Due on monday after Study Break, March 22nd.

This problem set is double the usual size. **Please start work early.**

If marked, at most half of the problem set (either 1-5 or 6-9) will be marked.

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1. A subatomic particle lives for 6×10^{-7} seconds, i.e., it decays 6×10^{-7} seconds after it is created, as seen from its own reference frame. This particle is created at a speed $v = \frac{4}{5}c$ with respect to earth.

   (a) [**4 pts.**] How long does it live, according to the earth frame? Specify clearly: in which frame is the lifetime the 'proper' time? Why?

   (b) [**4 pts.**] How far will the particle travel before it decays, as seen from the earth frame?

   (c) [**4 pts.**] Imagine that the Galilean transformations were true instead of the relativistic (Lorentz) transformations. If this were the case, how far would the particle travel before it decays, as seen from the earth frame?

2. Frame S' moves at speed v_1 with respect to frame S, and frame S'' moves at speed v_2 with respect to frame S', both in the common x, x', x'' direction. The Lorentz transformations are

$$x' = \gamma_1(x - v_1 t) \ , \quad t' = \gamma_1(t - v_1 x/c^2) \qquad \text{with} \quad \gamma_1 = \gamma(v_1) = \frac{1}{\sqrt{1 - v_1^2/c^2}}$$

$$x'' = \gamma_2(x' - v_2 t') \ , \quad t'' = \gamma_2(t' - v_2 x'/c^2) \qquad \text{with} \quad \gamma_2 = \gamma(v_2) = \frac{1}{\sqrt{1 - v_2^2/c^2}}$$

   (a) [**4 pts.**] Express x'' (the coordinate of the S'' frame) in terms of x and t, in the form $x'' = Ax + Bt$. Express A and B in terms of v_1, v_2, γ_1 and γ_2.

   (b) [**3 pts.**] If frame S'' moves at speed u with respect to frame S, you would have $x'' = \gamma(u)(x - ut)$. Comparing with $x'' = Ax + Bt$ and using your expressions for A and B, find u. Have you recovered the relativistic velocity addition formula?

   (c) [**3 pts.**] Verify that your constant A is really equal to $\gamma(u)$, with the u known from the velocity addition formula.

3. Some groups. This problem requires some searching and reading.

   Look up the terms $SO(2)$, $SO(3)$, $SO(3,1)$, $SU(2)$. Each of these groups can be thought of as a set of matrices.

   (a) **[2 pts]** What does 'S' in these names stand for?
   Which matrices are included in the group $O(3)$, but not in the group $SO(3)$?

   (b) **[3 pts]** Which physical transformations are represented by the groups $SO(2)$ and $SO(3)$? Are these groups abelian or non-abelian?

   (c) **[2 pts]** What is the physical interpretation of matrices which are included in the group $O(3)$, but not in the group $SO(3)$?

4. Spacetime diagrams (plots of ct versus x).

   (a) **[4 pts]** On a spacetime diagram, draw the worldline of a photon starting at the origin, the worldline of an object with speed $c/2$ starting at the same time at the origin, and the worldline of an object with speed $4c$ starting at the same time at the origin. Which worldline is unphysical?

   (b) **[3 pts]** On a spacetime diagram, draw the worldline of a photon starting at the origin, and the worldline of an object with velocity $c/2$ starting at the origin four seconds later.

5. Frame Σ' moves at speed v with respect to frame Σ, in the common x, x' direction. The two frames are aligned at time $t = t' = 0$.

   (a) **[6+3 pts.]** A particle has velocity $\vec{u}' = (0, c, 0)$ relative to Σ'. Use the velocity addition formulae to find the velocity of the particle relative to Σ.
   Explain how your result is consistent with the constancy of the speed of light.

   (b) **[5 pts.]** A stick is at rest in frame Σ. It lies in the xy plane and makes angle $\pi/4$ with the x-axis. Using the Lorentz transformations or the length contraction formula, find the angle that the stick makes with the x' axis, as observed from the Σ' frame.

6. **[7 pts]** Show that, if the 4×4 transformation matrix Λ acting on 4-vectors (ct, x, y, z) preserves the Minkowski norm $c^2t^2 - x^2 - y^2 - z^2$, then it must satisfy $\Lambda^T g \Lambda = g$. Here g is the metric tensor defined in Problem Set 05.

7. Frame Σ' moves at speed v with respect to frame Σ, in the common x, x' directions. The two frames are aligned at time $t = t' = 0$.

   (a) **[3 pts]** Write down the Galilean transformation as a matrix transforming the Σ-frame coordinates (ct, x) to the Σ' coordinates (ct', x'). You can ignore the transverse directions and work with 2×2 matrices.

   (b) **[4 pts]** Write down the Lorentz transformation matrix. Take the limit $c \to \infty$, and show that this fails to reproduce the Galilean transformation matrix.

   (c) **[6 pts.]** The reason for the failure above is that in the $c \to \infty$ limit, the variables ct and ct' lose meaning.

   So, write down the Galilean transformation as a matrix transforming the Σ-frame coordinates (t, x) to the Σ' coordinates (t', x'). i.e., the first coordinate does not have the c factor. Write down the Lorentz transformation the same way, as a matrix transforming (t, x) to (t', x').

   Now show that the Lorentz transformation matrix reduces to the Galiliean transformation matrix in the limit $c \to \infty$.

   (d) **[5 pts.]** Considering two successive Galilean transformations in the same x direction, with velocities v_1 and v_2, show by matrix multiplication that their combination is also a Galilean transformation, with velocity $v_1 + v_2$.

8. This problem continues from question 3.

   (a) **[1 pt]** How many matrices does $SO(2)$ contain?

   (b) **[6 pts.]** The group $SO(3,1)$ is the proper Lorentz group.
   What is the size of matrices belonging to this group?
   These matrices are not orthogonal, but must instead satisfy a similar equation. Write down the equation that defines membership in the Lorentz group.
   Under what condition is a member of this group called *orthochronous*?

   (c) **[2 pts.]** What is the size of matrices belonging to the group $SO(1,1)$? Which subset of Lorentz transformations forms this group?

9. Some more spacetime diagrams.

   (a) **[3 pts]** On a spacetime diagram, depict the situation that a light source at $x = 0$ is turned on for a few seconds, and then turned off again. (Show the worldlines of photons.)

   (b) **[3 pts]** A railgun located at the origin shoots projectiles at time intervals of T, alternately in the positive-x and negative-x directions. The projectiles in the positive x direction have speed $c/10$ and those in the negative x direction have speed $c/2$. Draw the situation on a spacetime diagram, showing the worldlines of a number of successive projectiles.

   (c) **[3 pts]** Draw the worldline of a tachyon. (A tachyon is a hypothetical particle that moves faster than the speed of light.)

   (d) **[7 pts]** Sketch the worldline of a particle subject to a constant force in the positive x direction. The particle starts from rest at the origin at time $t = 0$.
   What are the initial speed and the asymptotic (late-time) speed of the particle? Both should be clear from your sketch.