Due on Monday, April 12th.

In case you introduce your own notation, please clearly define the symbols you are introducing.

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1. Consider the set of real 4×4 matrices Λ which satisfy the relation

$$\Lambda^T g \Lambda = g\,,$$

where g is the metric tensor.

   (a) **[7 pts]** Show that this set is a group under matrix multiplication.

   (b) **[2 pts]** Write down the matrix transforming (ct, x, y, z) to (ct', x', y', z'), for a standard Lorentz boost in the common x, x' direction. Find out whether this matrix belongs to the above set.

   (c) **[3 pts]** Write down the matrix transforming (ct, x, y, z) to (ct', x', y', z'), for a rotation of the coordinate frame around the common z, z' axis. (There is no relative velocity between the frames). Find out whether this rotation matrix belongs to the above set.

2. **[4+3 pts]** Relative to frame Σ, frame Σ' has velocity v in the common x, x' direction. A photon has velocity

$$\vec{u'} \;=\; (\frac{3}{5}c,\ 0,\ \frac{4}{5}c)$$

relative to Σ'. Find the velocity of the photon relative to Σ.

Find out whether your result is consistent with the constancy of the speed of light.

3. **[6 pts]** A neutral pion has mass M and is traveling with speed v when it decays into two photons. The photons are seen to emerge at equal angles θ on either side of the original velocity. Show that $v = c\cos\theta$.

(As usual with decay/reaction problems, first draw separate pictures showing the situation BEFORE and AFTER the decay. Then write equations for conservation of energy and momentum, choosing the variables more suitable for the problem [express in terms of speeds and γ's, or in terms of momenta?]. Then work with those equations.)

4. A particle of rest mass M, while at rest in the laboratory, decays into a particle of mass m and speed v, and a photon of frequency f moving in opposite direction. Relativistic momentum and energy is conserved in this process.

   (a) [**0 pts**] Draw the situations before and after the decay process, as seen in the laboratory frame. Please indicate the velocity of each particle.

   (Advice/warning: always draw SEPARATE pictures, one showing the situation before and one showing the situation after. Merging the before- and after- situations in one picture will likely cause confusions.)

   (b) [**4 pts**] Write down the equations for momentum conservation and energy conservation (in the laboratory frame).

   (c) [**3 pts**] Use these equations to calculate m as a function of M and v.

   (d) [**3 pts**] Show that the photon frequency is given by

   $$hf = \frac{Mvc^2}{c+v}$$

   in the lab frame.

   (e) [**2 pts**] Now work in the frame moving at speed v; in this frame the particle of mass m is at rest after the decay. Draw the situations before and after the decay process, as seen in this frame. Do indicate the speeds.

   (f) [**4 pts**] Write down the equations of momentum conservation and energy conservation in this frame.

   (g) [**3 pts**] Solve for m (as a function of M and v) to show that you get the same expression for m as you did by working in the laboratory frame.

   Why should you expect the same expression?

   (h) [**3 pts**] Solve for the photon frequency to show that you get a *different* frequency, compared to the lab frame.

   (i) [**3 pts**] The photon frequency is different, when observed from a different frame! Explain the factor between observed frequencies using the longitudinal Doppler effect.