No guarantees, be careful about typos; I didn't proofread carefully.
As usual, I am very grateful when informed about typos or other errors!

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1. A subatomic particle lives for 6×10^{-7} seconds, i.e., it decays 6×10^{-7} seconds after it is created, as seen from its own reference frame. This particle is created at a speed $v = \frac{4}{5}c$ with respect to earth,

(a) How long does it live, according to the earth frame? Specify clearly: in which frame is the lifetime the 'proper' time?

(Partial) Solution/Hint →

The rest frame of the particle, i.e., the frame in which the particle is at rest, is the frame where the lifetime is the 'proper' lifetime. This is the frame where the two events (creation of the particle and the destruction of the particle) happen at the same position.

Time intervals measured in other frames are always dilated compared to the proper lifetime. Thus, in the earth frame, which is moving with speed $v = \frac{4}{5}c$ with respect to the rest frame of the particle, the lifetime will appear to be dilated to

$$\begin{aligned} \gamma_v \times \text{proper lifetime} &= \frac{1}{\sqrt{1-(4/5)^2}} \times 6 \times 10^{-7}\text{seconds} \\ &= \frac{5}{3} \times 6 \times 10^{-7}\text{seconds} \;=\; 10^{-6}\text{seconds} \end{aligned}$$

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(b) How far will the particle travel before it decays, as seen from the earth frame?

(Partial) Solution/Hint →

Relative to the earth frame, the speed is $v = \frac{4}{5}c$ and the lifetime is 10^{-6} seconds. Hence the distance covered is

$$\begin{aligned} \text{speed} \times \text{lifetime} &= \frac{4}{5}c \times 10^{-6}\text{s} \;=\; 8 \times 10^{-7}\text{s} \times c \\ &\approx 8 \times 10^{-7}\text{s} \times 3 \times 10^{8}\text{m/s} \;=\; 240\text{m} \end{aligned}$$

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(c) Imagine that the Galilean transformations were true instead of the relativistic (Lorentz) transformations. If this were the case, how far would the particle travel before it decays, as seen from the earth frame?

(Partial) Solution/Hint →

If the Galilean transformations were correct, there would be no time dilation, so the lifetime would be 6×10^{-7} seconds in the earth frame as well. Hence the distance traveled would have been

$$\text{speed} \times \text{lifetime} \;=\; \frac{4}{5}c \times 6 \times 10^{-7}\text{s} \;=\; 4.8 \times 10^{-7}\text{s} \times c$$
$$\approx\; 4.8 \times 10^{-7}\text{s} \times 3 \times 10^{8}\text{m/s} \;=\; 144\text{m}$$

Galilean transformations predict a larger distance compared to the prediction of Lorentz transformations. This observation is the basis for one of the early experimental verifications of special relativity. See the wikipedia article on "*Experimental testing of time dilation*." The first section (titled "*Atmospheric Testing*") explains in some detail.

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2. Frame S' moves at speed v_1 with respect to frame S, and frame S'' moves at speed v_2 with respect to frame S', both in the common x, x', x'' direction. The Lorentz transformations are

$$x' = \gamma_1(x - v_1 t)\ , \quad t' = \gamma_1(t - v_1 x/c^2) \qquad \text{with} \quad \gamma_1 = \gamma(v_1) = \frac{1}{\sqrt{1 - v_1^2/c^2}}$$
$$x'' = \gamma_2(x' - v_2 t')\ , \quad t'' = \gamma_2(t' - v_2 x'/c^2) \qquad \text{with} \quad \gamma_2 = \gamma(v_2) = \frac{1}{\sqrt{1 - v_2^2/c^2}}$$

(a) Express x'' (the coordinate of the S'' frame) in terms of x and t, in the form $x'' = Ax + Bt$. Express A and B in terms of v_1, v_2, γ_1 and γ_2.

(Partial) Solution/Hint →

$$x'' = \gamma_2(x' - v_2 t') = \gamma_2\left[\gamma_1(x - v_1 t) - v_2\gamma_1(t - v_1 x/c^2)\right]$$
$$= \left(\gamma_1\gamma_2 + \frac{\gamma_1\gamma_2 v_1 v_2}{c^2}\right)x + (-\gamma_1\gamma_2 v_1 - \gamma_1\gamma_2 v_2)t$$

Thus
$$A = \gamma_1\gamma_2\left(1 + \frac{v_1 v_2}{c^2}\right) \qquad \text{and} \qquad B = -\gamma_1\gamma_2(v_1 + v_2)$$

— * —

(b) If frame S'' moves at speed u with respect to frame S, you would have $x'' = \gamma(u)(x - ut)$. Comparing with $x'' = Ax + Bt$ and using your expressions for A and B, find u. Have you recovered the relativistic velocity addition formula?

(Partial) Solution/Hint →

Comparing $x'' = \gamma(u)(x - ut)$ and $x'' = Ax + Bt$, we see that

$$A = \gamma(u) \qquad \text{and} \qquad u = -\frac{B}{A}$$

Using the expressions for B and A we found above,

$$u = -\frac{B}{A} = -\frac{-\gamma_1\gamma_2(v_1 + v_2)}{\gamma_1\gamma_2\left(1 + \frac{v_1 v_2}{c^2}\right)} = \frac{v_1 + v_2}{\left(1 + \frac{v_1 v_2}{c^2}\right)}$$

Thus we have recovered the relativistic velocity addition formula.
Suggestion: explain carefully, perhaps with a sketch: why should u be given by the relativistic addition of velocities v_1 and v_2?

— * —

(c) Verify that your constant A is really equal to $\gamma(u)$, with the u known from the velocity addition formula.

(Partial) Solution/Hint →

We have found that $A = \gamma_1\gamma_2\left(1 + \frac{v_1 v_2}{c^2}\right)$. We have to show that this is the same as

$$\gamma(u) = \gamma\left(\frac{v_1 + v_2}{\left(1 + \frac{v_1 v_2}{c^2}\right)}\right)$$

Let's make life easier by using $c = 1$ units. Then

$$\gamma(u) \;=\; \gamma\left(\frac{v_1+v_2}{1+v_1v_2}\right) \;=\; \frac{1}{\sqrt{1-\left(\frac{v_1+v_2}{1+v_1v_2}\right)^2}}$$

$$= \frac{1+v_1v_2}{\sqrt{(1+v_1v_2)^2-(v_1+v_2)^2}} \;=\; \frac{1+v_1v_2}{\sqrt{1+v_1^2v_2^2-v_1^2-v_2^2}}$$

$$= \frac{1+v_1v_2}{\sqrt{(1-v_1^2)(1-v_2^2)}} \;=\; \frac{1}{\sqrt{(1-v_1^2)}}\frac{1}{\sqrt{(1-v_2^2)}}(1+v_1v_2) \;=\; \gamma_1\gamma_2(1+v_1v_2) \;=\; A$$

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3. Some groups. This problem requires some searching and reading.

   Look up the terms $SO(2)$, $SO(3)$, $SO(3,1)$, $SU(2)$. Each of these groups can be thought of as a set of matrices.

(a) What does 'S' in these names stand for?
Which matrices are included in the group $O(3)$, but not in the group $SO(3)$?

(Partial) Solution/Hint →

'S' stands for Special. For example, $SO(2)$ means special orthogonal matrices of size 2×2. Orthogonal matrices can have determinant $+1$ or -1. Special means only those with determinant $+1$ are included.

Orthogonal 3×3 matrices with determinant -1 are included in the group $O(3)$, but not in the group $SO(3)$.

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(b) Which physical transformations are represented by the groups $SO(2)$ and $SO(3)$? Are these groups abelian or non-abelian?

(Partial) Solution/Hint →

$SO(2)$ represents proper 2D rotations, i.e., rotations which do not include reflections. This is an abelian group, as we have found in a previous assignment.

$SO(3)$ represents proper 3D rotations, i.e., rotations which do not include reflections. This is a non-abelian group. Two arbitrary 3D rotations, if performed in opposite order, give different final results.

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(c) What is the physical interpretation of matrices which are included in the group $O(3)$, but not in the group $SO(3)$?

(Partial) Solution/Hint →

These represent rotations plus a reflection of axes. Due to the reflection, the resulting system does not have a right-handed coordinate system.

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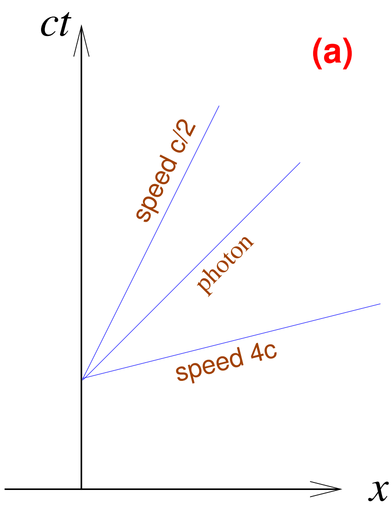
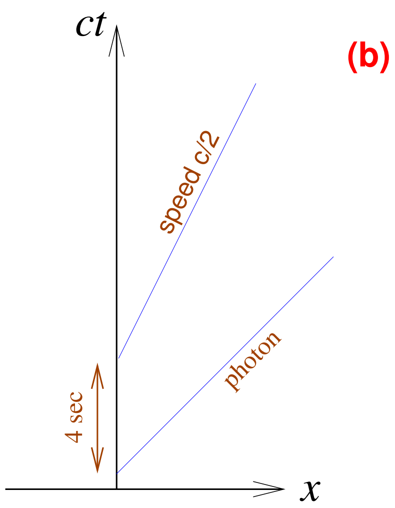
4. Spacetime diagrams (plots of ct versus x).

(a) On a spacetime diagram, draw the worldline of a photon starting at the origin, the worldline of an object with speed $c/2$ starting at the same time at the origin, and the worldline of an object with speed $4c$ starting at the same time at the origin. Which worldline is unphysical?

(b) On a spacetime diagram, draw the worldline of a photon starting at the origin, and the worldline of an object with velocity $c/2$ starting at the origin four seconds later.

(Partial) Solution/Hint →





In (a), the worldline with slope less than 1 (speed = $4c$ larger than c) is unphysical.

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5. Frame Σ' moves at speed v with respect to frame Σ, in the common x, x' direction. The two frames are aligned at time $t = t' = 0$.

(a) A particle has velocity $\vec{u}' = (0, c, 0)$ relative to Σ'. Use the velocity addition formulae to find the velocity of the particle relative to Σ.
Explain how your result is consistent with the constancy of the speed of light.

(Partial) Solution/Hint →

The velocity components relative to Σ are $\left(v, \dfrac{c}{\gamma_v}, 0\right)$.
The particle moves with the speed of light relative to Σ', hence is a photon. For consistency with the postulates of relativity, its speed should be c relative to Σ as well. Using the velcity components, the speed relative to Σ is seen to be

$$\sqrt{v^2 + \left(\frac{c}{\gamma_v}\right)^2} = \sqrt{v^2 + c^2\left(1 - \frac{v^2}{c^2}\right)^2} = \sqrt{c^2} = c$$

Hence consistent with the postulate of constancy of the speed of light.

— * —

(b) A stick is at rest in frame Σ. It lies in the xy plane and makes angle $\pi/4$ with the x-axis. Using the Lorentz transformations or the length contraction formula, find the angle that the stick makes with the x' axis, as observed from the Σ' frame.

(Partial) Solution/Hint →

If the length of the stick is L, it's ends can be taken to be at $(0, 0)$ and $(L/\sqrt{2}, L/\sqrt{2})$ in frame Σ. In other words, the displacement between one end and other end has x-component $L/\sqrt{2}$ and y-component $L/\sqrt{2}$. Note this is the rest frame of the stick, and these lengths are proper lengths.
Measured from Σ', the y-component is unchanged, but the x-component is

$$\frac{L/\sqrt{2}}{\gamma_v} = L/(\sqrt{2}\gamma_v)$$

using the length contraction formula. Hence the angle with the x' axis is

$$\tan^{-1}\left(\frac{y\text{-component}}{x\text{-component}}\right) = \tan^{-1}\left(\frac{L/\sqrt{2}}{L/(\sqrt{2}\gamma_v)}\right) = \tan^{-1}\gamma_v$$

Since $\gamma_v > 1$, the angle is larger than $\pi/4$.

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6. Show that, if the 4×4 transformation matrix Λ acting on 4-vectors (ct, x, y, z) preserves the Minkowski norm $c^2t^2 - x^2 - y^2 - z^2$, then it must satisfy $\Lambda^T g \Lambda = g$. Here g is the metric tensor defined in Problem Set 05.

(Partial) Solution/Hint →

First note that

$$c^2t^2 - x^2 - y^2 - z^2 \;=\; \begin{pmatrix} ct & x & y & z \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}$$

or, calling the 4-vector X,

$$c^2t^2 - x^2 - y^2 - z^2 \;=\; X^T g X$$

If this quantity stays invariant under the transformation $X' = \Lambda X$, then for any 4-vector X we must have

$$\begin{aligned} X'^T g X' &= X^T g X \\ \Longrightarrow \quad (\Lambda X)^T g (\Lambda X) &= X^T g X \\ \Longrightarrow \quad (X^T \Lambda^T) g (\Lambda X) &= X^T g X \\ \Longrightarrow \quad X^T (\Lambda^T g \Lambda) X &= X^T g X \end{aligned}$$

Since this is true for every four-vector X, we must have

$$\boxed{\Lambda^T g \Lambda \;=\; g}$$

This important relationship defines the Lorentz group, any 4×4 matrix which satisfies this is a member of the Lorentz group. However, this group includes some transformations which are not very physical.

In addition, this important equation is also the **definition of a Lorentz transformation**: any transformation matrix Λ satisfying this relation represents a Lorentz transformation.

Exercise: Using this definition,
(1) Figure out whether a rotation matrix is a Lorentz transformation.
(2) Show that the standard boost is a Lorentz transformation.

— * —

7. Frame Σ' moves at speed v with respect to frame Σ, in the common x, x' directions. The two frames are aligned at time $t = t' = 0$.

(a) Write down the Galilean transformation as a matrix transforming the Σ-frame coordinates (ct, x) to the Σ' coordinates (ct', x'). You can ignore the transverse directions and work with 2×2 matrices.

(Partial) Solution/Hint →

Galilean transformations:

$$t' = t\,; \quad x' = x - vt\,; \quad y' = y\,; \quad z' = z\,. \qquad \text{Omit latter two.}$$

Writing in terms of ct, ct' instead of t, t':

$$ct' = ct\,; \quad x' = x - \frac{v}{c}(ct)\,.$$

In matrix form

$$\begin{pmatrix} ct' \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -v/c & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \end{pmatrix}$$

— * —

(b) Write down the Lorentz transformation matrix. Take the limit $c \to \infty$, and show that this fails to reproduce the Galiliean transformation matrix.

(Partial) Solution/Hint →

The LT matrix is

$$\begin{pmatrix} \gamma_v & -\gamma_v\,(v/c) \\ -\gamma_v\,(v/c) & \gamma_v \end{pmatrix}$$

In the limit $c \to \infty$, we get $\gamma_v \to 1$ and $\gamma_v(v/c) \to 0$. So

$$\begin{pmatrix} \gamma_v & -\gamma_v\,(v/c) \\ -\gamma_v\,(v/c) & \gamma_v \end{pmatrix} \xrightarrow{c\to\infty} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Thus taking the limit turns the LT matrix into the unit matrix, not the Galilean transformation matrix.

The problem is that one of the coordinates (ct or ct') does not make sense in this limit.

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(c) The reason for the failure above is that in the $c \to \infty$ limit, the variables ct and ct' lose meaning.

So, write down the Galilean transformation as a matrix transforming the Σ-frame coordinates (t, x) to the Σ' coordinates (t', x'). i.e., the first coordinate does not have the c factor. Write down the Lorentz transformation the same way, as a matrix transforming (t, x) to (t', x').

Now show that the Lorentz transformation matrix reduces to the Galiliean transformation matrix in the limit $c \to \infty$.

(Partial) Solution/Hint →

In these variables (without the c factor for time), the GT is

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}$$

and the LT is

$$\begin{pmatrix} t' \\ x' \end{pmatrix} = \begin{pmatrix} \gamma_v & -\gamma_v v/c^2 \\ -\gamma_v v & \gamma_v \end{pmatrix} \begin{pmatrix} t \\ x \end{pmatrix}.$$

Taking the $c \to \infty$ limit now gives

$$\begin{pmatrix} \gamma_v & -\gamma_v v/c^2 \\ -\gamma_v v & \gamma_v \end{pmatrix} \xrightarrow{c\to\infty} \begin{pmatrix} 1 & 0 \\ -v & 1 \end{pmatrix}$$

i.e., the LT reduces to the GT as expected.

— * —

(d) Considering two successive Galilean transformations in the same x direction, with velocities v_1 and v_2, show by matrix multiplication that their combination is also a Galilean transformation, with velocity $v_1 + v_2$.

(Partial) Solution/Hint →

$$\begin{pmatrix} 1 & 0 \\ -v_2/c & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -v_1/c & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -(v_1+v_2)/c & 1 \end{pmatrix}$$

It looks simpler if we use $c = 1$ units:

$$\begin{pmatrix} 1 & 0 \\ -v_1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ -v_2 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -(v_1+v_2) & 1 \end{pmatrix}$$

— * —

8. This problem continues from question 3.

   (a) How many matrices does $SO(2)$ contain?

   (Partial) Solution/Hint →

   An infinite number. Since 2D rotation matrices, i.e., members of $SO(2)$, can be parametrized as

$$\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} ,$$

   we see that every value of $\theta \in [0, 2\pi)$ is a different matrix, i.e., a different element of $SO(2)$.

   Thus $SO(2)$ is an infinite group parametrized by one angle variable, i.e., a compact variable.

   Similarly, $SO(3)$ is an infinite group parametrized by three angle variables.

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   (b) The group $SO(3, 1)$ is the proper Lorentz group.

   What is the size of matrices belonging to this group?

   These matrices are not orthogonal, but must instead satisfy a similar equation. Write down the equation that defines membership in the Lorentz group.

   Under what condition is a member of this group called *orthochronous*?

   (Partial) Solution/Hint →

   These are 4×4 matrices.

   The equation is $\Lambda^T g \Lambda = g$.

   A Lorentz transformation, i.e., a member of the group $SO(3, 1)$, is called orthochronous if the time-time element, i.e. the diagonal element Λ_{00}, is positive. The elements of Lorentz transfo matrices are usually written with one superscript and one subscript, so the condition for being orthchronous is

$$\Lambda^0_0 \ > \ 0$$

   If $\Lambda^0_0 < 0$, the LT changes the direction/sign of the time coordinate, which is just strange and un-useful. So physical LT's are taken to be orthochronous.

— * —

(c) What is the size of matrices belonging to the group $SO(1,1)$?
Which subset of Lorentz transformations forms this group?

(Partial) Solution/Hint →

The group $SO(1,1)$ consists of 2×2 matrices Λ which satisfy

$$\Lambda^T g_2 \Lambda = g_2$$

The equation looks the same as the 4×4 matrix equation that defining the full Lorentz group, but here g_2 is the 2×2 version of the metric tensor, i.e., $g_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

The matrices of this set have the form $\begin{pmatrix} \cosh\phi & \sinh\phi \\ \sinh\phi & \cosh\phi \end{pmatrix}$, as derived in class. They represent Lorentz boosts in a single direction, for example, the set of boosts in the x direction.

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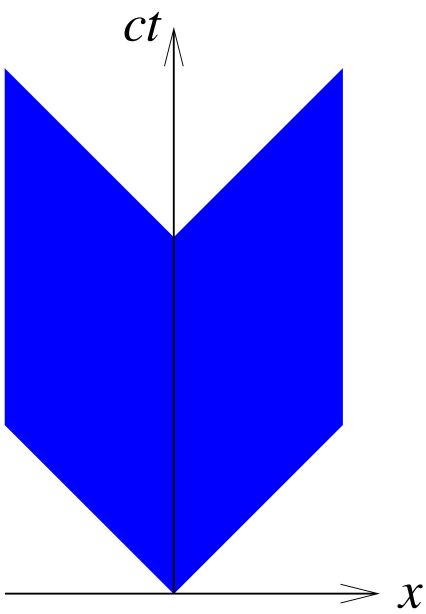
9. Some more spacetime diagrams.

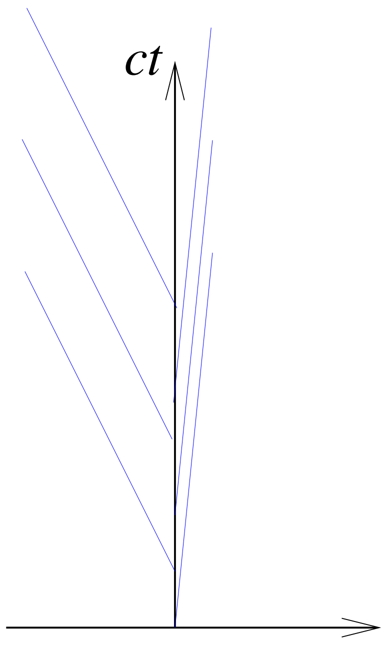
(a) On a spacetime diagram, depict the situation that a light source at $x = 0$ is turned on for a few seconds, and then turned off again. (Show the worldlines of photons.)

(b) A railgun located at the origin shoots projectiles at time intervals of T, alternately in the positive-x and negative-x directions. The projectiles in the positive x direction have speed $c/10$ and those in the negative x direction have speed $c/2$. Draw the situation on a spacetime diagram, showing the worldlines of a number of successive projectiles.

(Partial) Solution/Hint →

"Turned on for a few seconds" means that it is not just one photon, but a stream of photons. Each photon has its own worldline. Together, these lines cover a surface, shown below left.





In your 'railgun' diagram, it should be clear that *smaller* speeds have *larger* slopes. This is because time is on the vertical axis, and space is on the horizontal axis.

This is opposite of how you would choose axes if you were thinking of x as a function of t (which you might have done in introductory mechanics classes). In that case, smaller speeds have smaller slopes.

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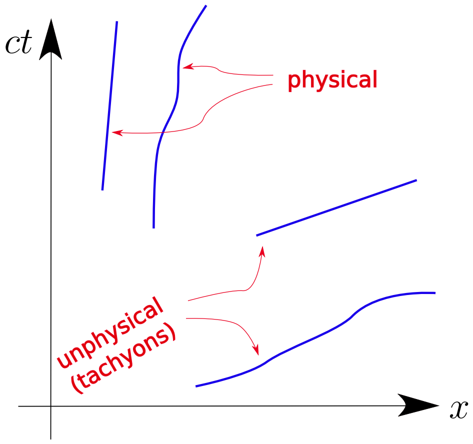
(c) Draw the worldline of a tachyon. (A tachyon is a hypothetical particle that moves faster than the speed of light.)

(Partial) Solution/Hint →



The figure shows a couple of tachyonic worldlines as well as a couple of physical worldlines.

Physical worldlines have slope > 1, reflecting speeds less than c. Tachyonic worldlines have slope < 1, reflecting speeds larger than c.

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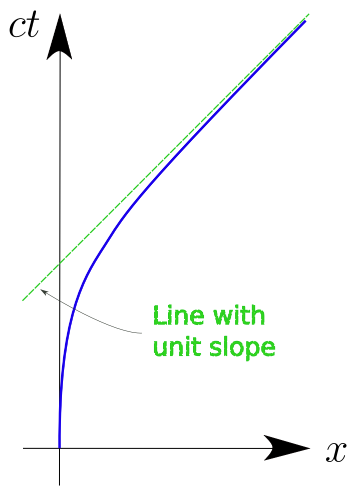
(d) Sketch the worldline of a particle subject to a constant force in the positive x direction. The particle starts from rest at the origin at time $t = 0$.

What are the initial speed and the asymptotic (late-time) speed of the particle? Both should be clear from your sketch.

(Partial) Solution/Hint →

Since the initial speed is zero, the initial slope should be infinity — the worldline starts out vertical.

At very long times, since the particle continues to be accelerated, it approaches (but never reaches) the speed of light, hence its slope approaches but never reaches unit slope, i.e., the slope of a diagaonal (45°) line.



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