As usual: my proofreading has been minimal; beware of typos and errors.

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1. Consider a set of inertial frames which are aligned at the commonly defined zero time, but might be moving in different relative directions.

(a) Write down the Lorentz boost matrices for relative velocity v_1 between frames in the x-direction, and for relative velocity v_2 in the y-direction.

(Partial) Solution/Hint →

The Lorentz boost matrix in the x direction is

$$L_1 = \begin{pmatrix} \gamma_{v_1} & -\gamma_{v_1}\frac{v_1}{c} & 0 & 0 \\ -\gamma_{v_1}\frac{v_1}{c} & \gamma_{v_1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \xrightarrow{\text{using ``}c=1\text{'' units}} \begin{pmatrix} \gamma_1 & -\gamma_1 v_1 & 0 & 0 \\ -\gamma_1 v_1 & \gamma_1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

using the notational simplifications suggested in the next problem. This is the 'standard' LT we have mostly been dealing with.
Boost matrix in the y direction is

$$L_2 = \begin{pmatrix} \gamma_{v_2} & 0 & -\gamma_{v_2}\frac{v_2}{c} & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma_{v_2}\frac{v_2}{c} & 0 & \gamma_{v_2} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma_2 & 0 & -\gamma_2 v_2 & 0 \\ 0 & 1 & 0 & 0 \\ -\gamma_2 v_2 & 0 & \gamma_2 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

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(b) Find out by matrix multiplication the result of successive application of these two transformations. Consider both possible orderings: applying the x-direction boost first and then the y-direction boost, and vice versa.

This will look cumbersome, so I suggest that you use $c = 1$ units and the notation $\gamma_1 = \gamma(v_1)$, $\gamma_2 = \gamma(v_2)$. Another hint: do you really need to multiply 4×4 matrices?

(Partial) Solution/Hint →

Both the matrices involved keep the z component unchanged. Hence we could simply work with 3×3 matrices that operate on (ct, x, y).

Let's first calculate L_2L_1., i.e., the case where one first operates with L_1, and then with L_2.

$$L_2L_1 = \begin{pmatrix} \gamma_2 & 0 & -\gamma_2 v_2 \\ 0 & 1 & 0 \\ -\gamma_2 v_2 & 0 & \gamma_2 \end{pmatrix} \begin{pmatrix} \gamma_1 & -\gamma_1 v_1 & 0 \\ -\gamma_1 v_1 & \gamma_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma_1\gamma_2 & -\gamma_1\gamma_2 v_1 & -\gamma_2 v_1 \\ -\gamma_1 v_1 & \gamma_1 & 0 \\ -\gamma_1\gamma_2 v_2 & \gamma_1\gamma_2 v_1 v_2 & \gamma_2 \end{pmatrix}$$

We now consider first applying the y-direction boost, and then applying the x-direction boost:

$$L_1L_2 = \begin{pmatrix} \gamma_1 & -\gamma_1 v_1 & 0 \\ -\gamma_1 v_1 & \gamma_1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \gamma_2 & 0 & -\gamma_2 v_2 \\ 0 & 1 & 0 \\ -\gamma_2 v_2 & 0 & \gamma_2 \end{pmatrix} = \begin{pmatrix} \gamma_1\gamma_2 & -\gamma_1 v_1 & -\gamma_1\gamma_2 v_2 \\ -\gamma_1\gamma_2 v_1 & \gamma_1 & \gamma_1\gamma_2 v_1 v_2 \\ -\gamma_2 v_2 & 0 & \gamma_2 \end{pmatrix}$$

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(c) Do the two Lorentz boost matrices commute? Compare with Lorentz boosts in the same direction.

(Partial) Solution/Hint →

Comparing the two products we calculated, we see $L_1L_2 \neq L_2L_1$. Thus L_1 and L_2 don't commute.

(Question: is this because the boosts in the two directions were with different speeds? Would the boosts commute for $v_1 = v_2$?)

We found previously that when the two boosts are both standard LT's in the x-direction, they commute.

Thus we've learned: boosts in the same direction commute, boosts in different directions don't.

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(d) Consider any of the matrices resulting from successive application of the two boosts. Is the matrix symmetric?

Remembering from Problem Set 5 that Lorentz boost transformations in arbitrary directions are symmetric matrices, interpret this observation.

(Partial) Solution/Hint →

Neither of the two resulting matrices (L_1L_2 or L_2L_1) are symmetric. Transposing L_1L_2 gives L_2L_1, not itself.

Now we found that the most general Lorentz boost (a Lorentz boost in an arbitrary direction) is represented by a symmetric matrix. This means that two Lorentz boosts (if in different directions) combine to form a transformation that is not a Lorentz boost! The set of Lorentz boosts in all directions do not form a group as this set does not satisfy closure.

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2. The kinetic energy of motion of a particle is the relativistic total energy minus the rest energy.

(a) A particle has rest mass M and speed v. If $v \ll c$, then show that the kinetic energy due to motion is approximated by the well-known non-relativistic expression for the kinetic energy.

(Partial) Solution/Hint →

$$\text{The kinetic energy due to motion } = \\ = \begin{pmatrix}\text{energy of}\\ \text{moving particle}\end{pmatrix} - \begin{pmatrix}\text{energy of}\\ \text{particle at rest}\end{pmatrix} = \gamma(v)Mc^2 - Mc^2 \\ = \left(\frac{1}{\sqrt{1-v^2/c^2}} - 1\right)Mc^2$$

When $v \ll c$, we can expand as a power series in $(v/c)^2$:

$$T = Mc^2\left(1-\frac{v^2}{c^2}\right)^{-1/2} - Mc^2 \\ = Mc^2\left(1 + (-\tfrac{1}{2})\left(-\frac{v^2}{c^2}\right) + \frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2}\left(-\frac{v^2}{c^2}\right)^2 + \cdots\right) - Mc^2 \\ = \left(Mc^2 + \tfrac{1}{2}Mv^2 + \cdots\right) - Mc^2 = \tfrac{1}{2}Mv^2 + \cdots$$

The leading term of this expression for small v/c is the nonrelativistic expression for kinetic energy, $\frac{1}{2}mv^2$.

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(b) Derive the leading correction to the non-relativistic expression.

(Partial) Solution/Hint →

The leading correction is the first term that we dropped in the above expansion:

$$\text{leading correction} \; = \; Mc^2\left(\frac{(-\frac{1}{2})(-\frac{1}{2}-1)}{2}\left(-\frac{v^2}{c^2}\right)^2\right) \; = \; \frac{3}{8}Mv^4/c^2$$

which means that a better approximation to the kinetic energy is

$$T \;\approx\; \frac{1}{2}Mv^2 + \frac{3}{8}Mv^4/c^2$$

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(c) What is a particle's speed if its kinetic energy T is equal to its rest energy? What is a particle's speed if its kinetic energy is n times its rest energy?

(Partial) Solution/Hint →

If $T = Mc^2$, then

$$\gamma(v)Mc^2 - Mc^2 \;=\; Mc^2$$
$$\implies \quad \gamma(v) = 2$$
$$\implies \quad \frac{1}{\sqrt{1-(v/c)^2}} = 2$$
$$\implies \quad (v/c) \;=\; \sqrt{1-\frac{1}{2^2}} \;=\; \frac{\sqrt{3}}{2} \quad \implies \quad v = \frac{\sqrt{3}}{2}c$$

Similarly, if $T = nMc^2$, then

$$\gamma(v) = n+1$$
$$\implies \quad (v/c)^2 \;=\; 1 - \frac{1}{(n+1)^2} \;=\; \frac{n^2+2n}{(n+1)^2}$$
$$\implies \quad v \;=\; \frac{\sqrt{n^2+2n}}{(n+1)}\,c$$

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(d) Express the momentum p of the particle as a function of the mass M and the kinetic energy of motion, which we will call T. Your expression should contain M and T, not v or γ_v.
Hint: use the relation between E and p that does not involve v or γ_v.

(Partial) Solution/Hint →

Using $E^2 = p^2c^2 + M^2c^4$ and $E = Mc^2 + T$, we get

$$\begin{aligned} (Mc^2 + T)^2 &= p^2c^2 + M^2c^4 \\ \Longrightarrow \quad & 2TMc^2 + T^2 = p^2c^2 \\ \Longrightarrow \quad p &= \frac{1}{c}\sqrt{2TMc^2 + T^2} \end{aligned}$$

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3. Relative velocities, longitudinal & transverse.

(a) A rocket traveling at speed $\frac{1}{2}c$ with respect to frame Σ shoots forward bullets at speed $\frac{3}{4}c$ relative to the rocket. What is the speed of the bullets relative to Σ?

(Partial) Solution/Hint →

This is simple one-dimensional relativistic velocity addition, no transverse directions are involved. The result is

$$\frac{\frac{1}{2}c + \frac{3}{4}c}{1 + \frac{\frac{1}{2}c \times \frac{3}{4}c}{c^2}} = \frac{\frac{5}{4}c}{11/8} = \frac{10}{11}c$$

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(b) Now imagine that the bullets are fired *perpendicular* to the direction of rocket motion, as perceived from the rocket. What is the speed of the bullets relative to Σ?

(Partial) Solution/Hint →

Relative to Σ, the bullet has velocity component $\frac{1}{2}c$ in the direction of rocket motion, i.e., the same velocity component as the rocket.
Perpendicular to the rocket motion, relative to Σ, the bullet has velocity component

$$\frac{\text{velocity relative to rocket}}{\gamma(\text{rocket speed relative to } \Sigma)} = \frac{\frac{3}{4}c}{\gamma(\frac{1}{2}c)} = \frac{\frac{3}{4}c}{1/\sqrt{1-(1/2)^2}}$$

$$= \frac{\frac{3}{4}c}{2/\sqrt{3}} = \frac{3\sqrt{3}}{8}c$$

If it's not clear how these two components were obtained, take the rocket to be the Σ' frame, and use the velocity addition formulae for longitudinal and transverse velocity components.
Using the velocity components, the speed of the bullet relative to Σ is

$$\sqrt{\left(\frac{1}{2}c\right)^2 + \left(\frac{3\sqrt{3}}{8}c\right)^2} = c\sqrt{\frac{1}{4} + \frac{27}{64}} = \frac{\sqrt{43}}{8}c$$

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(c) An X-ray beam is sent out from the rocket, perpendicular to the direction of rocket motion. Relative to Σ, what are the components of the X-ray beam velocity parallel and perpendicular to the direction of rocket motion?

Calculate from your velocity components the speed of the X-ray beam relative to Σ. Explain why the speed relative to Σ could have been expected.

(Partial) Solution/Hint $\rightarrow$

An X-ray beam is an electromagnetic wave, i.e., consists of photons, and hence has speed c relative to every frame.

In the direction of the rocket motion, the velocity component relative to Σ is the rocket velocity $\frac{1}{2}c$.

In the perpendicular direction, the velocity component is

$$\frac{\text{velocity relative to rocket}}{\gamma(\text{rocket speed relative to }\Sigma)} = \frac{c}{\gamma(\frac{1}{2}c)} = \frac{c}{2/\sqrt{3}} = \frac{\sqrt{3}}{2}c$$

The speed (relative to Σ) is

$$\sqrt{\left(\frac{1}{2}c\right)^2 + \left(\frac{\sqrt{3}}{2}c\right)^2} = \sqrt{c^2} = c$$

This is expected as electromagnetic waves have speed c relative to every frame.

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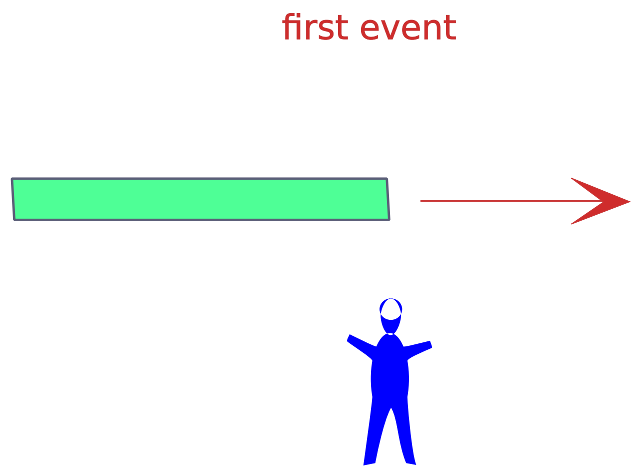
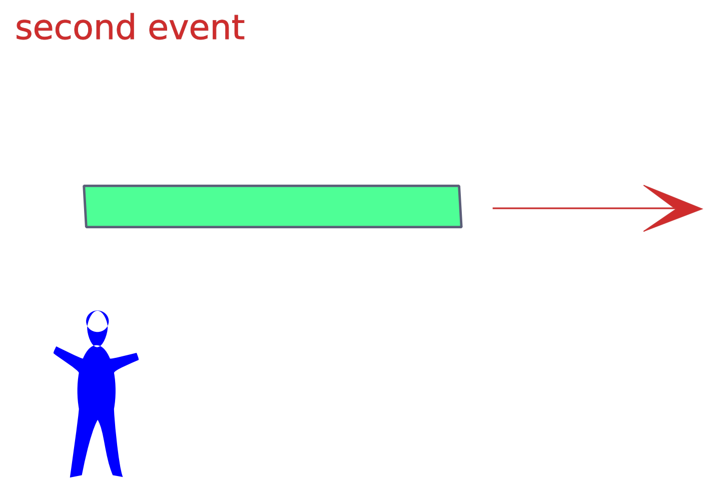
4. A stick of proper length L moves past you at speed v. There is a time interval between the front end passing you (first event) and the back end passing you (second event). We are interested in this time interval and in the length of the stick, as seen from various frames.

(a) Draw snapshots of the two events as seen from your frame, marking with an arrow what is moving (you or the stick).

(Partial) Solution/Hint →





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(b) Find the length of the stick as seen from your frame, and the time between the two events, as seen from your frame.

(Partial) Solution/Hint →

The proper length of the stick, i.e., length measured in its own frame, is L. Hence measured from your frame it is length-contracted and is measured to be L/γ_v. Since the speed is v, the time taken for length L/γ_v to pass by is $\dfrac{L/\gamma_v}{v} = \dfrac{L}{\gamma_v v}$.

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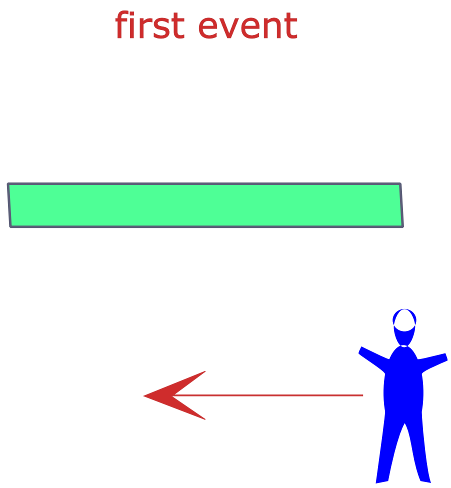
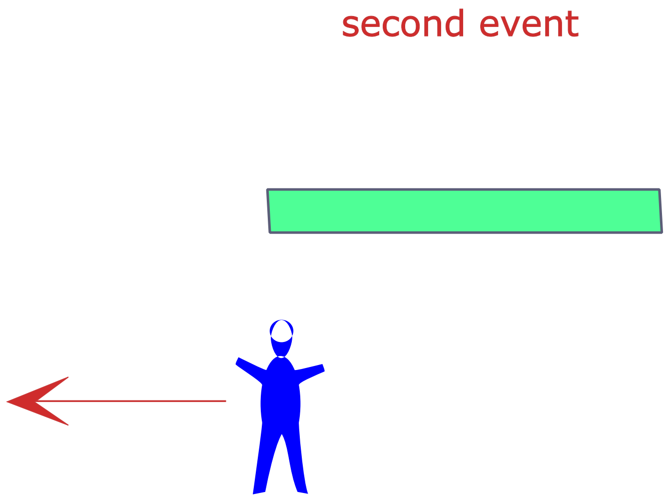
(c) Draw snapshots of the two events as seen from the frame of the stick, marking with an arrow what is moving, as seen from this frame.

(Partial) Solution/Hint →





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(d) Find the length of the stick and the time between the two events, as seen from the frame of the stick.

(Partial) Solution/Hint →

By definition, the rest frame of the stick (frame relative to which the stick is at rest, i.e., frame attached to the stick) is the frame which measures the length to be the proper length L.

Relative to the frame of the stick, you move with speed v in the opposite direction, and cover a distnace L during this time. Hence the time between the two events is L/v.

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(e) In which frame is the time interval the *proper* time interval: your frame or the stick's frame? Explain why.

(Partial) Solution/Hint →

Proper time interval is measured in the frame where the two events happen in the same position. In the stick frame, the events are at different positions, one in the front end and one in the back end. In your frame, the two events are both at your position, hence at the same position. Thus your frame measures the time interval to be the proper time interval.

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