Partial solutions to problem set 09.

If you spot a typo, please let me know.

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1. $A = (A^0, A^1, A^2, A^3)$ and $B = (B^0, B^1, B^2, B^3)$ are 4-vectors. Under a Lorentz transformation from frame Σ to $\tilde{\Sigma}$, the components of A transform as

$$\tilde{A}^0 = \gamma_v \left(A^0 - \frac{v}{c} A^1 \right) , \quad \tilde{A}^1 = \gamma_v \left(A^1 - \frac{v}{c} A^0 \right) ,$$
$$\tilde{A}^2 = A^2 \cos\theta + A^3 \sin\theta , \quad \tilde{A}^3 = -A^2 \sin\theta + A^3 \cos\theta .$$

(a) Write down the transformations for the components of B.

(Partial) Solution/Hint →

Exactly the same transformations for the components of B:

$$\tilde{B}^0 = \gamma_v \left(B^0 - \frac{v}{c} B^1 \right) , \quad \tilde{B}^1 = \gamma_v \left(B^1 - \frac{v}{c} B^0 \right) ,$$
$$\tilde{B}^2 = B^2 \cos\theta + B^3 \sin\theta , \quad \tilde{B}^3 = -B^2 \sin\theta + B^3 \cos\theta .$$

Why? Because that's the definition of four-vectors: they all transform the same way as x^μ under Lorentz transformations, and hence they all transform the same way as each other.

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(b) Show that the inner product, defined as

$$A \star B = A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3 \,,$$

is invariant under this transformation.

(Partial) Solution/Hint →

$$\tilde{A} \star \tilde{B} \;=\; \tilde{A}^0 \tilde{B}^0 - \tilde{A}^1 \tilde{B}^1 - \tilde{A}^2 \tilde{B}^2 - \tilde{A}^3 \tilde{B}^3$$

Consider the first two terms first:

$$\begin{aligned}
&\tilde{A}^0 \tilde{B}^0 - \tilde{A}^1 \tilde{B}^1 \\
&= \gamma_v \left(A^0 - \frac{v}{c} A^1 \right) \gamma_v \left(B^0 - \frac{v}{c} B^1 \right) - \gamma_v \left(A^1 - \frac{v}{c} A^0 \right) \gamma_v \left(B^1 - \frac{v}{c} B^0 \right) \\
&\qquad = A^0 B^0 \gamma_v^2 \left(1 - v^2/c^2 \right) + A^0 B^1 \left(-\gamma_v \frac{v}{c} + \gamma_v \frac{v}{c} \right) \\
&\qquad + A^1 B^0 \left(-\gamma_v \frac{v}{c} + \gamma_v \frac{v}{c} \right) + A^1 B^1 \gamma_v^2 \left(v^2/c^2 - 1 \right) \\
&\qquad\qquad = A^0 B^0 + 0 + 0 - A^1 B^1 \;=\; A^0 B^0 - A^1 B^1
\end{aligned}$$

Now consider the next two terms:

$$\begin{aligned}
&- \tilde{A}^2 \tilde{B}^2 - \tilde{A}^3 \tilde{B}^3 \\
&\qquad = \; - (A^2 \cos\theta + A^3 \sin\theta)(B^2 \cos\theta + B^3 \sin\theta) \\
&\qquad\quad - (-A^2 \sin\theta + A^3 \cos\theta)(-B^2 \sin\theta + B^3 \cos\theta) \\
&= \; - A^2 B^2 (\cos^2\theta + \sin^2\theta) - A^2 B^3 (\cos\theta \sin\theta - \sin\theta \cos\theta) \\
&\quad - A^3 B^2 (\sin\theta \cos\theta - \cos\theta \sin\theta) - A^3 B^3 (\cos^2\theta + \sin^2\theta) \\
&\qquad = \; - A^2 B^2 - 0 - 0 - A^3 B^3 \;=\; - A^2 B^2 - A^3 B^3
\end{aligned}$$

Put together:

$$\begin{aligned}
\tilde{A} \star \tilde{B} \;&=\; \tilde{A}^0 \tilde{B}^0 - \tilde{A}^1 \tilde{B}^1 - \tilde{A}^2 \tilde{B}^2 - \tilde{A}^3 \tilde{B}^3 \\
&= \; A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3 \;=\; A \star B
\end{aligned}$$

proving the invariance.

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The Minkowski inner product provides a Minkowski scalar, and scalars are invariant. Ordinary scalars (3-scalars) are invariant under rotations, whereas Minkowski scalars are invariant under Lorentz transformations. Another way of saying this: $A^\mu A_\mu = A_\mu A^\mu$ is a scalar (tensor of zero rank) and hence transforms as a scalar, i.e., is invariant.

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(c) The Lorentz transformation described here consists of a boost and a rotation. Describe in words how frame $\tilde{\Sigma}$ is moving and oriented with respect to frame Σ.

(Partial) Solution/Hint →

The transformation matrix is

$$\begin{pmatrix} \gamma_v & -\gamma_v \dfrac{v}{c} & 0 & 0 \\ -\gamma_v \dfrac{v}{c} & \gamma_v & 0 & 0 \\ 0 & 0 & \cos\theta & \sin\theta \\ 0 & 0 & -\sin\theta & \cos\theta \end{pmatrix}$$

We recognize the two sectors: ct and x are mixed according to a Lorentz boost while y and z are mixed according to a rotation.

The frame $\tilde{\Sigma}$ moves with velocity v relative to the frame Σ in the common x, $\tilde{x}$ direction. The y and z directions are not aligned with the $\tilde{y}$, $\tilde{z}$ directions. The $\tilde{y}$, $\tilde{z}$ directions are obtained from the y, z directions by rotating around the x (or $\tilde{x}$) axis.

Notice $\tilde{\Sigma}$ is rotated relative to Σ; however it is NOT *rotating* relative to Σ. The rotation angle is time-independent. If there was time-dependent rotation, then the transformation would not be a Lorentz transformation.

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2. In the lab frame, particle B moves to the right with speed u, and particle C moves to the left with speed v. In the frame of C, particle B is seen to move to the right with speed w, while particle C itself is of course at rest.

   Of course, w can be written down in terms of u and v using the velocity addition formula, but we will re-derive this formula below using 4-velocities.

   I encourage you to use $c = 1$ units for this problem, to make the calculations less messy. (But you can choose, of course.)

   > When you read this problem, I hope you immediately sketched the situation as seen from the lab frame and, separately, as seen from the frame of C.

   (a) In the lab frame, write down the 4-velocities for particle B and for particle C.

   (Partial) Solution/Hint →

   Let's choose the rightward direction to be the positive x axis. You could make another choice, but you should say so clearly, if you do.

   $$(\gamma_u c, \gamma_u u, 0, 0) \qquad (\gamma_v c, -\gamma_v v, 0, 0)$$

   Note the minus sign in one case!!
   (Exercise: imagine that u and v were not speeds but velocities. How would the expressions above change?)

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   (b) In C's frame, write down the 4-velocities for B and C.

   (Partial) Solution/Hint →

   $$(\gamma_w c, \gamma_w w, 0, 0) \qquad (\gamma_0 c, 0, 0, 0) = (c, 0, 0, 0)$$

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(c) The inner product of the two 4-velocities should be invariant. Write down an equation equating the inner product in the two frames.

(Partial) Solution/Hint →

$$(\gamma_u c)(\gamma_v c) - (\gamma_u u)(-\gamma_v v) \;=\; (\gamma_w c)(c) \qquad \Longrightarrow \qquad \gamma_u\gamma_v c^2 + \gamma_u\gamma_v uv = \gamma_w c^2$$

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(d) From the inner product invariance, derive an expression for γ_w in terms of γ_u, γ_v, u and v. (This equation should be familiar.)

(Partial) Solution/Hint →

$$\gamma_w \;=\; \gamma_u\gamma_v\left(1 + \frac{uv}{c^2}\right)$$

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(e) From the equation relating γ's, derive an expression for w in terms of u and v. You should have recovered a familiar formula.

(Partial) Solution/Hint →

It's getting hairy so I switch to $c = 1$ units.

$$\begin{aligned}
\frac{1}{\sqrt{1-w^2}} \;&=\; \frac{1}{\sqrt{1-u^2}}\frac{1}{\sqrt{1-v^2}}(1+uv) \\
\Longrightarrow \quad 1 - w^2 \;&=\; \frac{(1-u^2)(1-v^2)}{(1+uv)^2} \\
\Longrightarrow \qquad w^2 \;&=\; \frac{(1+uv)^2 - (1-u^2)(1-v^2)}{(1+uv)^2} \;=\; \frac{(u+v)^2}{(1+uv)^2}
\end{aligned}$$

Thus we have obtained the velocity addition formula $w = \dfrac{u+v}{1+uv}$.

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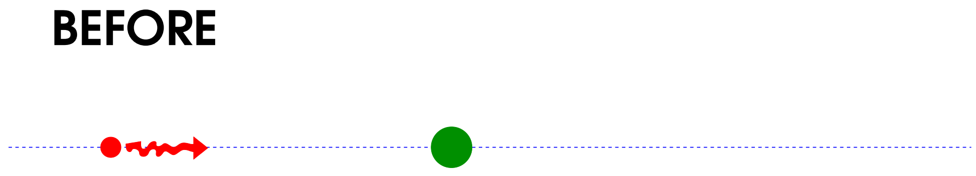
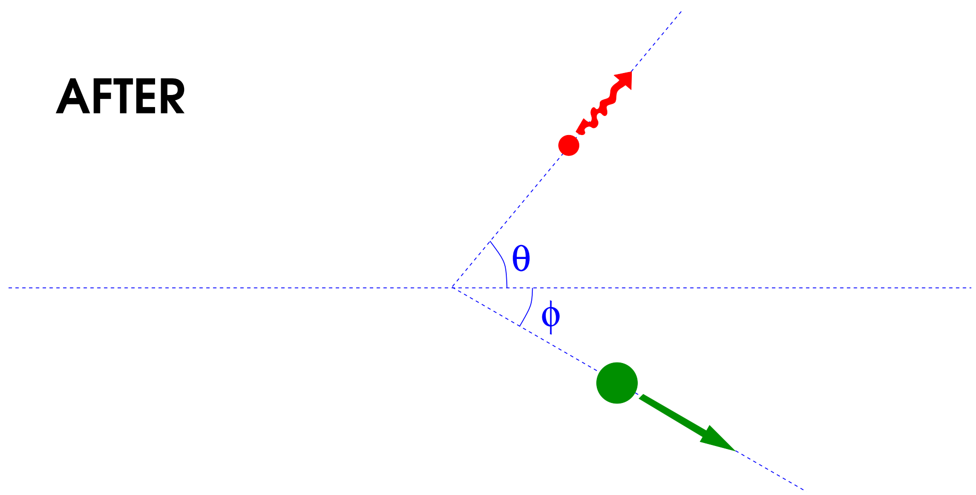
3. (Compton scattering.) A photon of wavelength λ collides with a stationary electron. After the collision, the photon scatters at an angle θ with respect to the incident direction, and has wavelength λ'. The electron moves with momentum p_e after the collision, in a direction making angle ϕ with the incident direction of the photon.

   (a) Draw the situations before and after, clearly showing everything relevant.

   (Partial) Solution/Hint →





We have denoted by ϕ the angle at which the electron scatters.

Note the electron and photon scatter on opposite sides of the incident direction. This is necessary because otherwise the momentum perpendicular to the incident direction cannot be conserved. For most of us, this is intuitively 'obvious'. If one takes both to be on the same side, when writing the momentum conservation equations and solving later on, one would find a minus for the angle.

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(b) Write down the equation for energy conservation, and two equations for momentum conservation. (Since momentum is a vector, you have one equation for each relevant direction.)

(Partial) Solution/Hint →

We denote by ϕ the angle at which the electron scatters.

$$\text{Energy conservation:} \qquad \frac{hc}{\lambda} + mc^2 = \frac{hc}{\lambda'} + \sqrt{p_e^2 + m^2c^4} +$$

$$\text{Momentum conservation 1:} \qquad \frac{h}{\lambda} = \frac{h}{\lambda'}\cos\theta + p_e\cos\phi$$

$$\text{Momentum conservation 2:} \qquad 0 = \frac{h}{\lambda'}\sin\theta - p_e\sin\phi$$

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(c) Show that

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta)$$

(Partial) Solution/Hint →

The energy equation gives

$$p_e^2 c^2 + m^2 c^4 \;=\; \left(\frac{hc}{\lambda} \;-\; \frac{hc}{\lambda'} \;+\; mc^2\right)^2$$

$$\Longrightarrow\; p_e^2 c^2 \;=\; \left(\frac{hc}{\lambda}\right)^2 + \left(\frac{hc}{\lambda'}\right)^2 - \frac{2h^2c^2}{\lambda\lambda'} \;+\; 2\left(\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right) mc^2$$

while the momentum equations give (after getting rid of ϕ):

$$p_e^2 \;=\; \left(\frac{h}{\lambda}\right)^2 + \left(\frac{h}{\lambda'}\right)^2 - \frac{2h^2}{\lambda\lambda'}\cos\theta$$

Equating the two expressions for p_e^2 gives

$$\frac{2h^2}{\lambda\lambda'}(1 - \cos\theta) \;=\; 2\left(\frac{hc}{\lambda} - \frac{hc}{\lambda'}\right) m$$

which gives the desired expression

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta)$$

Note how we completely avoided velocity, and worked in terms of the momentum and energy only.

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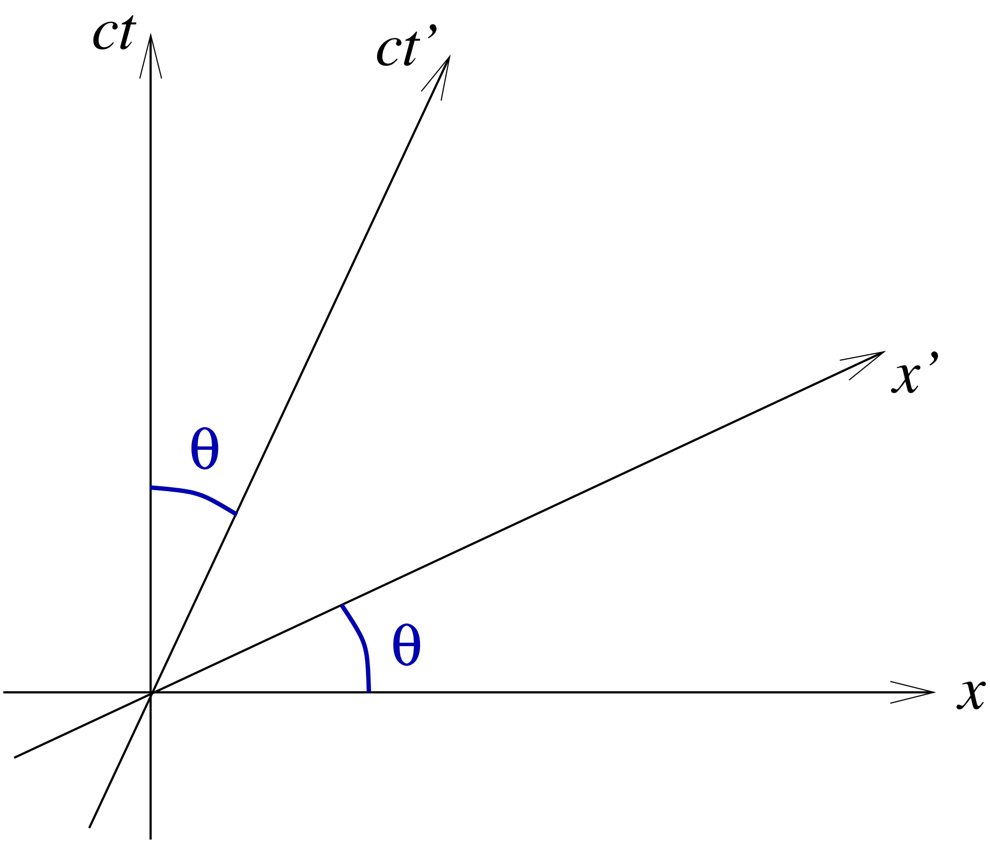
4. Frames Σ and Σ' are aligned at $t = t' = 0$, and Σ' moves with velocity v relative to Σ in the common x, x' direction.

   (a) Represent (draw) both frames on a common spacetime diagram such that the ct and x axes are perpendicular. What angles are made by the ct' and x' axes? How are the units for x' related to the units for x?

   (Partial) Solution/Hint →



The angle between the ct and ct' axis is the same as the angle between the x and x':

$$\tan\theta = \frac{v}{c}$$

One way to derive this is to use the LT to calculate the (ct, x) coordinates corresponding to $(ct', x') = (0, 1)$ or $(ct', x') = (1, 0)$.

The same calculation would show that

$$\text{One } x' \text{ unit} = \sqrt{\frac{c^2 + v^2}{c^2 - v^2}} \times \text{One } x \text{ unit}$$

That means the Euclidean distance (on the plane of the spacetime diagram) representing unit Σ'-distance along the x' direction is $\sqrt{\dfrac{c^2+v^2}{c^2-v^2}}$ times larger than the Euclidean distance on the spacetime diagram representing unit Σ-distance along the x direction.

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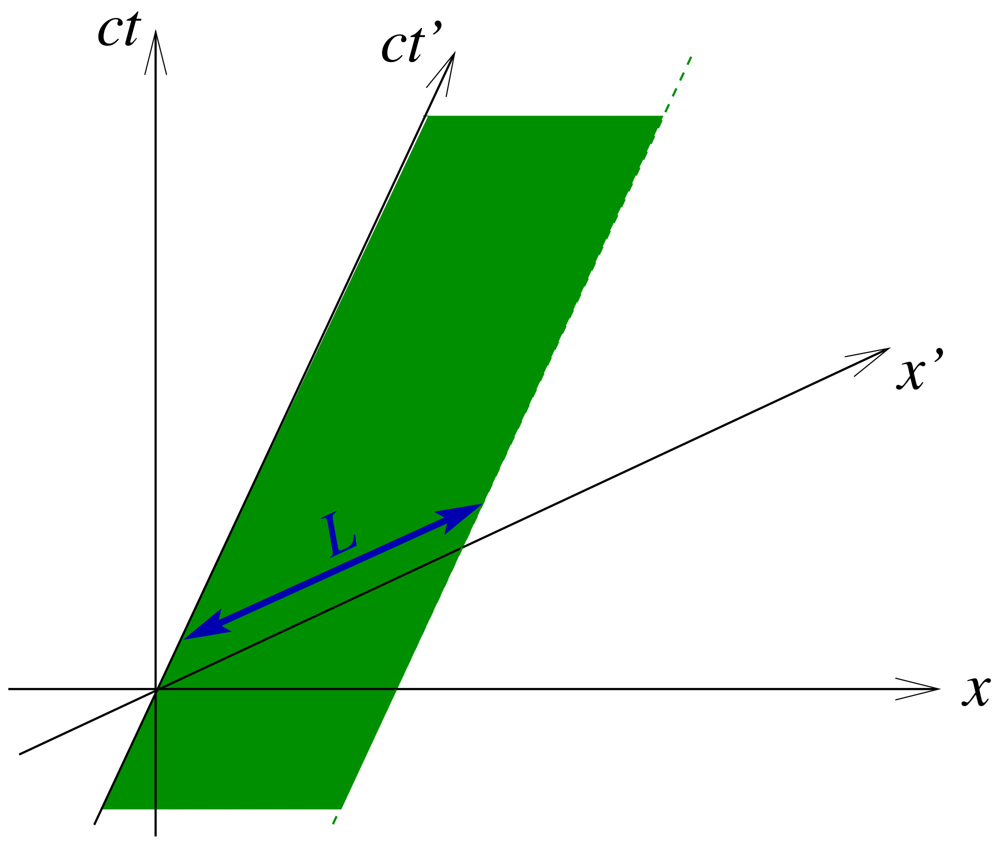
(b) A stick of length L lies at rest in the Σ' frame, with one end at the origin of the Σ' frame. Draw the world sheet of the stick.

(Partial) Solution/Hint →



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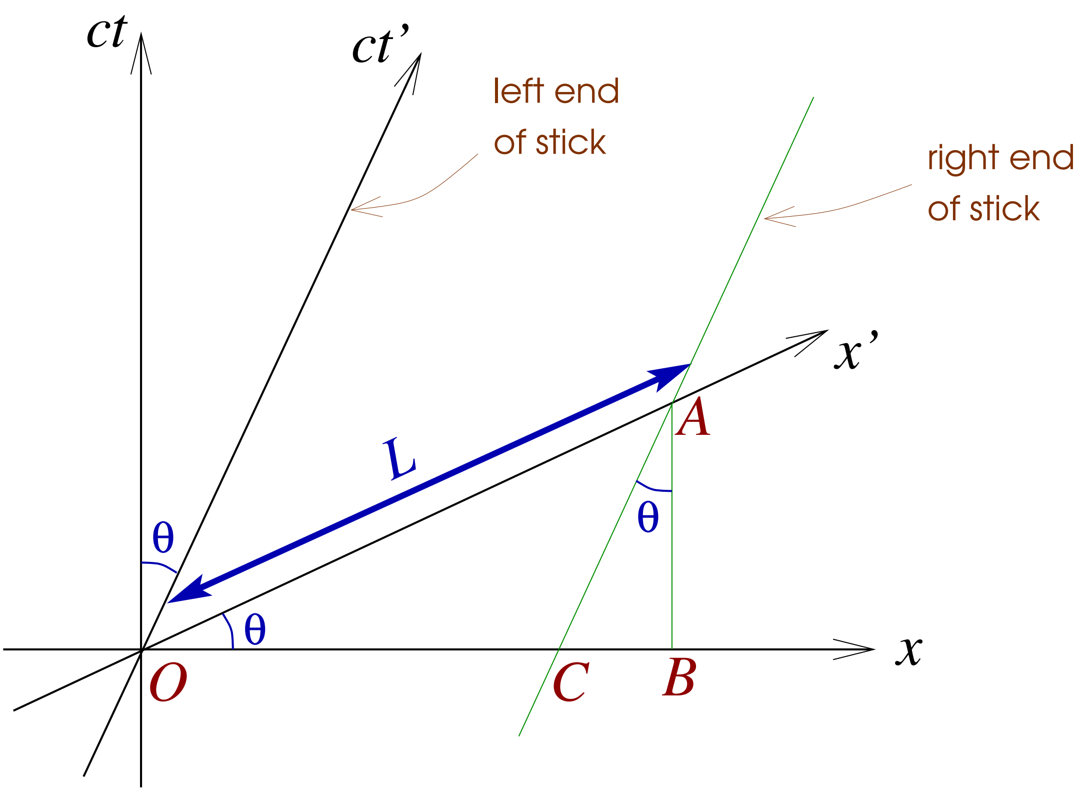
(c) Calculate geometrically the length of the stick measured in the Σ frame. (This will require careful drawings, so please do not submit your first attempt — first work out on rough paper what you want to present, and then produce neat drawings.)

(Partial) Solution/Hint →



The length of the stick as measured from Σ is the distance between the two ends of the stick simultaneously in Σ. Simultaneous in Σ means two points (events) on the spacetime diagram which are on the same horizontal line. Hence the distance between points marked O and C is the length as measured from Σ. We can geometrically relate this to the length $|OA|$ as follows:

$$\begin{aligned} |OC| \;&=\; |OB| - |CB| \;=\; |OA|\cos\theta - |AB|\tan\theta \\ &=\; |OA|\cos\theta - |OA|\sin\theta\tan\theta \;=\; |OA|\cos\theta\left(1-\tan^2\theta\right) \end{aligned}$$

> **NOTE!!!** It might be tempting to write $|OC| = |OA| \sin\theta$ from looking at the triangle OCA. This would be completely wrong, because OCA is NOT a right-angled triangle.
> On the other hand, OBA and CBA are nice right-angled triangles, which is what we used to write $|OB| = |OA| \cos\theta$ and $|CB| = |AB| \tan\theta$.

Remembering $\tan\theta = \frac{v}{c}$,

$$\begin{aligned}\cos\theta\left(1-\tan^2\theta\right) &= \frac{1}{\sec\theta}\left(1-\tan^2\theta\right) = \frac{1}{\sqrt{1+\tan^2\theta}}\left(1-\tan^2\theta\right) \\ &= \frac{1}{\sqrt{1+(v/c)^2}}\left(1-(v/c)^2\right)\end{aligned}$$

Thus

$$|OC| = |OA|\,\frac{1-(v/c)^2}{\sqrt{1+(v/c)^2}}$$

We know that the length $|OA|$ is equal to L in x' units, hence it is

$$L \times \sqrt{\frac{c^2+v^2}{c^2-v^2}} = L\sqrt{\frac{1+(v/c)^2}{1-(v/c)^2}} \qquad \text{in } x \text{ units.}$$

Therefore the required length in Σ, i.e., the distance $|OC|$ in x units, is

$$L\sqrt{\frac{1+(v/c)^2}{1-(v/c)^2}} \times \frac{1-(v/c)^2}{\sqrt{1+(v/c)^2}} = L\sqrt{1-(v/c)^2} = \frac{L}{\gamma_v}$$

which is the length contraction formula.

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