1. As measured in frame Σ, the coordinates of events A and B are respectively (ct_A, x_A) and (ct_B, x_B). The interval between the events are spacelike.

   There is then a frame Σ' in which the two events are simultaneous.

   (a) Plot the two events on a spacetime diagram, taking the Σ frame to have ct and x axes perpendicular to each other. Plot also the axes of the Σ' frame, so that it is clear that A and B are simultaneous in this frame. You only get credits if this last feature is evident in your drawing.

   (Partial) Solution/Hint →

   Simultaneity in the Σ' frame requires that the x' frame should be parallel to the line joining the two points representing events A and B. I won't try drawing this electronically.

   — * —

   (b) Using the spacetime diagram, calculate the speed v of the Σ' frame relative to the Σ frame, in terms of t_A, t_B, c, x_A, x_B.

   (Partial) Solution/Hint →

   The slope of the line joining the two points is

$$\frac{ct_A - ct_B}{x_A - x_B}$$

   Since this is parallel to the x' axis, it needs to have the same slope. The slope of the x' axis is

$$\tan\theta = \frac{v}{c}$$

   Equating the two slopes, we get

$$v = c^2 \frac{t_A - t_B}{x_A - x_B}$$

   — * —

(c) Now calculate the same expression for v using the Lorentz transformations.

(Partial) Solution/Hint →

Using the Lorentz transformations:

$$ct'_A = \gamma_v \left(ct_A - \frac{v}{c} x_A \right) \qquad ct'_B = \gamma_v \left(ct_B - \frac{v}{c} x_B \right)$$

Imposing $t'_A = t'_B$ (simultaneity in the S' frame) gives us

$$\frac{v}{c} = \frac{ct_A - ct_B}{x_A - x_B}$$

— * —

2. The 4-velocity and 4-acceleration of a particle are given by

$$V \;=\; (\gamma_v c,\; \gamma_v \vec{v})\;, \qquad A \;=\; \left(\frac{\gamma_v^4}{c} v\dot{v},\; \frac{\gamma_v^4}{c^2} v\dot{v}\vec{v} + \gamma_v^2 \vec{a}\right)\;,$$

where $\vec{v}$ is the 3-velocity, $v = \sqrt{\vec{v}\cdot\vec{v}}$ is the speed, and $\dot{v} = \frac{dv}{dt}$ is NOT the 3-acceleration $\vec{a}$. Show that the 4-velocity and 4-acceleration are 'orthogonal' to each other, i.e., have zero inner product.

From problem set 1, we know $v\dot{v} = \vec{v}\cdot\vec{a}$.

<u>**(Partial) Solution/Hint →**</u>

$$\begin{aligned}
V \star A \;&=\; (\gamma c)\left(\frac{\gamma^4}{c} v\dot{v}\right) - (\gamma \mathbf{v})\cdot\left(\frac{\gamma^4}{c^2} v\dot{v}\mathbf{v} + \gamma^2\mathbf{a}\right) \\
&=\; \gamma^5 v\dot{v} - \frac{\gamma^5}{c^2} v\dot{v}(\mathbf{v}\cdot\mathbf{v}) - \gamma^3(\mathbf{v}\cdot\mathbf{a}) \\
&=\; \gamma^5 v\dot{v} - \gamma^5\frac{v^2}{c^2} v\dot{v} - \gamma^3 v\dot{v} \qquad\qquad \text{using } \mathbf{v}\cdot\mathbf{a} = v\dot{v} \\
&=\; \gamma^5 v\dot{v}\left(1 - \frac{v^2}{c^2} - \gamma^{-2}\right) \;=\; 0 \qquad\qquad \text{since } \gamma^{-2} = 1 - \frac{v^2}{c^2}
\end{aligned}$$

Note above that we used $v\dot{v} = \vec{v}\cdot\vec{a}$ from Problem Set 1.

Also, obviously, γ means γ_v; I dropped the subscript to avoid clutter. But you should do this ONLY when you are sure there is only one speed in the problem, so there is no confusion which speed γ refers to .

Alternate! Since inner products are invariant, we could work in any frame. The problem becomes simple if we work in the instantaneous rest frame of the particle. Since we are thinking of accelerating particles, the rest frame itself changes, so we need to think of the instantaneous rest frame. In this frame $v = 0$ and $\gamma_v = \gamma_0 = 1$; hence

$$V \;=\; (c, 0)\;, \qquad A \;=\; (0, \vec{a})$$

Hence

$$V \star A \;=\; 0 - 0 \;=\; 0$$

Notation comment. Above, I've used the notation $\star$ for the Minkowski inner product. This is not really standard. Sometimes authors simply use a dot, $V \cdot A$, but then it's not immediately clear that you don't mean the 'usual' (Euclidean) dot product between 3-vectors. If you write $g_{\mu\nu}V^{\mu}A^{\nu}$ or $V_{\mu}A^{\mu}$, there should be no mistake that you are referring to the Minkowski inner product.

Of course, the Minkowski inner product itself depends on the metric convention. In this module we have used the convention $(+,-,-,-)$, in which case you get the inner product used here. In the 'other' convention, the inner product is the negative of what we used here. If you stick to our convention, you should be safe for this module. If you look up a book or continue with theoretical physics, you will have to be able to translate between the two conventions.

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3. Four-momenta.

(a) Show that the four-momentum of a particle with nonzero mass is always timelike.

(Partial) Solution/Hint →

4-momentum is

$$(E/c, p_x, p_y, p_z)$$

The norm-square is

$$\begin{aligned} p_\mu p^\mu &= (E/c)^2 - p_x^2 - p_y^2 - p_z^2 \;=\; (E/c)^2 - p^2 \\ &\qquad\qquad\qquad\qquad = \; (p^2c^2 + m^2c^4)/c^2 - p^2 \;=\; m^2/c^2 > 0 \end{aligned}$$

Norm-squared being positive indicates that the 4-vector is time-like.

— * —

(b) Write down the four-momentum of a photon moving in the y direction. The frequency of the photon is denoted by f.

(Partial) Solution/Hint →

The energy is hf and the momentum is $(hf/c)\vec{e}_y$. The 4-momentum is thus

$$(hf/c, 0, hf/c, 0)$$

— * —

(c) Find out whether the 4-momentum of a photon is timelike, spacelike, or lightlike/null.

(Partial) Solution/Hint →

For the photon 4-momentum we have written down above, the norm-squared is

$$(hf/c)^2 - 0^2 - (hf/c)^2 - 0^2 = 0$$

and it is also zero for any other direction of the photon motion. Hence the 4-momentum of a photon is light-like or null.

— * —

4. Four-velocities.

   (a) Write down the four-velocity of a particle with nonzero mass m moving in the y direction with speed v. Show that this four-vector is timelike.

   (Partial) Solution/Hint →

$$(\gamma_v c, 0, \gamma_v v, 0)$$

   The four-vector is timelike if it's norm-squared in the $(+,-,-,-)$ metric is positive:

$$v_\mu v^\mu \;=\; (\gamma_v c)^2 - 0^2 - (\gamma_v v)^2 - 0^2 \;=\; \gamma_v^2(c^2 - v^2) \;=\; \frac{c^2}{c^2 - v^2}(c^2 - v^2) \;=\; c^2$$

   which is obviously positive; hence the 4-velocity is a timelike 4-vector.

   If the other convention is used, the sign of the norm-squared is flipped and in that case negative norm-squared indicates timelike 4-vector.

   In either way, the 4-velocity of a particle with nonzero mass is found to be timelike. This is something to remember. The same is true for the 4-momentum of a particle with nonzero mass (always timelike), but not for the 4-acceleration (always spacelike).

— * —

(b) Find the four-velocity of a particle with nonzero mass m and velocity $\vec{u} = (c/2, c/2, c/2)$. Show that this four-vector is timelike.
Is this particle massless?

(Partial) Solution/Hint →

The speed of the particle is

$$\sqrt{\left(\frac{c}{2}\right)^2 + \left(\frac{c}{2}\right)^2 + \left(\frac{c}{2}\right)^2} \;=\; \frac{\sqrt{3}}{2}\,c$$

The γ factor corresponding to this speed is

$$\gamma\left(\frac{\sqrt{3}}{2}\,c\right) \;=\; \frac{1}{\sqrt{1-(3/4)}} \;=\; 2$$

Hence the 4-velocity is

$$2(c, c/2, c/2, c/2) \;=\; (2c, c, c, c)$$

The norm-squared is

$$(2c)^2 - c^2 - c^2 - c^2 \;=\; c^2 > 0$$

Hence the four-velocity is a timelike four-vector.
Massless particles move at the speed of light. The particle here has speed $\frac{\sqrt{3}}{2}c$ which is less than c. Thus the particle is not massless.

— * —

(c) What is the four-velocity of the particle with mass m in its own rest frame, i.e., in the frame where it is at rest?

(Partial) Solution/Hint →

The speed v is zero, so that $\gamma_v = 1$. The 4-velocity is

$$(\gamma_v c, 0, 0, 0) = (c, 0, 0, 0)$$

— * —

(d) Try writing down the four-velocity of a photon moving in the y direction. Is there a problem?

(Partial) Solution/Hint →

One could try writing

$$(\gamma_v c, 0, \gamma_v c, 0)$$

But then we notice that $\gamma_v = \gamma_c = \infty$. The 4-vector thus has two infinite components.

$\longrightarrow$ 4-velocities are only well-defined for massive particles.

— * —

5. Collision/decay problems.

   (a) A π-meson (a.k.a. pion) of mass M at rest disintegrates into a μ-meson (a.k.a. muon) of mass m_μ and a neutrino of effectively zero mass. Show that the kinetic energy of motion of the muon is

$$T = \frac{(M - m_\mu)^2}{2M} c^2$$

As usual: first draw (separately!) the situations before and after the reaction/decay, write down equations for energy and momentum conservation in terms of the most useful variables, and then manipulate these equations as required.

You should decide early whether it is helpful to work with speed and γ, or whether to avoid these and work with momentum instead.

You probably want to use $c = 1$ units for this problem.

(Partial) Solution/Hint $\rightarrow$

I am using $c = 1$ units.

Momentum conservation requires that the two created particles move in opposite directions after being created. Let's take the direction of the muon (μ-meson) to be positive, and refer to its momentum as p. Since the neutrino has an effectively zero mass, we can treat it like a photon, with energy hf and momentum $hf/c = hf$.

Energy conservation: $M = \sqrt{p^2 + m_\mu^2} + hf$

Momentum conservation: $0 = p - hf$

Notice: I made the choice to use p and $\sqrt{p^2 + m_\mu^2}$ for the muon momentum and energy, instead of expressing these in terms of speed/velocities, i.e., instead of using $\gamma_v m_\mu v$ and $\gamma_v m_\mu c^2 = \gamma_v m_\mu$. If you want, you can work in terms of v and γ_v, this will be messier but should give the same result in the end. This is also worked out below.

Eliminating hf,

$$M = \sqrt{p^2 + m_\mu^2} + p = (T + m_\mu) + \sqrt{T^2 + 2Tm_\mu}$$
$$\implies \sqrt{T^2 + 2Tm_\mu} = M - T - m_\mu$$

Squaring and collecting terms give

$$2TM = M^2 + m_\mu^2 - 2Mm_\mu = (M - m_\mu)^2$$

which is what we are asked to prove, in $c = 1$ units.

Alternate! We could alternately write the momentum and energy of the μ-meson in terms of the speed. In terms of v and γ_v, the kinetic energy is $T = \gamma_v m_\mu - m_\mu = (\gamma_v - 1)m_\mu$.
Energy conservation: $M = \gamma_v m_\mu + hf$
Momentum conservation: $0 = \gamma_v m_\mu v - hf$
Eliminating hf, we obtain

$$M \;=\; \gamma_v m_\mu \;+\; \gamma_v m_\mu v$$

Noting that

$$\gamma_v = \frac{1}{\sqrt{1-v^2}} \qquad \Longrightarrow \qquad v = \sqrt{1-\frac{1}{\gamma_v^2}} = \frac{\sqrt{\gamma_v^2-1}}{\gamma_v}$$

we obtain

$$M \;=\; \gamma_v m_\mu \;+\; \gamma_v m_\mu \frac{\sqrt{\gamma_v^2-1}}{\gamma_v} \qquad \Longrightarrow \qquad M - \gamma_v m_\mu \;=\; m_\mu\sqrt{\gamma_v^2-1}$$

Squaring both sides,

$$M^2 - 2M\gamma_v m_\mu \;=\; -\,m_\mu^2 \qquad \Longrightarrow \qquad \gamma_v \;=\; \frac{M^2+m_\mu^2}{2Mm_\mu}$$

$$\Longrightarrow \qquad T \;=\; (\gamma_v-1)m_\mu \;=\; \frac{M^2+m_\mu^2-2Mm_\mu}{2Mm_\mu}m_\mu \;=\; \frac{(M-m_\mu)^2}{2M}$$

The calculation in terms of v and γ_v looks somewhat more messy to me, but it's a matter of taste. You should be able to do it either way.

— * —

(b) A neutral pion of rest mass m, while moving in the positive x direction, decays into two photons. One of the photons is emitted in the same direction as the original pion, and the other in the opposite direction. If the first photon has three times the energy of the second, find the pion's original speed.

(Partial) Solution/Hint →

You should really reall draw a before picture and an after picture. The following should then make sense.

Let the frequency of the second photon (moving in the negative x direction) be f, then its energy is hf and momentum is $-hf/c$.

The energy of the first photon (moving in the positive x direction) is then $3hf$, hence its frequency is $3f$ and momentum is $+3hf/c$.

Energy conservation implies:

$$\gamma_v mc^2 \;=\; 3hf + hf \;=\; 4hf$$

Momentum conservation implies:

$$\gamma_v mv \;=\; \frac{3hf}{c} - \frac{hf}{c} \;=\; \frac{2hf}{c}$$

Dividing, we get

$$\frac{\gamma_v mv}{\gamma_v mc^2} = \frac{2hf/c}{4hf} \qquad \Longrightarrow \qquad \boxed{v = \tfrac{c}{2}}$$

NOTE! It might be tempting to write, e.g. from the energy equation

$$\gamma_v \;=\; \frac{4hf}{mc^2} \qquad \Longrightarrow \qquad v \;=\; c\sqrt{1-\left(\frac{mc^2}{4hf}\right)^2}$$

and present this as the answer. Why is this a non-answer?

This ‘answer’ expresses v in terms of f, a symbol introduced during solution and not defined in the problem, i.e., it is an unknown quantity. Thus we are then giving the answer in terms of an unknown quantity.

It’s as silly as saying, let’s define the quantity u to be half the pion speed, then the speed of the pion is $2u$. There is nothing wrong with the statement, but it does not address the given question at all.

When providing an answer, make sure that you’ve eliminated any quantities you introduced yourself. Symbols h and c are okay because they are fundamental constants, but the frequency of the photon is not known unless determined in terms of the quantities given.

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