Due on Monday, March 8th

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1. Consider inertial frames Σ and Σ' that are coincident at time $t = t' = 0$. The relative velocity of Σ' with respect to Σ is $\vec{v}$, not necessarily aligned with one of the axes. The transformation from $(t, \vec{r})$ to $(t', \vec{r}')$ is

$$t' \;=\; \gamma_v \left(t - \frac{\vec{v}\cdot\vec{r}}{c^2} \right) \;; \qquad \vec{r}' \;=\; \vec{r} \;+\; \alpha_v(\vec{v}\cdot\vec{r})\vec{v} - \gamma_v \vec{v} t$$

where $v = |\vec{v}|$ and $\alpha_v = \dfrac{\gamma_v - 1}{v^2} = \dfrac{\gamma_v^2/c^2}{\gamma_v + 1}$.

   (a) **[9 pts.]** (May be difficult. If unsure, please move on to next questions, which don't depend on having done this one.)
   Derive these transformation equations, by generalizing the standard form (for $\vec{v} = v\hat{i}$) used previously. It is helpful to decompose the position vector as $\vec{r} = \vec{r}_\parallel + \vec{r}_\perp$, parallel and perpendicular to $\vec{v}$, so that $\vec{r}_\parallel = \dfrac{(\vec{v}\cdot\vec{r})}{v^2}\vec{v}$, and then to start by writing the equations for $(\vec{r}_\parallel)'$ and $(\vec{r}_\perp)'$ separately.

   (b) **[SELF]** Show that $c^2t^2 - \vec{r}\cdot\vec{r}$ is invariant under this transformation.

   (c) **[5 pts.]** Expressing $\vec{v} = (v_i, v_2, v_3)$ and $\vec{r} = (x, y, z)$ in Cartesian components, express the transformation as a matrix equation, i.e., find the matrix Λ that transforms from (ct, x, y, z) to (ct', x', y', z').

   (d) **[SELF]** Find out whether the matrix Λ is *symmetric* or not. Contrast with rotation matrices.

   (e) **[6 pts.]** Show that the transformation matrix satisfies

$$\Lambda^T g \Lambda = g \;, \qquad \text{where} \quad g = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

   is called the metric tensor or Minkowski metric. This is somewhat messy, so it's okay if you prove an easier version by setting $v_3 = 0$.

2. The relationship $\Lambda^T g \Lambda = g$ is regarded as the definition of Lorentz transformations. (A matrix Λ satisfying this relationship describes a Lorentz transformation.)

(a) **[5 pts.]** If the matrix Λ is a Lorentz transformation, show that it has determinant of unit magnitude.

(b) **[2 pts.]** Look up the topic of sign conventions for the Minkowski metric. Write down the matrix g in the other common convention.

3. The following problems are about relativistic velocity addition.

(a) **[5 pts.]** Two rockets approach each other, as observed from earth, each with speed u. What is the relative speed of one rocket as seen from the other?

(b) **[0 pts.]** Sketch the situation considered in 3a from the earth's frame and from the frame of one of the rockets. In both figures, mark each object (earth, first rocket, second rocket) with an arrow and expression, representing the direction and magnitude of the velocity of that object.

(c) **[SELF]** A car moves leftward at speed v, while a light pulse moves rightward with speed c toward the car. Use the velocity addition formula to find out, from the perspective of the car driver, how fast the light pulse approaches. Explain why the answer is expected.

(d) **[SELF]** Two rockets P and Q approach each other on a collision course, moving (relative to the moon) at speeds $\frac{3}{5}c$ and $\frac{2}{5}c$ respectively. Find the speed of rocket Q as observed by an occupant of P.

(e) **[4 pts.]** Spacecrafts A and B are traveling in the *same* direction with speeds $4c/5$ and $3c/5$ respectively, as seen from earth. (A is chasing B from behind.) Find the speed of A as seen from B.

(f) **[7 pts.]** Following from previous problem (3e): Spacecraft C is between A and B, and traveling in the same direction. The passengers on C see the other two spacecrafts approaching C from opposite directions at the same speed. What is the speed of C as seen from earth? [Hint: You will need to solve a quadratic equation. Only one solution of the equation is physically acceptable; mention why.]

(g) **[7 pts.]** A rocket M moves at speed u directly away from earth. It emits a probe N at speed v perpendicular to the direction of its motion. Measured with respect to earth, what are the velocity components of the probe N? Measured with respect to earth, what is the speed of the probe?