Due on Monday, April 19th.

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1. $A = (A^0, A^1, A^2, A^3)$ and $B = (B^0, B^1, B^2, B^3)$ are 4-vectors. Under a Lorentz transformation from frame Σ to $\tilde{\Sigma}$, the components of A transform as
$$\tilde{A}^0 = \gamma_v \left(A^0 - \frac{v}{c} A^1 \right) \; , \quad \tilde{A}^1 = \gamma_v \left(A^1 - \frac{v}{c} A^0 \right) \; ,$$
$$\tilde{A}^2 = A^2 \cos\theta + A^3 \sin\theta \; , \quad \tilde{A}^3 = -A^2 \sin\theta + A^3 \cos\theta \; .$$

   (a) [**2 pts**] Write down the transformations for the components of B.

   (b) [**6 pts**] Show that the inner product, defined as
$$A \star B = A^0 B^0 - A^1 B^1 - A^2 B^2 - A^3 B^3 \, ,$$
   is invariant under this transformation.

   (c) [**2 pts**] The Lorentz transformation described here consists of a boost and a rotation. Describe in words how frame $\tilde{\Sigma}$ is moving and oriented with respect to frame Σ.

   (d) [**SELF**] Write the given transformations (for the components of A) as a matrix equation.

2. In the lab frame, particle B moves to the right with speed u, and particle C moves to the left with speed v. In the frame of C, particle B is seen to move to the right with speed w, while particle C itself is of course at rest.

   Of course, w can be written down in terms of u and v using the velocity addition formula, but we will re-derive this formula below using 4-velocities.

   I encourage you to use $c = 1$ units for this problem. (But you can choose.)

   (a) [**3 pts**] In the lab frame, write down the 4-velocities for particle B and for particle C.

   (b) [**3 pts**] In C's frame, write down the 4-velocities for B and C.

   (c) [**3 pts**] The inner product of the two 4-velocities should be invariant. Write down an equation equating the inner product in the two frames.

   (d) [**3 pts**] From the inner product invariance, derive an expression for γ_w in terms of γ_u, γ_v, u and v. (This equation should be familiar.)

   (e) [**3 pts**] From the equation relating γ's, derive an expression for w in terms of u and v. You should have recovered a familiar formula.

3. (Compton scattering.) A photon of wavelength λ collides with a stationary electron. After the collision, the photon scatters at an angle θ with respect to the incident direction, and has wavelength λ'. The electron moves with momentum p_e after the collision, in a direction making angle ϕ with the incident direction of the photon.

   (a) **[3 pts]** Draw the situations before and after, clearly showing everything relevant.

   (b) **[6 pts]** Write down the equation for energy conservation, and two equations for momentum conservation. (Since momentum is a vector, you have one equation for each relevant direction.)

   (c) **[6 pts]** Show that

$$\lambda' = \lambda + \frac{h}{mc}(1 - \cos\theta)$$

   Hint: you don't need to introduce velocity. If you need to transform between energy and momentum, use the relation $E^2 = p^2c^2 + m^2c^4$.

4. Frames Σ and Σ' are aligned at $t = t' = 0$, and Σ' moves with velocity v relative to Σ in the common x, x' direction.

   (a) **[2 pts]** Represent (draw) both frames on a common spacetime diagram such that the ct and x axes are perpendicular. What angles are made by the ct' and x' axes? How are the units for x' related to the units for x?

   (b) **[SELF]** A stick of length L lies at rest in the Σ' frame, with one end at the origin of the Σ' frame. Draw the world sheet of the stick.

   (c) **[8 pts]** Calculate geometrically the length of the stick measured in the Σ frame. (This will require careful drawings, so please do not submit your first attempt — first work out on rough paper what you want to present, and then produce neat drawings.)