Here are some hints or partial answers for problem set 11.
As usual, these were not proofread carefully, so please watch out for typographical or other errors.

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1. Collision/decay problems.

(a) Two photons have the same energy E_γ. The collide at an angle θ and create a single particle. Calculate the mass M of the final particle in terms of E_γ and θ.

(Partial) Solution/Hint →

By symmetry, the final particle must have momentum in a direction bisecting the trajectories of the two particles. In other words, the two photon momenta must each have angle $\theta/2$ with the direction of motion of the final particle.
(This can be worked out more elaborately by starting with an unknown angle for the final particle motion and using momentum conservation in two perpendicular directions to calculate this angle.)
Based on the before-&-after figures (which you MUST have drawn, surely?), we will write the energy and momentum conservation. In this case, there is no clear advantage in working with the speed formulae or with the momentum formulae, because the question doesn't ask to express anything in terms of the final speed or the final momentum. Either choice should work. Both possible calculations are shown below.

Using speeds.
Let's first try working with the speed formulae, denoting the speed of the final particle as v. Energy and momentum conservation give:

$$\begin{aligned} \text{Energy:} \qquad E_\gamma + E_\gamma &= \gamma_v M c^2 \\ \text{Momentum:} \qquad \frac{E_\gamma}{c}\cos(\theta/2) + \frac{E_\gamma}{c}\cos(\theta/2) &= \gamma_v M v \end{aligned}$$

Dividing, we get

$$v = c\cos(\theta/2)$$

$$\implies \gamma_v = \frac{1}{\sqrt{1-(v/c)^2}} = \frac{1}{\sqrt{1-\cos^2(\theta/2)}} = \frac{1}{\sin(\theta/2)}$$

Therefore
$$M = \frac{2E_\gamma}{\gamma_v c^2} = \frac{2E_\gamma}{c^2}\sin(\theta/2)$$

Using momenta.
Alternately, we could work with the momentum. Denoting the final particle momentum by p, the energy and momentum conservation equations are

$$\text{Energy:} \qquad E_\gamma + E_\gamma = \sqrt{p^2c^2 + M^2c^4}$$
$$\text{Momentum:} \qquad \frac{E_\gamma}{c}\cos(\theta/2) + \frac{E_\gamma}{c}\cos(\theta/2) = p$$

Substituting from the second equation into the first, we obtain

$$2E_\gamma = \sqrt{4E_\gamma^2\cos^2(\theta/2) + M^2c^4}$$
$$\Longrightarrow \quad M^2c^4 = 4E_\gamma^2 - 4E_\gamma^2\cos^2(\theta/2) = 4E_\gamma^2\sin^2(\theta/2)$$
$$\Longrightarrow \quad M = \frac{2E_\gamma}{c^2}\sin(\theta/2)$$

Thus the required expression can be obtained working either in terms of speed ($E = \gamma_v Mc^2$, $p = \gamma_v Mv$) or in terms of momentum ($E = \sqrt{p^2c^2 + M^2c^2}$).

NOTE! If you find yourself trying to solve such problems without drawing the before-&-after pictures, something is seriously wrong. You will almost certainly not get the correct answer.

Reminder.
For massive particles, we can work either using the speed v or the momentum p.
If using the speed v, the formulae are

$$\text{energy} = \gamma_v(\text{mass})c^2\ , \qquad \text{momentum} = \gamma_v(\text{mass})v\ .$$

If using the momentum p, the formulae are

$$\text{energy} = \sqrt{p^2c^2 + (\text{mass})^2c^4}\ , \qquad \text{momentum} = p\ .$$

Make sure you know all these expressions. In the problem above, either choice works: working with speed or working with momentum. In general, you need to make a choice based on what is asked for or given in the question.

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(b) Two balls of equal mass (m each) approach each other with equal but opposite velocities of magnitude v. Their collision is perfectly inelastic, so they stick together and form a single body of mass M. What is the velocity of the final body and what is its mass M?

Find the mass M in the specific cases of $v = 0.01c$, $v = 0.5c$, and $v = 0.9c$.

(Partial) Solution/Hint →

The final velocity is zero from symmetry. Alternately, you could assume a final velocity V_f, and write down the momentum conservation equation:

$$\gamma(v)mv - \gamma(v)mv \;=\; \gamma(V_f)MV_f$$

which gives $V_f = 0$.

NOTE! Above, the γ on the right side is different from the γ's on the left side. When writing a γ, it is essential to be clear which speed it corresponds to.

The energy conservation equation gives

$$\gamma(v)mc^2 + \gamma(v)mc^2 \;=\; \gamma(0)Mc^2 \quad\Longrightarrow\quad M \;=\; 2\gamma(v)m \;=\; \frac{2m}{\sqrt{1 - v^2/c^2}}$$

Using $v = 0.01c$ yields

$$M = \frac{2}{\sqrt{0.99}}m \;\approx\; 2.01m$$

Using $v = 0.5c$ yields

$$M = \frac{2}{\sqrt{0.75}}m \;\approx\; 2.31m$$

Using $v = 0.9c$ yields

$$M = \frac{2}{\sqrt{0.19}}m \;\approx\; 4.59m$$

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2. The four-vector A^μ is found to be timelike in one inertial frame.

   (a) Write an inequality expressing the fact that A^μ is timelike.

   (Partial) Solution/Hint →

   Timelike means that the zero ('time') component of the 4-vector dominates over the magnitude of the 3-vector part ('space' part). In other words

$$\left(A^0\right)^2 > \left(A^1\right)^2 + \left(A^2\right)^2 + \left(A^3\right)^2$$

   or

$$\left(A^0\right)^2 - \left(A^1\right)^2 - \left(A^2\right)^2 - \left(A^3\right)^2 > 0$$

   Clearly, this can be expressed in terms of the norm-squared.

   In the metric convention we have been using in class (diagonal elements of metric are $+1$, -1, -1, -1), this is expressed as

$$A_\mu A^\mu > 0 \qquad \text{or} \qquad g_{\mu\nu} A^\mu A^\nu > 0$$

   i.e., timelike means that the norm-squared is positive. This is because, in the $+---$ convention, the norm-squared is

$$g_{\mu\nu} A^\mu A^\nu \;=\; \left(A^0\right)^2 - \left(A^1\right)^2 - \left(A^2\right)^2 - \left(A^3\right)^2$$

   Note 1: this is the norm-squared, not the norm. It's best not to talk about the norm itself, because that can be either real or imaginary.

   Note 2: the notation $|A^\mu|$ is generally NOT used for Minkowski norms. This notation is associated too strongly with a positive real quantity, Please avoid! In fact, just avoid talking about the norm and stick to norm-squared.

   The other convention: If the metric is taken to have diagonal elements -1, $+1$, $+1$, $+1$ (as is common in general relativity), then the norm-squared has opposite sign, i.e.,

$$g_{\mu\nu} A^\mu A^\nu \;=\; -\left(A^0\right)^2 + \left(A^1\right)^2 + \left(A^2\right)^2 + \left(A^3\right)^2$$

   This means, in the other convention, timelike corresponds to negative norm-squared, i.e.,

$$\text{Other convention:} \qquad \begin{cases} g_{\mu\nu} A^\mu A^\nu < 0 & \text{timelike} \\ g_{\mu\nu} A^\mu A^\nu > 0 & \text{spacelike} \end{cases}$$

   Irrespective of convention, timelike still means that the 0 component dominates, and spacelike means that the 123 components dominate.

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(b) Explain why A^μ has to be timelike in any inertial frame.

(Partial) Solution/Hint →

The norm of a 4-vector is invariant under Lorentz transformations, i.e., the same in every inertial frame. Therefore if the norm-squared is positive in one reference frame, it will be positive in every reference frame. i.e., if A^μ is timelike in one intertial frame, it has to be timelike in any inertial frame.

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3. The mass m and the charge q of a particle are four-scalars. Explain why the combination

$$(m, q, m, q)$$

is not a four-vector.

(Partial) Solution/Hint →

Since m and q do not change under Lorentz transformations, the elements of the above quadruplet do not transform like a 4-vector should, i.e., they do not transform according to Lorentz transformations.

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4. Electromagnetism.

(a) Show that the tensor equation $\partial_\mu J^\mu = 0$ is equivalent to the continuity equation of electromagnetism. Here J^μ is the current density 4-vector.

(Partial) Solution/Hint →

The 4-vector is defined as

$$J^\mu = \left(c\rho, \vec{J}\right) = (c\rho, J_x, J_y, J_z) \qquad \text{or} \qquad J^\mu = \mu_0\,(c\rho, J_x, J_y, J_z)$$

For the purposes of this problem, a constant factor will not matter, so let's use the first definition.
The tensor equation $\partial_\mu J^\mu = 0$ becomes, after writing out the summation over the contracted indices,

$$\partial_0 J^0 + \partial_1 J^1 + \partial_2 J^2 + \partial_3 J^3 \;=\; 0$$

The zeroth term is

$$\partial_0 J^0 = \frac{\partial}{\partial x^0} J^0 = \frac{\partial}{\partial (ct)} J^0 = \frac{\partial}{\partial (ct)} J^0 = \frac{1}{c}\frac{\partial}{\partial t} J^0 = \frac{1}{c}\frac{\partial}{\partial t}(c\rho) = \frac{\partial \rho}{\partial t}$$

The other ('spatial') terms are

$$\begin{aligned}\partial_1 J^1 + \partial_2 J^2 + \partial_3 J^3 \;&=\; \frac{\partial}{\partial x^1} J^1 + \frac{\partial}{\partial x^2} J^2 + \frac{\partial}{\partial x^3} J^3 \\ &=\; \frac{\partial}{\partial x} J_x + \frac{\partial}{\partial y} J_y + \frac{\partial}{\partial z} J_z \;=\; \vec{\nabla}\cdot\vec{J}\end{aligned}$$

Thus in 3-vector notation the tensor equation becomes

$$\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\vec{J} \;=\; 0$$

which is the continuity equation.

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(b) The two inhomogeneous Maxwell's equations can be expressed as the tensor equation for the field tensor $F^{\mu\nu}$:

$$\partial_\mu F^{\mu\alpha} \;=\; J^\alpha$$

Derive the continuity equation from this equation, using the fact that the field tensor is antisymmetric.

(Partial) Solution/Hint →

The continuity equation involves a 4-gradient of the current 4-vector; hence we should use the equation $\partial_\mu F^{\mu\alpha} = J^\alpha$ to obtain a 4-gradient of J^α. This is obtained by applying the operator ∂_α to both sides, so that the index α is contracted (summed over):

$$\partial_\alpha \partial_\mu F^{\mu\alpha} \;=\; \partial_\alpha J^\alpha \tag{1}$$

The double sum on the left can be seen to be equal to its own negative, and hence zero, due to antisymmetry of the field tensor:

$$\partial_\alpha \partial_\mu F^{\mu\alpha} \;=\; \partial_\alpha \partial_\mu \left(-F^{\alpha\mu}\right) \;=\; -\,\partial_\mu \partial_\alpha F^{\alpha\mu} \;=\; -\,\partial_\alpha \partial_\mu F^{\mu\alpha}$$

In the last step, the indices have been switched, which we are allowed to do because they are dummy indices. Thus the quantity $\partial_\alpha \partial_\mu F^{\mu\alpha}$ is zero. The left side of Eq. (1) being zero, the right side should be zero as well; hence

$$\partial_\alpha J^\alpha \;=\; 0$$

which is the continuity equation.

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5. Minkowski tensors.

(a) Consider the Minkowski tensors

$$D^{\mu\nu}\ , \qquad D^{\mu\nu}B_\sigma\ , \qquad D^{\mu\sigma}B_\sigma\ , \qquad D^{\mu\sigma}D_{\mu\sigma}\ , \qquad C^\mu B_\mu\ , \qquad C^\mu B_\sigma\ .$$

Explain the rank of each tensor.
Which of these, if any, are scalars?

(Partial) Solution/Hint →

$D^{\mu\nu}$ is a tensor of rank 2. Both indices are contravariant; we could say it is a contravariant tensor of of rank 2.

$D^{\mu\nu}B_\sigma$ can be written in the form $D^{\mu\nu}B_\sigma = T^{\mu\nu}{}_\sigma$. It is a tensor of (total) rank 3, since there are three indices. It is neither fully contravariant nor fully covariant; sometimes called a mixed tensor. You could also refer to it as a tensor of rank $(2, 1)$.

$D^{\mu\sigma}B_\sigma = \sum_\sigma D^{\mu\sigma}B_\sigma$. Now the fun begins. The σ index is repeated and hence summed over, it's a dummy index. The result has only one index left. The quantity is a contravariant tensor of rank one, i.e., it's just a contravariant vector:

$$D^{\mu\sigma}B_\sigma = A^\mu \qquad \text{for some 4-vector } A^\mu$$

$D^{\mu\sigma}D_{\mu\sigma}$ is a scalar, as both indices are summed over and there are no free (not contracted) indices remaining.

$C^\mu B_\mu$ is a scalar, as μ is summed over and there are no un-contracted idices left. In other words, this is the inner product of two four-vectors and hence a scalar: $C^\mu B_\mu = g_{\mu\nu}C^\mu B^\nu$.

$C^\mu B_\sigma$ is a mixed tensor of rank 2:

$$C^\mu B_\sigma \ = \ U^\mu{}_\sigma$$

Among the examples above, only $D^{\mu\sigma}D_{\mu\sigma}$ and $C^\mu B_\mu$ are scalars.

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(b) Show how the tensor $T_\gamma^{\alpha\beta}$ transforms under the Lorentz transformation Λ.

(Partial) Solution/Hint →

Preliminaries.

Let's first review the transformation of 4-vectors, i.e., of Minkowski tensors of rank 1. After that we can extend to the case of the rank-3 tensor.

Since the transformation matrix is a *matrix*, it has two indices. These two indices are always arranged as one superscript and one subscript. The reason is that the transformation should take a contravariant vector to a contravariant vector:

$$\tilde{B}^\alpha = \Lambda_\mu^\alpha \, B^\mu ,$$

or a covariant vector to a covariant vector:

$$\tilde{C}_\alpha = \Lambda_\alpha^\mu \, C_\mu .$$

In each case, notice that the index summed over appears once as a superscript and once as a subscript. This is always true in index summation: the pair of indices which are summed over are one superscript and one subscript. In the transformations above, the remaining index than has the same type (contravariant or covariant) as the vector being transformed.

Note on horizontal placement of indices:

Above, we've written the Lorentz transformation matrices as Λ_α^μ, without a horizontal shift between the upper and lower index. Often, it is thought convenient to make clear which is the 'first' index and which is the 'second' index, thus ${\Lambda^\alpha}_\mu \, B^\mu$ or ${\Lambda^\mu}_\alpha$. If you think of the transformation as a matrix, then the upper index is the row index and the lower index is the column index. Thus you might see the form

$$\tilde{B}^\alpha = {\Lambda^\alpha}_\mu \, B^\mu \qquad \text{and} \quad \tilde{C}_\alpha = {\Lambda^\mu}_\alpha \, C_\mu = C_\mu \, {\Lambda^\mu}_\alpha$$

The second form for $\tilde{C}_\alpha$ (with C_μ written before Λ) is nice as it gives the feeling of C_μ as a row vector: a row vector multiplies a square matrix from the left, not from the right.

Below we might ignore this additional notational complexity and not bother with horizontal positioning of the indices. If the upper and lower indices do not have a horizontal displacement between them, assume the upstairs index to come first (leftmost) and the downstairs index to come last (rightmost).

Now, for a higher-rank tensor, one needs as many factors of the transformation matrix Λ as the rank. This makes sense because, e.g., a rank-2 tensor can be decomposed into two vectors, e.g,, $D^{\mu\nu} = B^\mu C^\nu$. Hence the transformation of $D^{\mu\nu}$ goes as follows:

$$\tilde{D}^{\alpha\beta} \;=\; \tilde{B}^\alpha \tilde{C}_\beta \;=\; \left(\Lambda^\alpha_\mu \, B^\mu\right)\left(\Lambda^\beta_\nu \, C^\nu\right) \;=\; \Lambda^\alpha_\mu \, \Lambda^\beta_\nu \, B^\mu \, C^\nu$$

Each vector brings with it one factor of transformation matrix Λ.

We could be careful about horizontal positioning:

$$\tilde{D}^{\alpha\beta} \;=\; \Lambda^\alpha{}_\mu \, \Lambda^\beta{}_\nu \, B^\mu \, C^\nu$$

Now the answer.

We have a rank three tensor. Hence

$$T^{\alpha\beta}_\gamma \quad\longrightarrow\quad \tilde{T}^{\alpha\beta}_\gamma \;=\; \Lambda^\alpha_\mu \, \Lambda^\beta_\nu \, \left(\Lambda^{-1}\right)^\sigma_\gamma \, T^{\mu\nu}_\sigma$$

Note: one needs to pair the indices such that they appear once as subscript and once as superscript, so that they get summed over through the Einstein summation convention.

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(c) A Minkowski tensor $U_{\mu\nu}$ is said to be symmetric if $U_{\mu\nu} = U_{\nu\mu}$. If a tensor is symmetric in one inertial frame, show that it is symmetric in all inertial frames.

(Partial) Solution/Hint →

In another inertial frame related by the transformation Λ, the tensor will be

$$\tilde{U}_{\alpha\beta} \;=\; \left(\Lambda^{-1}\right)^{\mu}{}_{\alpha} \left(\Lambda^{-1}\right)^{\nu}{}_{\beta}\, U_{\mu\nu}$$

The transpose is given by

$$\begin{aligned}
\tilde{U}_{\beta\alpha} &= \left(\Lambda^{-1}\right)^{\mu}{}_{\beta} \left(\Lambda^{-1}\right)^{\nu}{}_{\alpha}\, U_{\mu\nu} \\
&= \left(\Lambda^{-1}\right)^{\lambda}{}_{\beta} \left(\Lambda^{-1}\right)^{\sigma}{}_{\alpha}\, U_{\lambda\sigma} && \begin{cases}\text{changing dummy variables} \\ \text{(always allowed)}\end{cases} \\
&= \left(\Lambda^{-1}\right)^{\sigma}{}_{\alpha} \left(\Lambda^{-1}\right)^{\lambda}{}_{\beta}\, U_{\lambda\sigma} && \text{switching order of the two } \Lambda\text{'s} \\
&= \left(\Lambda^{-1}\right)^{\sigma}{}_{\alpha} \left(\Lambda^{-1}\right)^{\lambda}{}_{\beta}\, U_{\sigma\lambda} && \text{using the fact that } U \text{ is symmetric}
\end{aligned}$$

The last term is clearly $\tilde{U}_{\alpha\beta}$, Thus we have $\tilde{U}_{\beta\alpha} = \tilde{U}_{\alpha\beta}$, i.e., the tensor is symmetric in any inertial frame.

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6. Poincaré transformations. If Λ is a 4×4 matrix representing a Lorentz transformation, then transformations of the type

$$x' = \Lambda\, x + a$$

are known as Poincaré transformations. Here x and x' are column vectors 4×1 column vectors (4-vectors) representing spacetime coordinates of an event as seen from different frames, and a is a 4×1 column vector. In other words, a Poincaré transformation is a combination of a Lorentz transformation plus a possible shift of the space and time coordinates. We will denote this transformation as (Λ, a).

(a) Show that the result of two Poincaré transformations (Λ_1, a_1) and (Λ_2, a_2), applied successively, is the Poincaré transformation

$$(\Lambda_2\, \Lambda_1, \Lambda_2\, a_1 + a_2)$$

(Partial) Solution/Hint →

If

$$x' = \Lambda_1 x + a_1 \qquad \begin{cases} \text{Applying the} \\ \text{transformation } (\Lambda_1, a_1) \end{cases}$$

and if

$$x'' = \Lambda_2 x' + a_2 \qquad \begin{cases} \text{Applying the transformation } (\Lambda_2, a_2) \\ \text{on the result of the first transformation} \end{cases}$$

then we try relating the final four-vector to the original four-vector:

$$x'' = \Lambda_2 x' + a_2 = \Lambda_2 (\Lambda_1 x + a_1) + a_2 = (\Lambda_2 \Lambda_1)\, x + (\Lambda_2 a_1 + a_2)$$

Since Λ_1 and Λ_2 are Lorentz transformations, so is $\Lambda_2\, \Lambda_1$. Thus, x'' is obtained from x by applying the Poincaré transformation $(\Lambda_2\, \Lambda_1, \Lambda_2\, a_1 + a_2)$.

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(b) Are Poincaré transformations commutative?

(Partial) Solution/Hint →

Applying first (Λ_1, a_1) and then (Λ_2, a_2) gives us

$$(\Lambda_2, a_2) \star (\Lambda_1, a_1) \;=\; (\Lambda_2\,\Lambda_1, \Lambda_2\,a_1 + a_2)$$

as we have shown in the previous problem part.
Exchanging the roles of the two transformations gives us

$$(\Lambda_1, a_1) \star (\Lambda_2, a_2) \;=\; (\Lambda_1\,\Lambda_2, \Lambda_1\,a_2 + a_1)$$

Now Lorentz transformations are not commutative in general, as we have found in a previous problem set, using boosts in different directions. Thus $\Lambda_2\ \Lambda_1 \neq \Lambda_1\ \Lambda_2$ in general, and hence the two resulting Poincaré transformations obtained above are different.
Thus

$$(\Lambda_2, a_2) \star (\Lambda_1, a_1) \;\neq\; (\Lambda_1, a_1) \star (\Lambda_2, a_2)$$

in general, i.e., Poincaré transformations are not commutative.

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(c) Are Poincaré transformations associative?

(Partial) Solution/Hint →

The question is whether or not

$$(\Lambda_3, a_3) \star \Big[(\Lambda_2, a_2) \star (\Lambda_1, a_1)\Big]$$

is equal to

$$\Big[(\Lambda_3, a_3) \star (\Lambda_2, a_2)\Big] \star (\Lambda_1, a_1)$$

or not. We know how to combine two, so let's work out the result of combining three:

$$\begin{aligned}
&(\Lambda_3, a_3) \star \Big[(\Lambda_2, a_2) \star (\Lambda_1, a_1)\Big] \\
&\qquad = (\Lambda_3, a_3) \star (\Lambda_2\, \Lambda_1,\ \Lambda_2\, a_1 + a_2) \\
&\qquad = (\Lambda_3\, \Lambda_2\, \Lambda_1,\ \Lambda_3\, (\Lambda_2\, a_1 + a_2) + a_3) \\
&\qquad = (\Lambda_3\, \Lambda_2\, \Lambda_1,\ \Lambda_3\, \Lambda_2\, a_1 + \Lambda_3\, a_2 + a_3)
\end{aligned}$$

and combining three the other way:

$$\begin{aligned}
&\Big[(\Lambda_3, a_3) \star (\Lambda_2, a_2)\Big] \star (\Lambda_1, a_1) \\
&\qquad = (\Lambda_3\, \Lambda_2,\ \Lambda_3\, a_2 + a_3) \star (\Lambda_1, a_1) \\
&\qquad = (\Lambda_3\, \Lambda_2\, \Lambda_1,\ \Lambda_3\, \Lambda_2\, a_1 + (\Lambda_3\, a_2 + a_3))
\end{aligned}$$

Thus, the two combinations of the three transformations give the same final answer.

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(d) Does the set of all Poincaré transformations form a group?

(Partial) Solution/Hint →

Yes. Closure and associativity have been proven above.
To complete the proof, think separately about whether there is an identity, and given a Poincaré transformation (Λ_1, a_1), whether you can construct its inverse.

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