{r}' \;=\; \vec{r} \;+\; \alpha_v (\vec{v}\cdot\vec{r})\vec{v} - \gamma_v \vec{v} t$$

where $v = |\vec{v}|$ and $\alpha_v = \dfrac{\gamma_v - 1}{v^2} = \dfrac{\gamma_v^2/c^2}{\gamma_v + 1}$.

(a) Derive these transformation equations, by generalizing the standard form (for $\vec{v} = v\hat{i}$) used previously. It is helpful to decompose the position vector as $\vec{r} = \vec{r}_\parallel + \vec{r}_\perp$, parallel and perpendicular to $\vec{v}$, so that $\vec{r}_\parallel = \dfrac{(\vec{v}\cdot\vec{r})}{v^2}\vec{v}$, and then to start by writing the equations for $(\vec{r}_\parallel)'$ and $(\vec{r}_\perp)'$ separately.

(Partial) Solution/Hint →

Defining $\vec{r} = \vec{r}_\parallel + \vec{r}_\perp$, we note that $\vec{r}_\parallel$ should take the role of $x\hat{i}$ in the standard form. Thus the standard-form LT for time

$$t' = \gamma_v (t - vx/c^2) = \gamma_v (t - \vec{v}\cdot(x\hat{i})/c^2)$$

is replaced by

$$t' = \gamma_v (t - \vec{v}\cdot\vec{r}_\parallel/c^2) = \gamma_v (t - \vec{v}\cdot\vec{r}/c^2)$$

as claimed. The transformations for the spatial coordinates are

$$x' = \gamma_v (x - vt) \qquad \text{or} \qquad x'\hat{i} = \gamma_v (x\hat{i} - \vec{v}t)$$
$$y' = y \qquad z' = z$$

in the standard form. Therefore, for $\vec{v}$ in arbitrary direction we have

$$\vec{r}'_\parallel = \gamma_v (\vec{r}_\parallel - \vec{v}t)$$
$$\vec{r}'_\perp = \vec{r}_\perp$$

Adding, we get

$$\vec{r}' \;=\; \vec{r}_\perp + \gamma_v(\vec{r}_\parallel - \vec{v}t) \;=\; \vec{r} + (\gamma_v - 1)\vec{r}_\parallel - \gamma_v \vec{v}t$$
$$=\; \vec{r} + (\gamma_v - 1)\frac{(\vec{v}\cdot\vec{r})}{v^2}\vec{v} - \gamma_v\vec{v}t \;=\; \vec{r} + \alpha_v(\vec{v}\cdot\vec{r})\vec{v} - \gamma_v\vec{v}t$$

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(b) Show that $c^2t^2 - \vec{r}\cdot\vec{r}$ is invariant under this transformation.

(Partial) Solution/Hint →

I omit subscript v and write γ and α for this problem, since there is only one speed relevant for these functions. In problems involving multiple velocities, one should be more careful about the argument of γ.

$$
\begin{aligned}
& c^2t'^2 - \vec{r}'\cdot\vec{r}' \\
&\quad = c^2\gamma^2\left(t - \frac{\vec{v}\cdot\vec{r}}{c^2}\right)^2 - [\vec{r} + \alpha(\vec{v}\cdot\vec{r})\vec{v} - \gamma\vec{v}t]\cdot[\vec{r} + \alpha(\vec{v}\cdot\vec{r})\vec{v} - \gamma\vec{v}t] \\
&\quad = \left[c^2\gamma^2t^2 - 2\gamma^2(\vec{v}\cdot\vec{r})t + \frac{\gamma^2}{c^2}(\vec{v}\cdot\vec{r})^2\right] \\
&\quad - \left[\vec{r}\cdot\vec{r} + \alpha^2(\vec{v}\cdot\vec{r})^2v^2 + \gamma^2v^2t^2 + 2\alpha(\vec{v}\cdot\vec{r})^2 - 2\gamma(\vec{v}\cdot\vec{r})t - 2\alpha\gamma(\vec{v}\cdot\vec{r})v^2t\right] \\
&\quad = \left[\gamma^2(c^2-v^2)t^2 - \vec{r}\cdot\vec{r}\right] + \left[-\gamma^2 + \gamma + \alpha\gamma v^2\right]2(\vec{v}\cdot\vec{r})t \\
&\qquad + \left[\frac{\gamma^2}{c^2} - 2\alpha - \alpha^2v^2\right](\vec{v}\cdot\vec{r})^2
\end{aligned}
$$

The first group is equal to $c^2t^2 - \vec{r}\cdot\vec{r}$, because $\gamma^2(c^2-v^2) = c^2$. The other two terms disappear, because:

$$
-\gamma^2 + \gamma + \alpha\gamma v^2 \;=\; -(\gamma^2-\gamma) + \frac{\gamma^2-\gamma}{v^2}v^2 \;=\; 0
$$

and

$$
\begin{aligned}
& \frac{\gamma^2}{c^2} - 2\alpha - \alpha^2v^2 \;=\; \frac{\gamma^2}{c^2} - \frac{2\gamma^2/c^2}{\gamma+1} + \left(\frac{\gamma^2/c^2}{\gamma+1}\right)^2 c^2\left(1 - \frac{1}{\gamma^2}\right) \\
&= \frac{\gamma^2}{c^2}\left(1 - \frac{2}{\gamma+1}\right) - \frac{\gamma^4/c^2}{(\gamma+1)^2}\frac{\gamma^2-1}{\gamma^2} \;=\; \frac{\gamma^2}{c^2}\left(\frac{\gamma-1}{\gamma+1}\right) - \frac{\gamma^2}{c^2}\left(\frac{\gamma-1}{\gamma+1}\right) \;=\; 0
\end{aligned}
$$

Thus

$$
c^2t'^2 - \vec{r}'\cdot\vec{r}' \;=\; c^2t^2 - \vec{r}\cdot\vec{r}
$$

i.e.,

$c^2t^2 - \vec{r}\cdot\vec{r} \;=\; c^2t^2 - x^2 - y^2 - z^2$ is an invariant for Lorentz boosts in any direction, not only for boosts in the x-direction.

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(c) Expressing $\vec{v} = (v_i, v_2, v_3)$ and $\vec{r} = (x, y, z)$ in Cartesian components, express the transformation as a matrix equation, i.e., find the matrix Λ that transforms from (ct, x, y, z) to (ct', x', y', z').

(Partial) Solution/Hint →

Breaking up into components (shorthanding $\gamma = \gamma_v$ and $\alpha = \alpha_v$) gives

$$\begin{aligned} ct' &= \gamma(ct) - \gamma\left(\frac{v_1}{c}\right)x - \gamma\left(\frac{v_2}{c}\right)y - \gamma\left(\frac{v_3}{c}\right)z \\ x' &= x + \alpha(v_1 x + v_2 y + v_3 z)v_1 - \gamma v_1 t \\ &= -\gamma\left(\frac{v_1}{c}\right)(ct) + (1 + \alpha v_1^2)x + \alpha v_1 v_2 y + \alpha v_1 v_3 z \end{aligned}$$

and y' and z' can be written similarly. The matrix is (wriiten with $c = 1$)

$$\begin{pmatrix} \gamma & -\gamma v_1 & -\gamma v_2 & -\gamma v_3 \\ -\gamma v_1 & 1 + \alpha v_1^2 & \alpha v_1 v_2 & \alpha v_1 v_3 \\ -\gamma v_2 & \alpha v_1 v_2 & 1 + \alpha v_2^2 & \alpha v_2 v_3 \\ -\gamma v_3 & \alpha v_1 v_3 & \alpha v_2 v_3 & 1 + \alpha v_3^2 \end{pmatrix}$$

To put c back again, replace all γv_i by $\gamma \frac{v_i}{c}$.

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(d) Find out whether the matrix Λ is *symmetric* or not. Contrast with rotation matrices.

(Partial) Solution/Hint →

Symmetric.

This is the most general Lorentz boost. Hence Lorentz boost matrices are **always** symmetric.

In contrast, rotation matrices are NOT symmetric. (In fact, check that the matrix for rotations around the z axis are even anti-symmetric.)

The fact that boost matices are symmetric but rotation matrices are not, is helpful in showing that the set of all boosts does not satisfy **closure**. (Why?)

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(e) Show that the transformation matrix satisfies

$$\Lambda^T g \Lambda = g \; , \qquad \text{where} \quad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

is called the metric tensor or Minkowski metric. This is somewhat messy, so it's okay if you prove an easier version by setting $v_3 = 0$.

(Partial) Solution/Hint →

This is a long and messy, but straightforward, calculation. I will not type it up.
Note that, if you set $v_3 = 0$, you can do a multiplication of 3×3 matrices instead of 4×4 matrices.

The relation $\Lambda^T g \Lambda = g$ is important: it is the condition for preserving the invariant interval in the transformation. In fact, this equation is (usually) taken as the definition of Lorentz transformations, although this definition includes some non-physical transformations, such as improper rotations and non-orthochronous transformations.

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2. The relationship $\Lambda^T g \Lambda = g$ is regarded as the definition of Lorentz transformations. (A matrix Λ satisfying this relationship describes a Lorentz transformation.)

   (a) If the matrix Λ is a Lorentz transformation, show that it has determinant of unit magnitude.

   (Partial) Solution/Hint →

   Using the fact that $\det(g) = 1$ and $\det(M^T) = \det(M)$,

$$\Lambda^T g \Lambda = g \quad \Longrightarrow \quad \det(\Lambda^T g \Lambda) = \det(g)$$
$$\Longrightarrow \quad \det(\Lambda^T)\det(g)\det(\Lambda) = 1 \quad \Longrightarrow \quad \det(\Lambda)^2 = 1$$
$$\Longrightarrow \quad \det(\Lambda) = \pm 1$$

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(b) Look up the topic of sign conventions for the Minkowski metric. Write down the matrix g in the other common convention.

(Partial) Solution/Hint →

There are two common conventions. In the other convention, the Minkowski metric is

$$g = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & +1 \end{pmatrix}$$

One can describe our matric as the metric with trace -2 and the second metric as that with trace +2. One also reads about the $(+ - --)$ convention versus the $(- + ++)$ convention.

Our convention $(+---)$ means that we use $c^2dt^2 - d\vec{r}\cdot d\vec{r}$ as the invariant interval, while the other convention $(- + ++)$ corresponds to using $d\vec{r}\cdot d\vec{r} - c^2dt^2$ as the invariant interval. There is no physical difference.

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3. The following problems are about relativistic velocity addition.

   (a) Two rockets approach each other, as observed from earth, each with speed u. What is the relative speed of one rocket as seen from the other?

   (Partial) Solution/Hint →

   This is longitudinal (one-dimensional) velocity addition.

$$\frac{u+u}{1+\frac{u\cdot u}{c^2}} = \frac{2u}{1+u^2/c^2}$$

   Question: What if the speed of each rocket is $\frac{3}{4}c$. Would the relative speed be larger than the speed of light?

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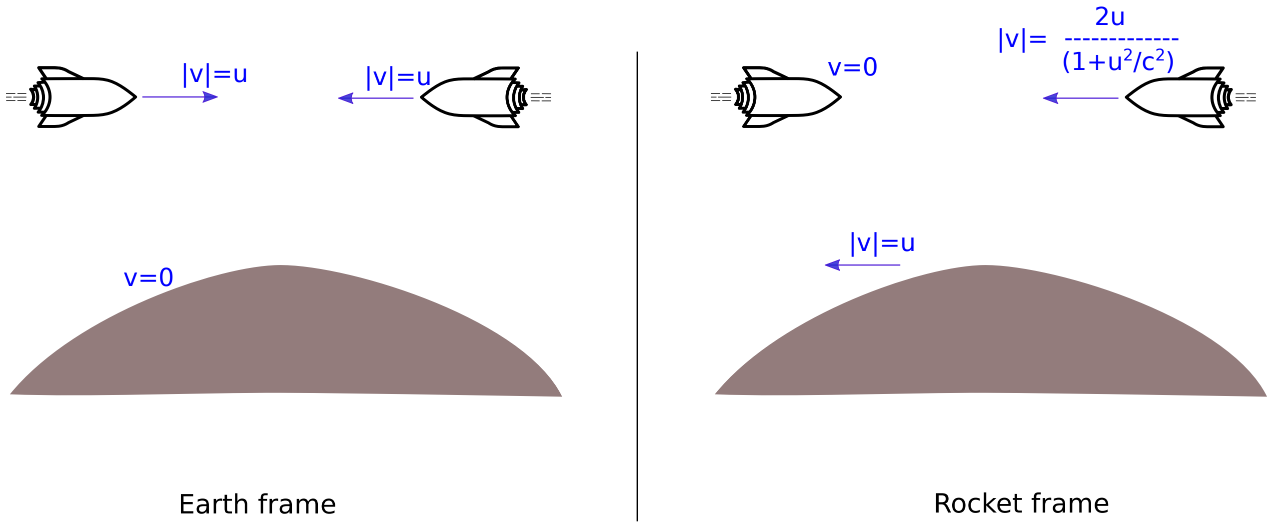
   (b) Sketch the situation considered in 3a from the earth's frame and from the frame of one of the rockets. In both figures, mark each object (earth, first rocket, second rocket) with an arrow and expression, representing the direction and magnitude of the velocity of that object.

   (Partial) Solution/Hint →



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(c) A car moves leftward at speed v, while a light pulse moves rightward with speed c toward the car. Use the velocity addition formula to find out, from the perspective of the car driver, how fast the light pulse approaches. Explain why the answer is expected.

(Partial) Solution/Hint →

Longitudinal (one-dimensional) velocity addition; no transverse components involved.

$$\frac{v+c}{1+\frac{v\cdot c}{c^2}} = \frac{v+c}{1+v/c} = c$$

This is expected, because the speed of light is supposed to be the same no matter which frame the observer is in.

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(d) Two rockets P and Q approach each other on a collision course, moving (relative to the moon) at speeds $\frac{3}{5}c$ and $\frac{2}{5}c$ respectively. Find the speed of rocket Q as observed by an occupant of P.

(Partial) Solution/Hint →

$$\frac{v_Q+v_P}{1+\frac{v_Q v_P}{c^2}} = \frac{\frac{3c}{5}+\frac{2c}{5}}{1+\frac{\frac{3c}{5}\frac{2c}{5}}{c^2}} = \frac{c}{1+6/25} = \frac{25}{31}c$$

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(e) Spacecrafts A and B are traveling in the *same* direction with speeds $4c/5$ and $3c/5$ respectively, as seen from earth. (A is chasing B from behind.) Find the speed of A as seen from B.

(Partial) Solution/Hint →

$$\frac{|v_A|-|v_B|}{1-\frac{|v_A v_B|}{c^2}} = \frac{\frac{4c}{5}-\frac{3c}{5}}{1-\frac{\frac{4c}{5}\frac{3c}{5}}{c^2}} = \frac{c/5}{1-12/25} = \frac{5}{13}c$$

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(f) Following from previous problem: Spacecraft C is between A and B, and traveling in the same direction. The passengers on C see the other two spacecrafts approaching C from opposite directions at the same speed. What is the speed of C as seen from earth? [Hint: You will need to solve a quadratic equation. Only one solution of the equation is physically acceptable; mention why.]

(Partial) Solution/Hint →

$$v_{AC} = \frac{|v_A| - |v_C|}{1 - \dfrac{|v_A v_C|}{c^2}} = \frac{\dfrac{4c}{5} - |v_C|}{1 - \dfrac{\frac{4c}{5}|v_C|}{c^2}} = \left(\frac{\dfrac{4}{5} - \beta}{1 - \dfrac{4}{5}\beta}\right) c \qquad \text{where } \beta = |v_C| c$$

$$v_{BC} = \frac{|v_B| - |v_C|}{1 - \dfrac{|v_B v_C|}{c^2}} = \frac{\dfrac{3c}{5} - |v_C|}{1 - \dfrac{\frac{3c}{5}|v_C|}{c^2}} = \left(\frac{\dfrac{3}{5} - \beta}{1 - \dfrac{3}{5}\beta}\right) c = \left(\frac{3 - 5\beta}{5 - 3\beta}\right) c$$

C sees both A and B approach her at the same speed, hence $v_{AC} = -v_{BC}$:

$$\left(\frac{4 - 5\beta}{5 - 4\beta}\right) c = -\left(\frac{3 - 5\beta}{5 - 3\beta}\right) c$$

$$\implies (4 - 5\beta)(5 - 3\beta) + (5 - 4\beta)(3 - 5\beta) = 0$$

$$\implies 35\beta^2 - 74\beta + 35 = 0$$

$$\implies 35\beta^2 - 25\beta - 49\beta + 35 = 0$$

$$\implies 5\beta(7\beta - 5) - 7\beta(7\beta - 5) = 0$$

$$\implies (7\beta - 5)(5\beta - 7) = 0 \qquad \beta = \frac{5}{7} \text{ or } \beta = \frac{7}{5}$$

which means $v_C = \frac{5}{7}c$ or $v_C = \frac{7}{5}c$. The second possibility is unacceptable because physical speeds are always $< c$.

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(g) A rocket M moves at speed u directly away from earth. It emits a bullet at speed v perpendicular to the direction of its motion. Measured with respect to earth, what are the velocity components of the bullet? Measured with respect to earth, what is the speed of the bullet?

(Partial) Solution/Hint →

The bullet has velocity u in the direction of the rocket motion and $\dfrac{v}{\gamma_u}$ in the perpendicular direction. Hence speed is

$$\sqrt{u^2 + \left(\frac{v}{\gamma_u}\right)^2}$$

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