1. Consider the set of real 4×4 matrices Λ which satisfy the relation

$$\Lambda^T g \Lambda = g\,,$$

where g is the metric tensor.

(a) Show that this set is a group under matrix multiplication.

(Partial) Solution/Hint →

To be a group, the properties of Closure, Associativity, Existence of Identity, and Existence of Inverse must be satisfied.

Closure: If Λ_1 and Λ_2 are members of the set, then $\Lambda_1^T g \Lambda_1 = g$ and $\Lambda_2^T g \Lambda_2 = g$. Then

$$\Big(\Lambda_1\Lambda_2\Big)^T g \Big(\Lambda_1\Lambda_2\Big) \;=\; \Big(\Lambda_2^T\Lambda_1^T\Big) g \Big(\Lambda_1\Lambda_2\Big) \;=\; \Lambda_2^T\Big(\Lambda_1^T g\Lambda_1\Big)\Lambda_2 \;=\; \Lambda_2^T g \Lambda_2 \;=\; g$$

which means that $\Lambda_1\Lambda_2$ is also a member of the set.

Associativity: Matrix multiplication is known to be associative.

Identity: The 4×4 identity matrix

$$I = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

is a member of the set, because

$$I^T g I = I g = g$$

Hence the set contains an identity element.

Note that just pointing to the 4×4 identity matrix I is not enough: one has to explicitly show that I belongs to this particular set.

Inverse: If Λ is an element of the set, $\Lambda^T g \Lambda = g$ by definition. To show that the matrix inverse Λ^{-1} also belongs to the set, multiply both sides by $(\Lambda^{-1})^T$ on the left and by Λ^{-1} on the right:

$$\big(\Lambda^{-1}\big)^T \Big(\Lambda^T g \Lambda\Big) \Lambda^{-1} = \big(\Lambda^{-1}\big)^T g \Lambda^{-1}$$

The left side is

$$\Big(\big(\Lambda^{-1}\big)^T \Lambda^T\Big) g \Big(\Lambda\Lambda^{-1}\Big) \;=\; \Big(\Lambda\Lambda^{-1}\Big)^T g I \;=\; I^T g \;=\; g$$

so that we have obtained

$$g = \left(\Lambda^{-1}\right)^T g \Lambda^{-1}$$

i.e., the inverse of Λ satisfies the defining equation and hence belongs to the set.

These four properties show that the set defined by $\Lambda^T g \Lambda = g$ is a group under matrix multiplication.

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Note: the group of 4×4 matrices which satisfy $\Lambda^T g \Lambda = g$ is known as the LORENTZ GROUP. The results below will show that Lorentz boosts belong to this group (as expected), but also, that ROTATIONS belong to this group. The Lorentz group consists of all boosts, all rotations, and all combinations of these. It is denoted as $O(3,1)$ or $O(1,3)$.
In fact, the Lorentz group as defined by the condition $\Lambda^T g \Lambda = g$ also contains reflections (improper rotations) and time reversal (non-orthchronous transformations). We might regard these as unphysical, so it is common to add two more conditions:

$$\det[\Lambda] = +1 \qquad \text{and} \qquad \Lambda^0_0 > 0$$

With these two conditions, one obtains a subset of the Lorentz group, usually called the RESTRICTED LORENTZ GROUP. This set is denoted as $SO^{\uparrow}(3,1)$. The S stands for 'special' (positive determinant) and the $\uparrow$ indicates the orthochronous condition.

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(b) Write down the matrix transforming (ct, x, y, z) to (ct', x', y', z'), for a standard Lorentz boost in the common x, x' direction. Find out whether this matrix belongs to the above set.

(Partial) Solution/Hint →

The Lorentz boost matrix in the x direction is

$$B_x = \begin{pmatrix} \gamma_v & -\gamma_v \frac{v}{c} & 0 & 0 \\ -\gamma_v \frac{v}{c} & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \gamma_v & -\gamma_v v & 0 & 0 \\ -\gamma_v v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

where I use $c = 1$ for notational simplicity.
We need to find out whether this matrix satisfies

$$(B_x)^T \, g \, B_x = g$$

Since $(B_x)^T = B_x$, we have

$$(B_x)^T \, g \, B_x = B_x \, g \, B_x$$

$$= \begin{pmatrix} \gamma_v & -\gamma_v v & 0 & 0 \\ -\gamma_v v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma_v & -\gamma_v v & 0 & 0 \\ -\gamma_v v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} \gamma_v & +\gamma_v v & 0 & 0 \\ -\gamma_v v & -\gamma_v & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} \gamma_v & -\gamma_v v & 0 & 0 \\ -\gamma_v v & \gamma_v & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Using $\gamma_v = 1/\sqrt{1 - v^2}$, this turns in a couple lines of calculation to

$$(B_x)^T \, g \, B_x = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g$$

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(c) Write down the matrix transforming (ct, x, y, z) to (ct', x', y', z'), for a rotation of the coordinate frame around the common z, z' axis. (There is no relative velocity between the frames). Find out whether this rotation matrix belongs to the above set.

(Partial) Solution/Hint →

$$L_{\text{rot}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Does it belong to the set?

$$\begin{aligned}
(L_{\text{rot}})^T g L_{\text{rot}} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & +\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} g \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & \sin\theta & 0 \\ 0 & -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & +\sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -\cos\theta & -\sin\theta & 0 \\ 0 & +\sin\theta & -\cos\theta & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g
\end{aligned}$$

$\implies$ Yes it does

Although we have obtained the condition $\Lambda^T g \Lambda = g$ from the requirement of the invariant interval known from Lorentz boosts, it appears that we cannot ignore rotations as they also satisfy this condition.

In fact, boosts and rotations are strongly connected. We also learned previously that two successive boosts might not be a pure boost but rather a combination of a boost and a rotation.

Physically, the (restricted) Lorentz group $SO^{\uparrow}(3,1)$ or $SO^{\uparrow}(1,3)$ consists of all boosts, all rotations, and all combinations of boosts and rotations.

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2. Relative to frame Σ, frame Σ' has velocity v in the common x, x' direction. A photon has velocity
$$\vec{u'} = (\frac{3}{5}c,\ 0,\ \frac{4}{5}c)$$
relative to Σ'. Find the velocity of the photon relative to Σ.

Find out whether your result is consistent with the constancy of the speed of light.

(Partial) Solution/Hint →

I will use $c = 1$ units.

$$u_x = \frac{u'_x + v}{1 + u'_x v} = \frac{\frac{3}{5} + v}{1 + \frac{3}{5}v} = \frac{3 + 5v}{5 + 3v}$$

$$u_y = \frac{u'_y}{\gamma_v(1 + u'_x v)} = 0$$

$$u_z = \frac{u'_z}{\gamma_v(1 + u'_x v)} = \frac{\frac{4}{5}}{\gamma_v\left(1 + \frac{3}{5}v\right)} = \frac{4}{\gamma_v\,(5 + 3v)} = \frac{4\sqrt{1 - v^2}}{5 + 3v}$$

The speed of the photon should be $c = 1$ in any frame. It works as given for the Σ' frame, because $u'^2 = (\frac{3}{5})^2 + (\frac{4}{5})^2 = 1$. For the Σ frame,

$$\begin{aligned} u^2 &= \left(\frac{3 + 5v}{5 + 3v}\right)^2 + (0)^2 + \left(\frac{4\sqrt{1 - v^2}}{5 + 3v}\right)^2 \\ &= \frac{(3 + 5v)^2 + 16(1 - v^2)}{(5 + 3v)^2} = \frac{9 + 30v + 25v^2 + 16 - 16v^2}{(5 + 3v)^2} \\ &= \frac{25 + 30v + 9v^2}{(5 + 3v)^2} = \frac{(5 + 3v)^2}{(5 + 3v)^2} = 1 \end{aligned}$$

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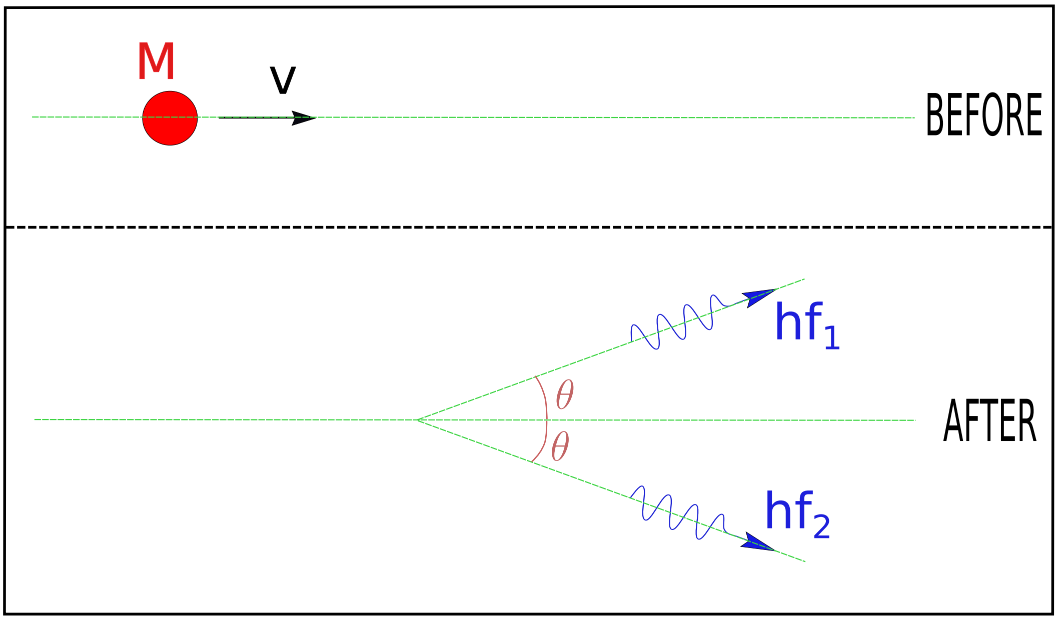
3. A neutral pion has mass M and is traveling with speed v when it decays into two photons. The photons are seen to emerge at equal angles θ on either side of the original velocity. Show that $v = c\cos\theta$.

   (As usual with decay/reaction problems, draw clear before & after pictures, write equations for conservation of energy and momentum, choosing the variables more suitable for the problem [express in terms of speeds and γ's, or in terms of momenta?], and then work with those equations.)

   (Partial) Solution/Hint →



It's important to draw SEPARATE pictures for the situation before and the situation after. If you try representing everything on the same picture, you are much more likely to confuse yourself when writing down the conservation equations.

If the pictures are drawn reasonably, you should be able to write down energy and momentum conservation equations. I write them in terms of speeds, since the desired result concerns the speed.

Energy conservation:

$$\gamma_v M c^2 \;=\; hf_1 + hf_2$$

I've introduced notation f_1 and f_2 for the frequencies of the two photons. When you introduce new notation in your assignments of exams, please do state clearly what you have introduced.

Momentum conservation in the original direction of pion motion:

$$\gamma_v M v \;=\; \frac{hf_1}{c}\cos\theta + \frac{hf_2}{c}\cos\theta$$

Momentum conservation in the direction perpendicular to the original velocity:

$$0 \;=\; \frac{hf_1}{c}\sin\theta - \frac{hf_2}{c}\sin\theta$$

The last equation gives $f_1 = f_2$. The first equation then yields $hf_1 = \frac{1}{2}\gamma_v Mc^2$, which, when put into the second equation, gives

$$\gamma_v M v \;=\; 2(\tfrac{1}{2}\gamma_v M c^2)\cos\theta \qquad \Longrightarrow \qquad v = c\cos\theta$$

Note: if we wanted the momentum of the pion instead of the speed, it would have been better to work in terms of momenta instead of in terms of speeds. We would then write the conservation equations as

$$\sqrt{p^2c^2 + m^2c^4} \;=\; hf_1 + hf_2 \qquad \text{(energy conservation)}$$

$$p \;=\; \frac{hf_1}{c}\cos\theta + \frac{hf_2}{c}\cos\theta \qquad \text{(momentum conservation)}$$

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4. A particle of rest mass M, while at rest in the laboratory, decays into a particle of mass m and speed v, and a photon of frequency f moving in opposite direction. Relativistic momentum and energy is conserved in this process.

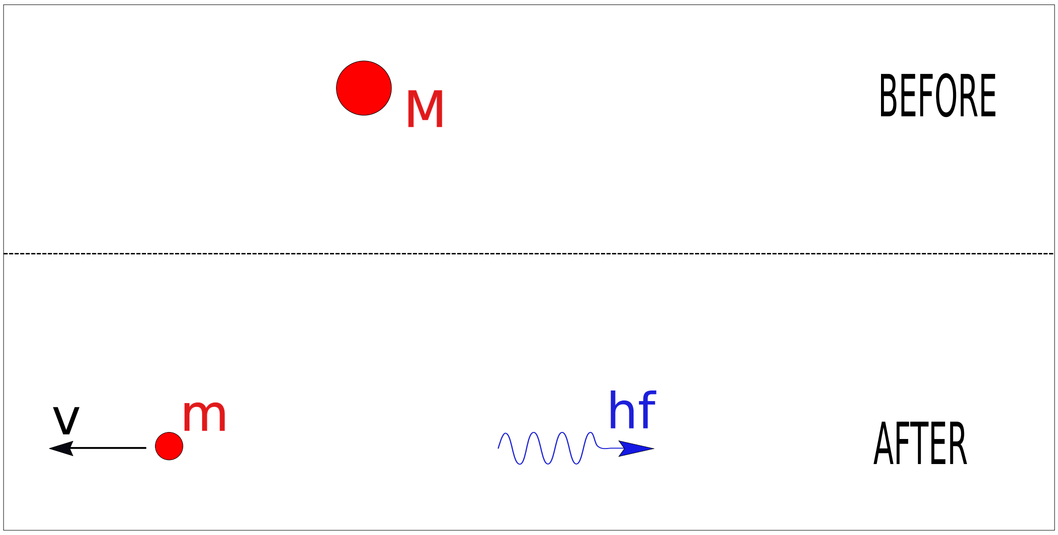
   (a) Draw the situations before and after the decay process, as seen in the laboratory frame. Please indicate the velocity of each particle.

   (Partial) Solution/Hint →

   Before: particle of mass M at rest

   After: particle of mass m moving leftward at speed v, photon of frequency f moving rightward at speed c.

   Or, equally correct if mass m moves rightward and photon moves leftward.



Warning (yet again): note how the before and after situations are two separate sketches. Trying to depict both before and after situations onto a single picture will likely just confuse.

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(b) Write down the equations for momentum conservation and energy conservation (in the laboratory frame).

(Partial) Solution/Hint →

$$\text{Momentum conservation:} \qquad 0 \;=\; \frac{hf}{c} - \gamma(v)mv$$

$$\text{Energy conservation:} \qquad Mc^2 \;=\; hf + \gamma(v)mc^2$$

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(c) Use these equations to calculate m as a function of M and v.

(Partial) Solution/Hint →

Eliminating hf yields

$$\begin{aligned} Mc^2 \;&=\; \gamma(v)mvc + \gamma(v)mc^2 \\ \Longrightarrow \qquad m \;&=\; \frac{Mc}{\gamma(v)\times(c+v)} \;=\; M\frac{c\sqrt{1-(v/c)^2}}{c+v} \;=\; M\sqrt{\frac{c-v}{c+v}} \end{aligned}$$

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(d) Show that the photon frequency is given by

$$hf = \frac{Mvc^2}{c+v}$$

in the lab frame.

(Partial) Solution/Hint →

From the momentum conservation equation,

$$hf \;=\; \gamma(v)mvc \;=\; \left(\frac{c}{\sqrt{c^2-v^2}}\right)\left(M\sqrt{\frac{c-v}{c+v}}\right)vc \;=\; \frac{Mvc^2}{c+v}$$

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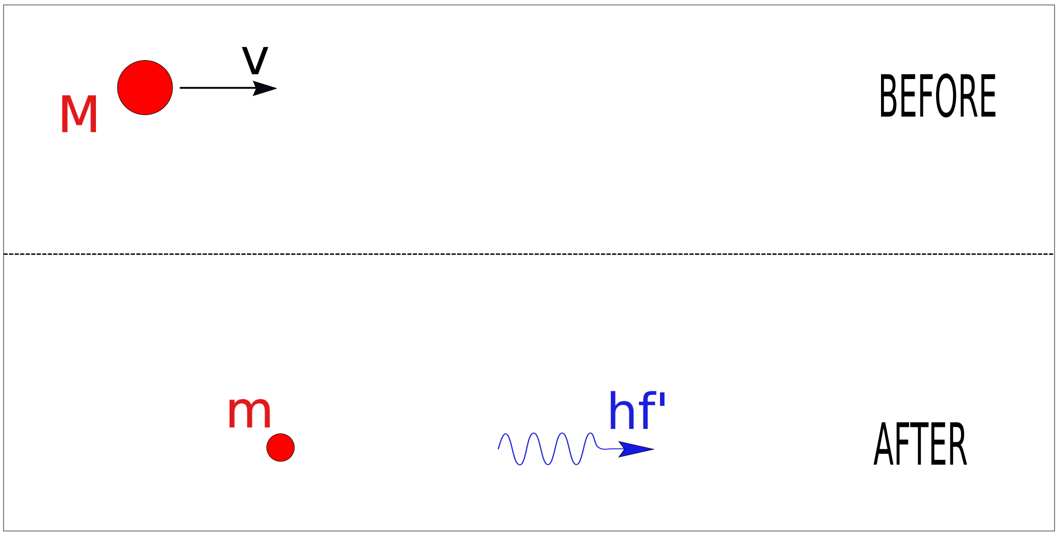
(e) Now work in the frame moving at speed v; in this frame the particle of mass m is at rest after the decay. Draw the situations before and after the decay process, as seen in this frame. Do indicate the speeds.

(Partial) Solution/Hint →

Before: particle of mass M moving rightward at speed v

After: particle of mass m at rest, photon of frequency f' moving rightward at speed c.



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(f) Write down the equations of momentum conservation and energy conservation in this frame.

(Partial) Solution/Hint →

Momentum:

$$\gamma_v M v = 0 + hf'/c$$

Energy:

$$\gamma_v M c^2 = mc^2 + hf'$$

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(g) Solve for m (as a function of M and v) to show that you get the same expression for m as you did by working in the laboratory frame.

Why should you expect the same expression?

(Partial) Solution/Hint →

Eliminating hf',

$$\gamma_v M c^2 \;=\; mc^2 + \gamma_v M v c$$
$$\Longrightarrow \quad m \;=\; \gamma_v M \;-\; \gamma_v M v/c \;=\; M\gamma_v\,(1 - v/c)$$
$$\Longrightarrow \quad m \;=\; M\frac{1 - v/c}{\sqrt{1 - v^2/c^2}} \;=\; M\sqrt{\frac{c - v}{c + v}}$$

Same expression as before obtained.

This is expected because the mass of objects is an invariant under Lorentz transformations and is not frame-dependent.

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(h) Solve for the photon frequency to show that you get a *different* frequency, compared to the lab frame.

(Partial) Solution/Hint →

$$hf' \;=\; \gamma_v M v c \;=\; \frac{Mvc}{\sqrt{1 - v^2/c^2}} \;=\; \frac{Mvc^2}{\sqrt{c^2 - v^2}}$$

which is quite different from $hf = \dfrac{Mvc^2}{c+v}$.

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(i) The photon frequency is different, when observed from a different frame! Explain the factor between observed frequencies using the longitudinal Doppler effect.

(Partial) Solution/Hint →

The source of the photon (the particle that decayed) is at rest in the lab frame. Hence, observing it from a frame moving in the opposite direction, the frequency will be observed to be larger by a factor $\sqrt{(c+v)/(c-v)}$.

Is this consistent with the frequency expressions we have calculated?

$$f'/f \;=\; \frac{\dfrac{Mvc}{\sqrt{1-v^2/c^2}}}{\dfrac{Mvc^2}{\sqrt{c^2-v^2}}} \;=\; \sqrt{\frac{c+v}{c-v}}.$$

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